4 Risk, Surplus and With-Profits Business

4.1 Risk and Surplus

Premium and valuation bases state **what future outcomes the actuary expects.**But the future is **uncertain:**

- The actuary does not know who will die, or when.
- Premiums will be invested in assets such as bonds and equities, many of which earn a rate of return that cannot be guaranteed in advance.
- Future expenses are hard to predict accurately, especially because some may be affected by inflation.

All such sources of **uncertainty**introduce **risk**. The only statement that can be made with certainty is that **the future will not be exactly as assumed in the actuary's basis.**Each introduces the possibility of a good or a bad outcome, once the future *experience* is known.

For **insurance business**the following are good outcomes:

Interest:	**Experience**	$>$	**Basis**
Mortality:	**Experience**	$<$	**Basis**
Expenses:	**Experience**	$<$	**Basis**

For **annuity business**the following are good outcomes:

Interest:	**Experience**	$>$	**Basis**
Mortality:	**Experience**	$>$	**Basis**
Expenses:	**Experience**	$<$	**Basis**

We know, from the recursive relationships between reserves (discrete model) or Thiele's equation (continuous model) that the following is true:

> **If the premium and valuation bases are the same, *and* the experience follows them exactly, then the assets the office holds will always be exactly equal to the policy value.**

It follows, since we know that the experience will *not* exactly follow the premium/valuation basis, that at any time the office will hold assets *not* equal in value to the policy value. The difference is called ***surplus***Positive surplus is good, negative surplus (a deficit) is bad.

For a portfolio of long-term contracts, the ultimate surplus cannot be known **until all the contracts have expired.**But we can measure how much surplus emerges each year (or other short period). We need to do this in order to:

(a) **pay bonuses to with-profit policyholders**

(b) **pay dividends to shareholders**

(c) **measure the change in working capital**

because none of these can wait, possibly for decades, until the policies have expired.

Surplus is measured using the valuation basis. In the discrete model it goes as follows:

Step 1: Choose a valuation basis and **calculate the policy values at the start of the year.**

Step 2: Assume the office holds assets equal to the policy values at the start of the year.

Step 3: Calculate the value of the assets at the end of the year, adding interest *actually* earned and premiums *actually* paid and deducting the *actual* amounts of claims and expenses.

Step 4: Calculate the **policy values for policies in force at the end of the year.**

- Step 5: The surplus **emerging**during the year is the difference between (3) and (4).

In the continuous model, we measure the instantaneous **rate**at which surplus is emerging, as follows:

Step 1: Choose a valuation basis and **calculate the policy values at time t.**

Step 2: Assume the office holds assets equal to the policy values at time t.

Step 3: By substituting the ***actual*** **forces of interest and mortality (and possibly rate of expense) into Thiele's equation, calculate the rate at which the value of the assets is changing.**

Step 4: Using Thiele's equation, calculate the ***expected***rate of change of the reserve.

Step 5: The rate at which surplus is emerging is the difference between (3) and (4).

We can measure the amount of surplus emerging over any period by integrating the rate at which it emerges.

Consider a whole life policy, issued to (x):

- SA **of \$1 payable immediately on death**
- Premiums **are payable continuously**
- No expenses
- Valuation basis δ and μ_{x+t}
- Actual experience δ' and μ'_{x+t}

On the valuation basis we expect the reserve to change at rate:

$$\frac{d}{dt}V(t) = V(t)\,\delta + \bar{P}_x - \mu_{x+t}(1 - V(t)).$$

However, it is actually changing at rate:

$$\begin{aligned}\frac{d}{dt}V'(t) &= V(t)\,\delta' + \bar{P}_x - \mu'_{x+t}(1 - V(t)) \\ &= \frac{d}{dt}V(t) + S_t\end{aligned}$$

where S_t is the rate at which **surplus is earned (or emerges).** Therefore:

$$S_t = (\delta' - \delta)\, V(t) + \left(\mu_{x+t} - \mu'_{x+t}\right)(1 - V(t)).$$

Note:

- $\delta' - \delta$ represents **the excess of actual interest earned over expected interest.**
- $\mu_{x_t} - \mu'_{x+t}$ represents **the excess of the expected mortality over the actual experience.**

Given the need to avoid negative surpluses (losses) we next consider **how likely it is** that this will happen.

