2 Policy Values and Reserves

[Handout to accompany this section: reprint of ‘Life Insurance’, from *The Encyclopaedia of Actuarial Science*, John Wiley, Chichester, 2004.]

2.1 The Life Office’s Balance Sheet

Every enterprise, life offices included, needs a balance sheet showing the values of its assets and its liabilities. If assets exceed liabilities in value, the company is solvent, otherwise it is insolvent.

What are the values of the assets and liabilities of a life office?

- The majority of the assets will be investments such as bonds, equities and property. They have been purchased with policyholders’ premiums and they will be held in a fund from which benefits will be paid out.
- The liabilities are the promises made to pay benefits in future, against which can be set

the policyholders' promises to pay premiums in future.

It is relatively easy to quantify the assets, since investments can be valued **(e.g. bonds and equities have a market value).**The question is, how do we quantify or *value* the liabilities?

The answer is: the actuary **makes estimates of future interest rates, mortality and possibly expenses — called a *valuation basis* — and using these, calculates the assets required to meet the expected future payments to policyholders.**The resulting number is called a *policy value*. The total of all the policy values is then the liability shown in the balance sheet.

The **reserve**is then the portfolio of assets held by an insurer in order to ensure that it can meet its future liabilities. The reserve must be at least equal to the total of policy values.

An insurance company needs to hold a reserve in respect of a life insurance policy because **the premium income does not coincide with the benefits and expenses outgo.**

The outgo usually **increases**with duration, whereas the premium income is usually **level.**

Example:

Consider a whole life policy issued to a life aged 30 with:

- Sum assured: \$100,000 payable at the end of the year of death
- Interest: 5% p.a.
- Mortality: A1967-70 ultimate mortality.

Then the level annual premium payable is **\$724.**

In year	Cost of cover	Prem Income
5	**100,000** $q_{34} = £79$	\$724
30	**100,000** $q_{59} = £1299$	\$724

By year 30 the premium income is **insufficient**to meet the cost of cover.

2.2 The Loss Random Variable, L_t, and its Mean

For a whole life policy issued to (x) with sum assured \$1 and level annual premiums, the annual premium payable given some mortality and interest assumptions (and ignoring expenses), is:

$$P_x = \frac{A_x}{\ddot{a}_x}.$$

In effect we determined the premium that makes:

$$\text{EPV[benefits]} = \text{EPV[premiums]}$$

$$\mathsf{E}[v^{K_x+1}] = \mathsf{E}[P_x \, \ddot{a}_{\overline{K_x+1}|}]$$

$$\Rightarrow \quad \mathsf{E}[v^{K_x+1}] - \mathsf{E}[P_x \, \ddot{a}_{\overline{K_x+1}|}] = 0$$

$$\Rightarrow \quad \mathsf{E}[v^{K_x+1} - P_x\, \ddot{a}_{\overline{K_x+1}|}] = 0.$$

Definition:

At any future time t, we define the random variable L_t, called the **loss at time t**, to be the difference between the present values, at time t, of future outgo and of future income:

$$L_t = \text{PV[Future Outgo} - \text{Future Income]}.$$

The expected value of this random variable is called **the *prospective policy value* of the contract at time t, and is denoted $V(t)$.**

As a specific example, the loss just before a premium payment date t is:

$$L_t = v^{K_{x+t}+1} - P_x \ddot{a}_{\overline{K_{x+t}+1}|}$$

Therefore the prospective policy value at that time is:

$$V(t) = \mathsf{E}[L_t] = A_{x+t} - P_x \ddot{a}_{x+t}.$$

Some notes on policy values:

(1) Convention: The policy value above was calculated just **before** the premium then due, at **an integer duration.**It turns out that policy values of this kind make the formal mathematics as simple as it is possible to make it, so it is often **assumed**that policy values are calculated **just before premium due dates**when working out the mathematics.

