3 Life Insurance and Differential Equations

3.1 Model specification

Our model was specified by the assumption that the remaining lifetime at age x was a random variable T_x with a continuous distribution. That distribution is completely described if we know *either*:

- the c.d.f. $F_x(t)$, **or**
- the p.d.f. $f_x(t)$, **or**
- the force of mortality μ_{x+t}.

One way is to assume that the first of these is true, and that the c.d.f. $F_x(t)$ is known (meaning that **it has been estimated using suitable data**). It is summarised numerically by the life table l_x, which is used to compute EPVs.

Here, we assume the third of these is true: we are given the force of mortality at all ages. We can visualise this with the following picture:

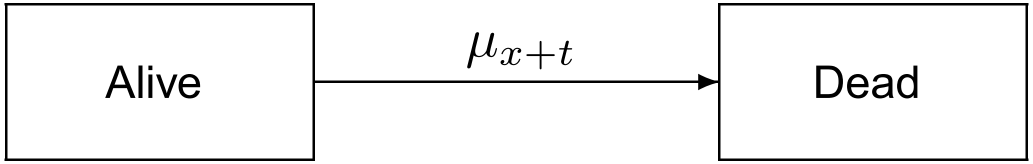

Figure 1: A model that can be used to represent a life insurance contract.

The problem then becomes: **given μ_{x+t}, how can we find all the probabilities and EPVs needed to find premiums and policy values?**

The answer lies in solving ordinary differential equations (ODEs). We have already seen the following ODE which allows us to find **survival probabilities:**

$$\frac{d}{dt}\,{}_tp_x = -\,{}_tp_x\,\mu_{x+t}.$$

We will call this the **Kolmogorov equation**for reasons that will become clear later. Combined with the initial condition ${}_0p_x = 1$,we wish to solve this for all $t > 0$.

We have also seen Thiele's differential equation, which for a whole-life contract was:

$$\frac{d}{dt}V(t) = V(t)\delta + \bar{P}_x - \mu_{x+t}(1 - V(t)).$$

This allows us to calculate policy values, but in fact the EPVs of *any* cashflows contingent upon the model pictured above can be found. The boundary condition in this case is a **terminal condition**rather than an **initial condition**(we will see this in Section 3.5).

Very rarely, μ_{x+t} has such a simple form (e.g. a constant) that we can find **explicit solutions, i.e. simple formulae**to one of both of these ODEs. Nearly always, however, we must solve them numerically, using a computer. We will do this using the simplest possible method, an **Euler scheme.**

3.2 A comment on this approach

At first sight, this approach seems more complicated than just using a life table. Indeed it is, moreover it is distinctly modern because before today's computer power became available, the numerical solution of ODEs was a formidable task.

The payoff will come later, when we consider more complicated contracts. Here is an advance look at a model that underlies **disability insurance**, in which someone who is too sick to work receives **a regular income to replace their lost earnings.**

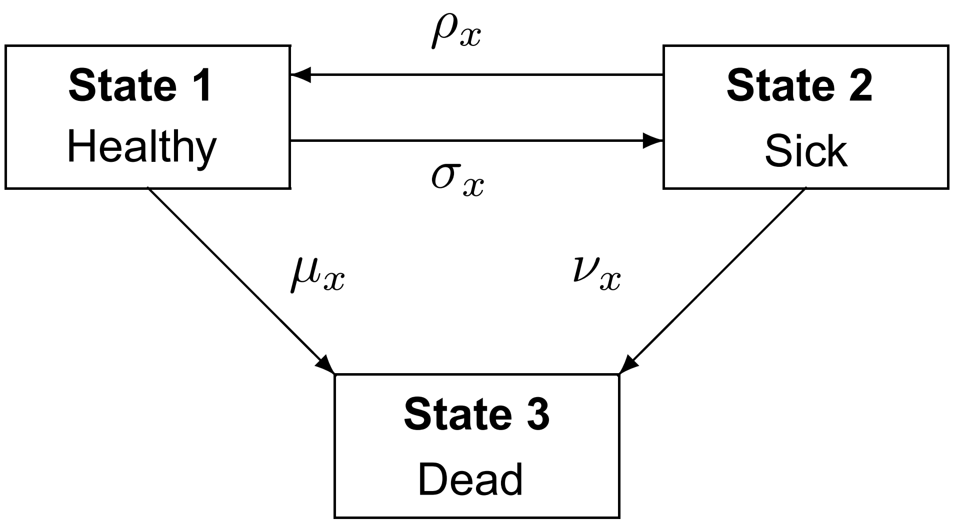

Figure 2: A model that can be used to represent a disability insurance contract.