4.2 Risk Reserves

Our starting point is a life insurance company that prices its policies using the equivalence principle. Suppose the probability that the insured event occurs is small (e.g. a term assurance).

If the company sold **very few policies**then:

- the probability of making a loss **is very small; but**
- if a loss occurs, the size of the loss **could greatly exceed the premiums received.**

The limiting case is a person who does not buy insurance: effectively they become **an insurance company with one policy.**

Consider a portfolio of N **insurance policies**issued to identical, but independent, lives. For policy i define:

$$L_i = \textbf{random loss at inception}$$

Then **the L_i are i.i.d. random variables,**and because the equivalence principle has been used **E**$[L_i] = 0$. Let σ be the standard deviation of each L_i, i.e. **Var**$[L_i] = \sigma^2$.

The total loss on the insurance portfolio is:

$$L = \sum_{i=1}^{N} L_i$$

and the Law of Large Numbers (LLN) means that:

$$\lim_{N\to\infty} \frac{L}{N} = \mathbf{E}[L_1].$$

This underlies the intuitive reason for insurance to exist — **pooling large numbers of small risks leads to a more certain outcome.** (The motto of the Institute of Actuaries is *certum ex incertis.*)

However, the LLN does not imply that all risk can be **eliminated**, merely that it can be **collectivised**. By the Central Limit Theorem, as $N \to \infty$:

$$\frac{\sum_{i=1}^{N}(L_i - \mathbf{E}[L_i])}{\sqrt{N}} \sim \textbf{Normal}\left(0, \sigma^2\right)$$

or:

$$\frac{L}{\sigma\sqrt{N}} \sim \textbf{Normal}\left(0, 1\right).$$

Therefore:

$$\mathbf{P}\left[\textbf{Loss on Portfolio}\right] = \mathbf{P}[L > 0] = \frac{1}{2}.$$

This means that if we:

- make **realistic assumptions**in the premium basis;
- price using **the equivalence principle**; and
- sell a **large number**of policies

then the probability P$[L > 0]$ of making a loss on the whole portfolio approaches $1/2$.This is unacceptable to all stakeholders — **policyholders, regulators and the owners of the insurance company.**

However, all is not lost. We will show that as N increases the size of any loss is likely to be **much lower**as a proportion of the premiums received. Suppose the premium per policy is P (regular or single, it does not matter). Then the loss on the ith policy, as a proportion of the premium, is L_i/P. The total loss, as a proportion of **all premiums, is L/NP.**Since:

$$\frac{L}{\sigma\sqrt{N}} \sim \text{Normal}(0,1)$$

We have:

$$\frac{L}{NP} = \frac{\sigma}{P\sqrt{N}}.\frac{L}{\sigma\sqrt{N}} \sim \text{Normal}\left(0, \frac{\sigma^2}{P^2 N}\right).$$

Hence as N increases, the variance of the **proportionate**loss decreases as $1/N$. Hence the probability of loss approaches $1/2$, but the **probability of a *large* loss approaches 0.**

To use these results we need to compute $\sigma^2 =$ **Var**$[L_i]$. In general Var$[L_i]$ must be calculated *numerically* (easy on a spreadsheet) except in a few cases.

Consider a whole life contract to (x) with:

- SA of \$1 payable at end of year of death
- level annual premiums
- interest i% p.a.
- no expenses.