It is important to realise, however, that in practice a life office will usually have to calculate the policy values for all its in-force business on **a fixed calendar date.**Only by coincidence will this be an integer policy duration, for any randomly chosen policy.

(2) Policy values and reserves: There is a distinction between **a policy value** and **a reserve**.

- A policy value is **a number calculated by an actuary on the basis of some assumptions about future mortality, interest, etc.**

- A reserve (now called a provision) is **a quantity of actual assets (*e.g.* equities, bonds) whose value at least equals the policy value.**

  This is the portfolio of assets that the company hopes is sufficient to meet the future liabilities.

(3) Applications of policy values and reserves:

- Reserves are needed to pay surrender values to policy-holders who surrender, if the policy terms allow the payment of a surrender benefit.

- Policy values and reserves are necessary for calculations related to **policy alterations and conversions.**

  For example, a policy-holder who has held a whole life policy for 8 years may request to change the cover to that of an endowment assurance, if the policy terms allow.

- Policy values and reserves are important for **demonstrating solvency.**

  On at least one fixed day each year life offices are required to show that they are solvent by demonstrating that they have sufficient assets to set aside reserves at least equal to the total of the policy values of their in-force business.

- For with-profit policies, policy values and reserves are used **in determining bonus levels.**

- They are also used by proprietary companies as part of the determination of **dividends to be paid to shareholders.**

(4) Valuation basis: The assumptions used to calculate a policy value — interest, mortality and possibly expenses — are collectively called the **valuation basis.**

(5) Premium calculation: The calculation of premiums is a special case of the calculation of policy values. We find the value of P for which **the expected future loss at outset is zero.**

For the above example, this means:

$$\mathsf{E}[L_0] = V(0) = A_x - P_x \ddot{a}_x = 0$$

This gives the required result:

$$P_x = \frac{A_x}{\ddot{a}_x}$$

(6) Boundary conditions for policy values: There are often simple and obvious boundary conditions for policy values at outset and at the expiry of a policy.

- If the basis used to calculate the policy values and that used to calculate the premium are the same, then:

$$V(0) = 0$$

- If a policy does not pay a benefit at maturity (e.g. an n-year term assurance policy), then:

$$V(n) = 0$$

- For a policy paying out a maturity benefit of S after the expiry at time n, then:

$$V(n) = S$$

2.3 Net premium policy values

There are several different types of policy value, suitable for different purposes. The simplest is the **net premium policy value.**We assume the valuation basis to be given.

(a) The premium used in the policy value calculation is ***not* the gross premium (or office premium) — we compute an *artificial* premium using the valuation basis.**This is called the valuation premium, the valuation net premium, or just the net premium.

(b) Expenses are **ignored**throughout, indeed there are no expenses in a net premium valuation basis.

(c) For with-profit policies, the benefits valued include any bonuses that have already been declared. Future bonuses that have not been declared are ignored.

Until recently, in most European countries, it was mandatory that policy values be calculated on a net premium basis.

Note: It is assumed that the difference between the “office premium” and the “net premium” will cover the expenses and, in the case of with-profit policies, future bonuses. It is therefore important to ensure that **the net premium calculated and used is *less* than the office premium being charged.**

2.4 Gross premium policy values

Net premiums policy values ignore certain features of the policy, such as the office premium, expenses and future bonuses. In contrast, a **gross premium policy value**allows for all these features.

(a) we use the **office premium (actual premium being paid)** in the policy value calculation.

(b) We include the EPV of future expenses.

(c) in the case of with-profit policies, the benefits valued should include **any bonuses already declared *and* some assumed level of future bonus declarations.**

Consequently, a gross premium valuation basis will include assumed future levels of bonus and expenses.