We *could* try to formulate this model in terms of the random times at which events take place, **the analogues of the random lifetime T_x.** This turns out to be extremely difficult and complicated, and computing probabilities and EPVs by this approach is a nightmare.

But, if we take as given the ‘forces’ governing transitions between the three states, **the analogues of the force of mortality μ_{x+t},** it turns out that versions of the Kolmogorov equation and Thiele’s equation can be written down very easily, and solving them numerically is just as easy as in the simple life-death model. That is why we take this approach.

3.3 An Euler scheme for solving an ODE

We show, below, how a simple recursive scheme **Euler scheme**can be used to solve an ODE, given an initial condition.

We are given the differential equation

$$f'(t) = g(f(t), t)$$

and an initial condition $f(0) = c$.

For example, put:

$$
\begin{aligned}
f(t) &= {}_tp_x \\
g(f(t),t) &= -f(t)\,\mu_{x+t} \\
f(0) &= 1.
\end{aligned}
$$

This sets up an Euler scheme to solve the Kolmogorov equation, conditional on the life being alive at age x.

First, we choose a suitable **step size, denoted h.** This should be as small as possible, consistent with the computing capacity available. In the tutorials there is an opportunity to experiment with different step sizes. For example, 0.1 year or 0.01 year might be chosen.

The Euler scheme advances the solution of the ODE in steps of length h, starting with the initial condition. It does so by assuming that **the function $f(t)$ is approximately linear.** This assumption gets more reasonable as the step size decreases.

Suppose the solution has been advanced by k steps, that is, we have found approximate values of

$$f(h), f(2h), \ldots, f(kh).$$

If $f(t)$ **actually was linear on** $[kh,(k+1)h]$**,**it would be a straight line with slope $f'(t)$, and the following would be *exactly* true:

$$f((k+1)h) = f(kh) + h\, f'(kh).$$

Although $f(t)$ is not, in general, linear anywhere (certainly not in the case of the Kolmogorov equation representing human mortality) if the step size h is small enough, the error we make by assuming $f(t)$ to be *approximately* linear may be small enough to be acceptable. Therefore, we apply Equation (3.3) successively, starting with the initial condition $f(0)$. This is the Euler scheme.

3.4 Solving the Kolmogorov equation

Applying Euler's method to the Kolmogorov equation we have:

$$\begin{aligned} {}_{h}p_x &\approx {}_{0}p_x + h\,\frac{d}{dt}{}_{t}p_x\bigg|_{t=0} \\ &= {}_{0}p_x + h\left[-{}_{0}p_x\,\mu_{x+0}\right] \\ &= 1 - h\,\mu_x \end{aligned}$$

and:

$$\begin{aligned} {}_{2h}p_x &\approx {}_{h}p_x + h\,\frac{d}{dt}{}_{t}p_x\bigg|_{t=h} \\ &= {}_{h}p_x + h\left[-{}_{h}p_x\,\mu_{x+h}\right] \\ &= 1 - h\,\mu_x - h\,(1 - h\,\mu_x)\mu_{x+h} \end{aligned}$$

and:

$$
\begin{aligned}
{}_{3h}p_x &\approx {}_{2h}p_x + h\, \frac{d}{dt}\, {}_t p_x \Big|_{t=2h} \\
&= {}_{2h}p_x + h \left[-{}_{2h}p_x\, \mu_{x+2h} \right]
\end{aligned}
$$

and so on. This can easily be programmed:

- in a spreadsheet (see tutorials)
- using any programming language (e.g. Visual Basic, C)
- using any standard maths package (e.g. Maple, Matlab, Mathematica).

In fact, any standard maths package will probably solve ODEs as a standard feature, so the user need not program the Euler scheme themselves.

Note that the Euler scheme is the simplest numerical method for solving ODEs (which is why we use it). It is also, however, among the slowest and least accurate methods. Many better methods are available, described in any good text on numerical analysis (any maths package will certainly use one of them). We would probably not use an Euler scheme in practice.