Then:

$$\begin{aligned} L_i &= v^{K_x+1} - P_x \ddot{a}_{\overline{K_x+1}|} \\ &= v^{K_x+1} - P_x \left(\frac{1 - v^{K_x+1}}{d} \right) \end{aligned}$$

$$= \left(1 + \frac{P_x}{d}\right) v^{K_x+1} - \frac{P_x}{d}.$$

This means that:

$$\begin{aligned} \text{Var}[L_i] &= \left(1 + \frac{P_x}{d}\right)^2 \text{Var}[v^{K_x+1}] \\ &= \left(1 + \frac{P_x}{d}\right)^2 \left({}^*A_x - (A_x)^2\right) \end{aligned}$$

where * indicates interest at $i^2 + 2i$,as usual.

We illustrate these ideas with an example.

Example

Whole life policies issued to lives aged 40:

- SA \$1 at end of year of death
- Mortality: A1967-70 ultimate

- Interest: 4% p.a.
- Given $A_{40} = 0.09422$ @ 8.16% p.a.

For the i^{th} policy:

$$\begin{aligned}\text{Var}[L_i] &= \left(1 + \frac{P_{40}}{d}\right)^2 \left({}^*A_{40} - (A_{40})^2\right) \\ &= 0.0370\end{aligned}$$

Now suppose we sell 10 such policies. Then:

$$\frac{L}{\sqrt{10(0.0370)}} \sim \text{Normal}(0, 1).$$

With this distribution we can calculate approximately some relevant probabilities. We might be interested in questions like:

(a) what loss will be exceeded 25% of the time?

(b) what loss will be exceeded 5% of the time?

Solution: We note that:

(a) $$\mathrm{P}\left[\frac{L}{\sqrt{10(0.0370)}} > 0.674\right] = 0.25$$

or **P**$[L > 0.410] = 0.25.$

(b) $$\mathrm{P}\left[\frac{L}{\sqrt{10(0.0370)}} > 1.645\right] = 0.05$$

or **P**$[L > 1.001] = 0.05.$

To put these in perspective, note that one year's premiums for these 10 policies is $10(0.01447) = 0.1447$, so there is:

(a) a 25% chance that the future loss will exceed **283% of one year's premiums**on the portfolio; and

(b) a 5% chance that the future loss will exceed **692% of one year's premiums**on the portfolio.

Similar calculations for larger portfolios are shown in Table 1.

Table 1: Relationship between probabilities of future loss and size of portfolio.

Portfolio Size N	Probability of 25% future loss exceeds Value	Probability of 25% future loss exceeds % 1 yr prems	Probability of 5% future loss exceeds Value	Probability of 5% future loss exceeds % 1 yrs prems
10	**0.410**	**283**	**1.001**	**692**
100	**1.296**	**89**	**3.164**	**219**
1000	**4.100**	**28**	**10.006**	**69**

Insurance depends on pooling large numbers of independent risks. But the insurer cannot alter the fact that P$[L > 0] \to 1/2$, it is a mathematical fact. The insurer has to find ways of managing or mitigating the risk of loss.

Suppose the company holds **extra assets of amount R, in addition to the reserve equal to the policy values.** This is called a **risk reserve.** Then instead of considering P$[L > 0]$, instead consider **P**$[L - R > 0]$. Effectively, the risk reserve allows the insurer to absorb losses up to R without being insolvent. Then:

(a) It is feasible for regulators to set out how big a risk reserve is needed, by specifying a suitably small **ruin probability.** For example, the regulator may require that:

$$\mathbf{P}[L - R > 0] \leq 0.01.$$

.

(b) The CLT shows that as the insurer sells more policies, the **risk reserve needed per policy** gets smaller.

Where does the risk reserve come from? The answer is it must come from capital. In fact it is the main reason why **insurance companies need capital.** Capital may be provided by investors, or by the policyholders themselves. Either way, the providers of capital

will expect to be rewarded.

Summary:

- For one policy, the probability of loss **may be small**but the size of the loss can be **much greater**than the size of the premium.
- The loss as a proportion of premiums can be **reduced**by selling more policies BUT this **increases**the probability of loss.
- The probability of loss can be mitigated by holding a risk reserve, available to meet losses up to a certain limit.
- The investors who supply the capital to set up the risk reserve must be rewarded.

This is an example of a technique of **risk management**, which is a large and important topic in its own right.