In the case of without-profit policies, the future loss is then:

$$PV_{\text{future loss}}$$

$$= PV_{\text{benefits}} + PV_{\text{expenses}} - PV_{\text{premiums}}$$

and its mean, the gross premium prospective policy value, is:

$$\mathrm{E}\left[\mathrm{PV}_{\text{future loss}}\right]$$

$$= \mathrm{E}\left[\mathrm{PV}_{\text{benefits}} + \mathrm{PV}_{\text{expenses}} - \mathrm{PV}_{\text{premiums}}\right]$$

For with-profit policies, the gross premium prospective policy value, allowing for declared and future bonuses, is:

$$\begin{aligned}\mathrm{E}\left[\mathrm{PV}_{\text{future loss}}\right] \quad = \quad & \mathrm{E}\left[\mathrm{PV}_{\text{basic benefit}}\right. \\ & + \mathrm{PV}_{\text{declared bonuses}} \\ & + \mathrm{PV}_{\text{future bonuses}} \\ & + \mathrm{PV}_{\text{expenses}} \\ & \left. - \mathrm{PV}_{\text{premiums}}\right]\end{aligned}$$

Policy values calculated in this way are called **bonus reserve policy values.**

Expenses

In the expression for gross premium policy value we note that the sum assured, the premiums and the bonuses are aspects of a policy that we have met before. When a company quotes a premium, it would have calculated the premium under some assumptions on the level and timing of future expenses. The company hopes that the actual expenses will not exceed those assumed.

Expenses fall into the broad groups of:

(a) **Initial expenses:** These are incurred at the outset or during the first few years and can be particularly heavy, often exceeding the first or second years' premiums in value. Initial expenses are largely due to costs of paying commission, and head office expenses like underwriting and setting up the policy on computer systems.

(b) **Renewal expenses:** These are incurred every year or month (perhaps except the first) and are typically due to renewal commission, premium collection costs and claims handling.

Expenses are expressed in one of the following ways:

(a) As per-policy expenses (e.g. $\$100$). Per-policy expenses are not related to the level of the benefit and may be subject to inflation.

(b) As a percentage of the premium.

(c) As a percentage of the benefit.

2.5 Notation

The standard actuarial notation for various kinds of policy may also be extended to policy values using the symbol ${}_tV$, where t is the duration, and appending the usual buscripts and superscripts. For example:

Policy	Premiums	Policy Value
Whole-life	Annual	${}_tV_x$
Whole-life	m-thly	${}_tV_x^{(m)}$
n-year endowment	Annual	${}_tV_{x:\overline{n}\rceil}$

and so on. However, we will just use the simpler notation $V(t)$, leaving it to the context to determine the type of policy, and modifying it whenever needed to distinguish between different policies.

2.6 Recursive relations between policy values

Consider a non-profit whole life policy issued to (x), with sum assured \$1 payable at the end of year of death.

Suppose premiums and reserves are calculated **on the same basis**and there are no expenses. Hence the annual premium is $P_x = A_x/\ddot{a}_x$.

At time t since inception while the policy is still in force, the policy value is $V(t)$.The office is assumed to hold a reserve of assets equal in value to $V(t)$.

If the life office earns interest on its assets at the rate assumed in the valuation basis, denoted i, then at time $t+1$, just before payment of any benefits then due, we have:

$$(V(t) + P_x)(1+i)$$

Now, during the year the policyholder can either:

(i) die — with probability q_{x+t}— in which case the policy pays **the sum assured of $1**

(ii) survive — with probability p_{x+t}— in which case the reserve should be sufficient to **satisfy the policy value at time $t+1$ i.e. $V(t+1)$.**

Therefore we should *expect* (for we have not yet proved it) that:

$$(V(t) + P_x)(1+i) = q_{x+t} + p_{x+t}V(t+1).$$

Formal proof of the recursive relationship for a whole life policy.

$$(V(t) + P_x)(1+i) = (A_{x+t} - P_x\, \ddot{a}_{x+t} + P_x)\,(1+i)$$

$$\begin{aligned}
&= [A_{x+t} - P_x(\ddot{a}_{x+t} - 1)]\,(1+i) \\
&= [A_{x+t} - P_x(\, a_{x+t})]\,(1+i) \\
&= \frac{vq_{x+t} + vp_{x+t}A_{x+t+1} - P_x(vp_{x+t}\ddot{a}_{x+t+1})}{v} \\
&= q_{x+t} + p_{x+t}\,(A_{x+t+1} - P_x\,\ddot{a}_{x+t+1}) \\
&= q_{x+t} + p_{x+t}\,V(t+1).
\end{aligned}$$

Importance of recursive relationships.