3.5 Solving Thiele's differential equation

We can solve Thiele's differential equation either:

(a) forwards from the ***initial***condition

$$V(0) = 0$$

(b) or backwards from ***terminal***conditions like:

$$V(n) = 0 \quad \text{or} \quad V(n) = 1.$$

Note: the direction of time is immaterial when solving ODEs. Suppose the function $f(t)$ satisfies the ODE:

$$f'(t) = g(f(t), t)$$

and satisfies the initial condition $f(0) = c_0$ and the terminal condition $f(T) = c_T$ for some time $T > 0$. Then we can either:

- choose a **positive step size h and advance the solution forward from $f(0)$,** or
- choose a **negative step size $-h$ and advance the solution backwards from $f(T)$.**

In the first case, when we reach time T we should obtain the correct value $f(T) = c_T$ (to within numerical error) and, in the second case, when we reach time 0 we should obtain the correct value $f(0) = c_0$.

In solving Thiele's equation, we might know an initial value, but often we do not. In fact the initial value is often the unknown quantity whose value we want to find. But **we almost always know a terminal value, because:**

- a policy with no maturity benefit (e.g. a term assurance) always has policy value 0 at expiry; and

- a policy with a maturity benefit (e.g. an endowment) always has policy value equal to the maturity benefit just before expiry.

Therefore, we will solve Thiele's equation backwards from such terminal conditions.

The following are the first few steps of an Euler scheme with step size $-h$ for a term assurance contract sold to a life age x, with sum assured \$1 payable immediately on death within n years and premium payable continuously at rate $\bar{P}$ per annum. The policy value at duration t is denoted $V(t)$ as usual.

$$
\begin{aligned}
f(t) &= V(t) \\
g(f(t), t) &= V(t)\delta + \bar{P} - \mu_{x+t}(1 - V(t)) \\
f(n) &= 0.
\end{aligned}
$$

Applying Euler's method with step size $-h$ we have:

$$
\begin{aligned}
V(n-h) &\approx V(n) - h\,\frac{d}{dt}V(t)\Big|_{t=n} \\
&= V(n) - h[V(n)\delta + \\
&\qquad \bar{P} - \mu_{x+n}(1 - V(n))]
\end{aligned}
$$

and:

$$
\begin{aligned}
V(n-2h) &\approx V(n-h) - h\,\frac{d}{dt}V(t)\Big|_{t=n-h} \\
&= V(n-h) - h[V(n-h)\delta + \\
&\qquad \bar{P} - \mu_{x+n-h}(1 - V(n-h))]
\end{aligned}
$$

and:

$$
\begin{aligned}
V(n-3h) &\approx V(n-2h) - h\,\frac{d}{dt}V(t)\Big|_{t=n-2h} \\
&= V(n-2h) - h[V(n-2h)\delta + \\
&\quad \bar{P} - \mu_{x+n-2h}(1 - V(n-2h))]
\end{aligned}
$$

and so on.

3.6 Thiele's equation as a general tool

We have introduced Thiele's equation as a method of finding **policy values.**But, if we consider ***any*** **future benefits**as being paid for by a single premium at outset, we have:

$$
\begin{aligned}
\text{Premium} &= \text{EPV[Benefits]} \\
&= \text{EPV[Benefits]} - \text{EPV[Future premiums]}
\end{aligned}
$$

$$= \quad \text{Policy value at outset}$$

because there are *no* future premiums. In other words, we can use Thiele's equation to compute the EPV of **any benefits at all,**not just policy values of contracts with regular premiums.

Example: Consider a non-profit endowment with term 10 years sold to (30). The sum assured of $50,000 is payable on maturity, or immediately on earlier death. The force of mortality is μ_{x+t} and the force of interest is δ, and there are no expenses. Find the annual rate of premium $\bar{P}$.

Method 1: Set up Thiele's equation for the policy value:

$$\frac{d}{dt}V(t) = V(t)\,\delta + \bar{P} - \mu_{x+t}(50,000 - V(t)).$$

The boundary condition is $V(10) = 50,000$, but $\bar{P}$ is unknown. Since this will all be done with a computer we can find $\bar{P}$ easily **by trial and error.**

Method 2: Set up Thiele's equation separately for a benefit of $1 and for an annuity of $1 per annum. These are, respectively:

$$\begin{aligned}
\frac{d}{dt}V_A(t) &= V_A(t)\,\delta + 0 - \mu_{x+t}(1 - V_A(t)) \\
\frac{d}{dt}V_a(t) &= V_a(t)\,\delta + 1 - \mu_{x+t}(0 - V_a(t))
\end{aligned}$$

with boundary conditions $V_A(10) = 1$ **and** $V_a(10) = 0$**.**Then by the usual equivalence principle:

$$\bar{P} = 50,000\,V_A(0)/V_a(0).$$