Next, we consider the keystone of the system which for over 200 years allowed **the policyholders themselves**to be the investors who provided the risk reserve.

4.3 Lidstone's Theorem

Lidstone's theorem states that, for whole life and endowment policies:

(i) as the rate of interest increases, **net premium policy values decrease.**

(ii) as the rate of mortality increases, **net premium policy values increase.**

We will discuss the result for the rate of interest.

This is not a trivial result since, as an example, for a whole life policy, the policy value at time t is:

$$V(t) = A_{x+t} - P_x \, \ddot{a}_{x+t}$$

and both the EPVs *and* the net premium on the right hand side are affected by the interest rate.

Proof of Lidstone's Theorem

We will prove the 'interest rate' result only for the special case of a whole life policy.

We consider the whole life policy with:

- Sum assured of \$1 payable immediately on death.
- Premiums payable continuously.

<u>Stage 1</u>

Suppose $\delta$and δ'represent two forces of interest ($\delta \neq \delta'$) with corresponding net premium policy values $V(t)$and $V'(t)$

We need to prove:

$$\delta < \delta' \implies V(t) > V'(t).$$

Now, given annual premium rates of $\bar{P}_x$ and $\bar{P}'_x$ (corresponding to the forces of interest δ and δ' respectively) and force of mortality μ_x, Thiele's
differential equations are:

$$\frac{d}{dt}V(t) = V(t)\,\delta + \bar{P}_x - \mu_{x+t}(1 - V(t)) \tag{4.3}$$

$$\frac{d}{dt}V'(t) = V'(t)\,\delta' + \bar{P}'_x - \mu_{x+t}(1 - V'(t)). \tag{4.4}$$

By adding and subtracting $V(t)\,\delta'$, equation (4.4) can be expressed as:

$$\begin{aligned}\frac{d}{dt}V'(t) =& V'(t)\,\delta' + V(t)\,\delta' - V(t)\,\delta' \\ &+ \bar{P}'_x - \mu_{x+t}(1 - V'(t)).\end{aligned} \tag{4.5}$$

Taking away equation **(4.5)**from equation **(4.3)**:

(4.6) $$\frac{d}{dt}\left(V(t)-V'(t)\right)=\left(\delta'+\mu_{x+t}\right)\left(V(t)-V'(t)\right) - C_t$$

where: $C_t = (\bar{P}'_x - \bar{P}_x) + (\delta' - \delta)V(t)$.Now let $f(t) = V(t) - V'(t)$. Then equation **(4.6)**can be written as:

(4.7) $$\frac{d}{dt}f(t)=\left(\delta'+\mu_{x+t}\right)f(t)-C_t.$$

Stage 2

Our aim was to prove:

$$\delta < \delta' \implies V(t) > V'(t)$$

which is equivalent to proving:

$$\delta < \delta' \implies f(t) > 0.$$

This is our new aim. By rewriting (4.7) as:

$$\frac{d}{dt} f(t) - (\delta' + \mu_{x+t}) f(t) = -C_t \tag{4.8}$$

we notice that this is a differential equation which can be solved using the integration factor method.

Reminder of the Integration Factor method

Recall that if we have an ordinary differential equation of the form:

$$\frac{d}{dt} y(t) - a(t) y(t) = b(t),$$

we define the integrating factor as:

$$IF(t) = \mathbf{exp}\left(-\int_0^t a(s)ds\right).$$

Then multiplying both sides of the differential equation by $IF(t)$, we get:

$$\frac{d}{dt}\left(y(t)IF(t)\right) = b(t)IF(t)$$

which can be solved by integration.

Continuing our proof of Lidstone's theorem, we consider equation (4.8):

$$\frac{d}{dt} f(t) - (\delta' + \mu_{x+t}) f(t) = -C_t$$

and define the integrating factor:

$$IF(t) = \textbf{exp} \left(- \int_0^t (\delta' + \mu_{x+s}) ds \right).$$

Multiplying both sides of equation (4.8) by $IF(t)$, we get:

$$\frac{d}{dt} (IF(t) f(t)) = -C_t IF(t). \tag{4.9}$$

Stage 3

Our aim from Stage 2 was to prove:

$$\delta < \delta' \implies f(t) > 0.$$

Since $IF(t) \geq 0$, this is equivalent to proving:

$$\delta < \delta' \implies IF(t)f(t) \geq 0.$$

This is our new aim. We note that:

- $f(0) = 0$
- $f(\omega - x) = 0$ **for some duration** ω**.**

Now integrating both sides of equation (4.9), we get:

$$\int_0^{\omega-x} \left[\frac{d}{dt} \left(IF(t) f(t) \right) \right] dt = - \int_0^{\omega-x} C_t IF(t) dt$$

which implies that:

$$IF(\omega - x) f(\omega - x) - IF(0) f(0)$$

$$= - \int_0^{\omega-x} C_t IF(t) dt.$$

Now, since $f(\omega - x) = 0$ and $f(0) = 0$, we have:

$$\int_{0}^{\omega-x} C_t IF(t) dt = 0.$$

Stage 4

Given that:

- $IF(t)$ is always ≥ 0 and
- C_t is always increasing

(Reminder: $C_t = (\bar{P}'_x - \bar{P}_x) + (\delta' - \delta)\, V(t)$)

the integral $\int_0^{\omega-x} C_t IF(t)dt$ can only be equal to zero if C_t changes sign at some duration.

Therefore, there exists a duration t_0 such that:

$$\begin{aligned} C_t < 0 \quad &\textbf{if} \quad t < t_0 \\ C_t \geq 0 \quad &\textbf{if} \quad t \geq t_0. \end{aligned}$$

Now consider $\dfrac{d}{dt}\,(IF(t)f(t)) = -C_t IF(t)$:

$$
\begin{aligned}
t < t_0 \quad &\Longrightarrow \quad C_t < 0 \\
&\Longrightarrow \quad -C_t IF(t) \geq 0 \\
&\Longrightarrow \quad \frac{d}{dt}\left(IF(t)f(t)\right) \geq 0.
\end{aligned}
$$

and:

$$
\begin{aligned}
t \geq t_0 \quad &\Longrightarrow \quad C_t \geq 0 \\
&\Longrightarrow \quad -C_t IF(t) \leq 0 \\
&\Longrightarrow \quad \frac{d}{dt}\left(IF(t)f(t)\right) \leq 0.
\end{aligned}
$$

We now have the following information on the function $IF(t)f(t)$:

(a) $IF(t)f(t) = 0$ **when** $t = 0$

(b) $IF(t)f(t) = 0$ **when** $t = \omega - x$

(c) $\dfrac{d}{dt}\left(IF(t)f(t)\right) \geq 0$ **when** $t < t_0$

(d) $\dfrac{d}{dt}\left(IF(t)f(t)\right) \leq 0$ **when** $t > t_0$.

Step 5

¿From these features we plot the graph of $IF(t)f(t)$ against t, which must look like Figure 3.

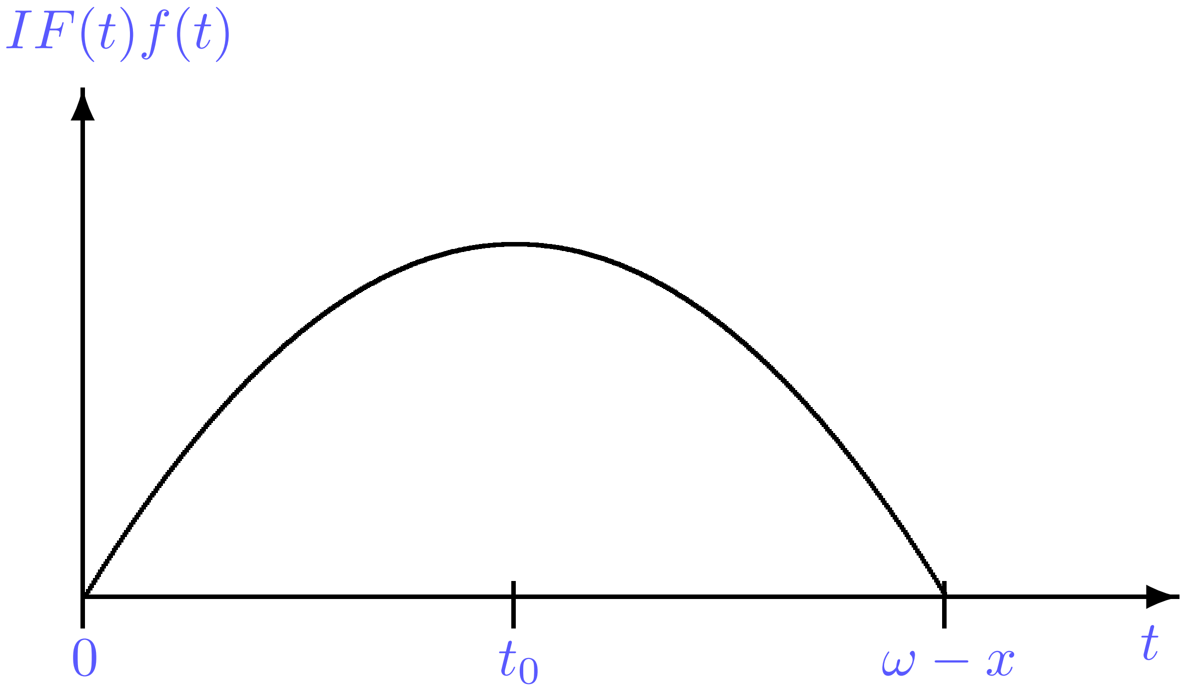

Figure 3: The graph of $IF(t)f(t)$.

Therefore:

$$
\begin{aligned}
\delta < \delta' \quad &\Longrightarrow \quad IF(t)f(t) > 0 \\
&\Longrightarrow \quad f(t) > 0
\end{aligned}
$$

$$\Longrightarrow \quad V(t) > V'(t)$$

which proves Lidstone's theorem.

Summary of Proof

Stage 1 Statement of Theorem to be proved:

$\delta < \delta' \implies V(t) > V'(t)$.

Stage 2 Reformulate theorem to give equivalent statement: $\delta < \delta' \implies f(t) > 0$,

where $f(t) = V(t) - V'(t)$.

Stage 3 Reformulate theorem again to give another equivalent statement:

$\delta < \delta' \implies IF(t)f(t) \geq 0$, where $IF(t)$ is an integrating factor:

$$IF(t) = \exp\left(-\int_0^t (\delta' + \mu_{x+s})ds\right).$$

Stage 4 Determine the characteristics of the

function $IF(t)f(t)$.

Stage 5 Plot $IF(t)f(t)$and conclude that:

$\delta < \delta' \implies IF(t)f(t) \geq 0$ and hence

$\delta < \delta' \implies V(t) > V'(t)$.

4.4 With-Profits Business

To re-cap, we have considered the following points so far:

- Most life insurance risks are **increasing** over time. If we insure risks over a long term with a level premium, we **build up a reserve.**
- We can use the **equivalence principle** to set premiums if we insure many independent risks (Law of Large Numbers).
- The CLT however shows that we need **an additional reserve (risk reserve) to manage or mitigate** the probability of overall loss on the insurance portfolio.
- Lidstone's theorem gives us a simple practical way of setting up additional reserves by basing all our calculations on **an artificially low rate of interest.**

Now, consider a policy which could be charged a premium of $\bar{P}'$ if a *realistic* force of interest δ' was used. However, to increase the reserves, a *lower* force of interest δ is used (Lidstone's theorem) such that the premium actually charged is $\bar{P}$.

Since:

$$\frac{d}{dt}V(t) = V(t)\,\delta + \bar{P} - \mu_{x+t}(1 - V(t))$$

and:

$$\frac{d}{dt}V(t) + S_t = V(t)\,\delta' + \bar{P} - \mu_{x+t}(1 - V(t))$$

then unless the experience is bad, surplus will emerge at rate S_t where:

$$S_t = (\delta' - \delta)\,V(t).$$

Finally, we note the following:

- This surplus has emerged because the policyholder was charged **an artificially higher premium.**

- It is fair that the surplus emerging be **returned to the policyholder.**
- Surplus can be returned to the policyholder by **declaring bonuses.**

Hence we have invented with-profits business.

Example: Faculty & Institute of Actuaries: Subject 105, April 2002.

100 people aged exactly 50 are each sold a 15-year endowment assurance policy with sum assured \$100,000. The premiums are paid annually in advance, and the sum assured is paid on maturity or at the end of the year of death.

The life insurance company's assumptions are:

Mortality: A1967–70 Ultimate, and the lives are independent with respect to mortality.

Interest: 6% per annum.

Expenses: Initial: \$300. Renewal: 2.5% of each premium, including the first.

Let P be the gross annual premium.

(1) State the gross future loss random variable for one policy at the outset.

(2) Using your answer to part (1) or otherwise, evaluate, in terms of P:

(a) the mean and variance of the loss (in present value terms) for a single policy at outset.

(b) the mean and variance of the loss (in present value terms) for the entire portfolio at outset.

Note: $A_{50:\overline{15}|}$ at 12.36% per annum = 0.20426

(3) Show what values the gross annual premium P can take if the company requires that the probability it incurs a loss (in present value terms) on the entire portfolio has to be less than 2.5%. Use the Normal approximation.

Solution: (1)

$$\begin{aligned} L_i &= PV[\text{Outgo}] - PV[\text{Income}] \\ &= 100K\, v^{\text{min}(K_{50}+1,15)} + 300 \\ &\qquad - 0.975\, P\, \ddot{a}_{\overline{\text{min}(K_{50}+1,15)}|} \end{aligned}$$

$$= \left(100K + \frac{0.975\,P}{d}\right) v^{\min(K_{50}+1,15)} + 300 - \frac{0.975\,P}{d}$$

(2)(a)

$$\begin{aligned} \mathsf{E}[L_i] &= 100K\, A_{50:\overline{15}|} + 300 - 0.975\, \ddot{a}_{50:\overline{15}|} \\ &= 44,695 - 9.577425\, P \end{aligned}$$

$$\begin{aligned} \mathsf{Var}[L_i] &= \left[100K + \frac{0.975\,P}{d}\right]^2 \mathsf{Var}\left[v^{\min(K_{50}+1,15)}\right] \\ &= \left[100K + \frac{0.975\,P}{d}\right]^2 \\ &\qquad \times \left\{ {}^{*}A_{50:\overline{15}|} - \left(A_{50:\overline{15}|}\right)^2 \right\} \end{aligned}$$

Where * indicates interest at $i^2 + 2\,i = 12.36\%$.

$$\Longrightarrow \; SD[L_i] = 0.084666\,[100K + 17.225\,P]$$

(2)(b) For 100 policies we have:

$$\begin{aligned} \text{Mean} &= 100\,\mathsf{E}[L_i] = \mu \\ \text{Variance} &= 100\,\mathsf{Var}[L_i] = \sigma^2 \\ \text{Std Deviation} &= 10\,SD[L_i] = \sigma. \end{aligned}$$

(3) We want: P $\{\text{Loss} > 0\} < 0.025$:

$$\Longrightarrow \; \mathsf{P}\left\{\frac{\text{Loss} - \mu}{\sigma} > \frac{0-\mu}{\sigma}\right\} < 0.025$$

$$\Longrightarrow \; \mathsf{P}\left\{z > -\frac{\mu}{\sigma}\right\} < 0.025$$

$$\implies \frac{\mu}{\sigma} < -1.96$$

$$\implies 100(44,695 - 9.577425\, P)$$

$$< -1.96(84,666 + 14.5837\, P)$$

$$\implies P \geq 4,988.86.$$