(a) They can be interpreted as a statement about the evolution of the life office's **balance sheet,** thus: If the office invests in assets that earn the rate of return assumed in the basis, and mortality and (possibly) expenses are also exactly as in the basis, then **at all times the office will hold assets exactly equal in value to the policy value, i.e. the liability.**

(b) These recursive relationships are valid for all forms of benefit, even benefits that may **depend on the reserve.**

(c) They define the policy values at **non-integer durations.** For example, consider a whole-life policy issued to (x), with benefit \$1 payable at the end of the year of death, and annual premiums:

(i) Given $V(t+1)$ evaluate $V(t+0.75)$:

$$V(t+0.75)(1+i)^{0.25}$$

$$= {}_{0.25}q_{x+t+0.75} + {}_{0.25}p_{x+t+0.75}\, V(t+1).$$

Note: For annual premium policies, premiums are payable at durations t and $t+1$, not $t + 0.75$.

(ii) Given $V(t)$ evaluate $V(t + 0.75)$:

$$(V(t) + P_x)(1 + i)^{0.75}$$

$$= {}_{0.75}q_{x+t} v^{0.25} + {}_{0.75}p_{x+t} V(t + 0.75).$$

Note: If the life dies between time t and time $t + 0.75$, the death benefit will be payable at time $t + 1$ (not time $t + 0.75$).

Example: Consider a 30-year endowment sold to a life age 30, with death benefit payable at the end of the year of death. Using A1967–70 ultimate mortality and 4% interest, evaluate:

(i) $V(4.5)$, assuming premiums are payable annually, given that $V(5) = 0.099342$.

(ii) $V(14.75)$, assuming premiums are payable quarterly.

Solution:

(i) No premium is paid between $t = 5$ and $t = 4.5$, hence:

$$V(4.5)(1+i)^{0.5} = {}_{0.5}q_{34.5} + {}_{0.5}p_{34.5}\, V(5).$$

How do we calculate ${}_{0.5}p_{34.5}$?

If we assume that the force of mortality is constant between integer ages, then:

$$\left({}_{0.5}p_{34.5}\right)^2 = p_{34}$$

¿From the tables, this gives us:

$${}_{0.5}p_{34.5} \approx 0.999605$$

and substitution gives us:

$$V(4.5) \approx 0.097762$$

(ii) Premiums are paid quarterly. Let $P^{(4)}_{30:\overline{30|}}$ be the annual amount of premium, then:

$$P^{(4)}_{30:\overline{30|}} = \frac{A_{30:\overline{30|}}}{\ddot{a}^{(4)}_{30:\overline{30|}}} = 0.01856$$

and:

$$V(15) = A_{45:\overline{15|}} - P^{(4)}_{30:\overline{30|}}\,\ddot{a}^{(4)}_{45:\overline{15|}} = 0.36284$$

Now the recursive relationship is:

$$\left(V(14.75) + 0.25\,P^{(4)}_{30:\overline{30|}}\right)(1+i)^{0.25}$$

$$= {}_{0.25}q_{44.75} + {}_{0.25}p_{44.75}\,V(15).$$

If we assume a constant force of mortality between exact ages 44 and 45 we can calculate

${}_{0.25}p_{44.75}$ as:

$$ {}_{0.25}p_{44.75} \approx (p_{44})^{\frac{1}{4}} = 0.99416 $$

Then substitution gives us:

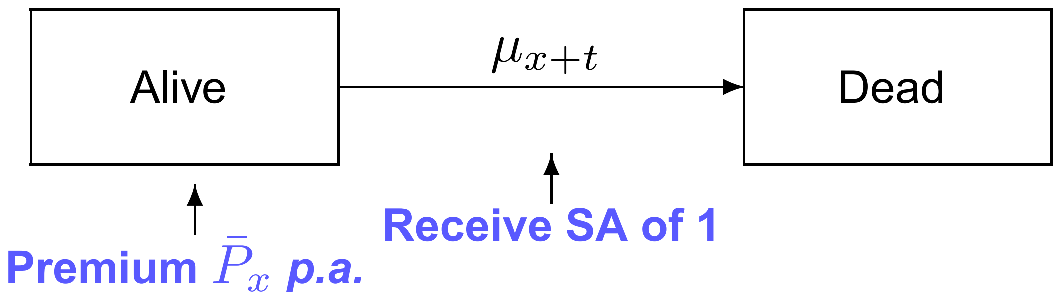
$$ V(14.75) = 0.35503. $$

2.7 Thiele's differential equation.

When benefits are payable immediately on death, and premiums are paid continuously, the recursive relationship between reserves becomes a differential equation, called **Thiele's differential equation.** We will derive it in the particular case of a whole-life non-profit insurance, **issued to (x), t years ago.** We assume:

- Sum assured of $\$1$ is paid immediately on death.
- Premiums at rate $\bar{P}_x$ *p.a.* are payable continuously.

This policy can be represented as:



We suppose that the premium and valuation bases are the same, and are given by force of interest δ and force of mortality μ_{x+t}. The policy value at time t years since policy inception is again denoted $V(t)$.

[Note: In standard actuarial notation, a bar is used to denote policy values in the continuous model, e.g. ${}_t\bar{V}_x$ and ${}_t\bar{V}_{x:\overline{n}|}$. We do not use this notation.]

In the small interval of time dt:

(i) The reserve earns interest of:

$$V(t)\,\delta\,dt.$$

(ii) Premium income is:

$$\bar{P}_x\,dt.$$

Therefore at time $t + dt$ we should have funds of:

$$V(t) + V(t)\,\delta\,dt + \bar{P}_x\,dt.$$

Applying the same reasoning as explained the recursive relationships, if all goes exactly as assumed in the bases, this should be just enough to cover the cost of paying expected death claims, and setting up the required reserve at time $t + dt$. That is, it should be equal to:

$$\mu_{x+t}\,dt + (1 - \mu_{x+t}\,dt)V(t + dt).$$

Rearranging this equation, we get:

$$\begin{aligned} V(t + dt) - V(t) \quad &= \quad V(t)\,\delta\,dt + \bar{P}_x\,dt \\ &\quad\quad -\mu_{x+t}(1 - V(t + dt))\,dt. \end{aligned}$$

Divide by dt to get:

$$\frac{V(t+dt) - V(t)}{dt} = V(t)\,\delta + \bar{P}_x - \mu_{x+t}(1 - V(t+dt)).$$

Taking the limit as $dt \to 0$, we obtain **Thiele's differential equation:**

$$\frac{d}{dt}V(t) = V(t)\delta + \bar{P}_x - \mu_{x+t}(1 - V(t)).$$

The above is intuitive, not a proof.

Proof of Thiele's equation (for a whole life policy)

We will need the following two results, whose proofs are tutorial questions:

$$(2.1) \qquad V(t) = 1 - \frac{\bar{a}_{x+t}}{\bar{a}_x}$$

$$(2.2) \qquad \frac{d}{dt}(\bar{a}_{x+t}) = \mu_{x+t}\,\bar{a}_{x+t} - \bar{A}_{x+t}.$$

Proof of Thiele:

$$\frac{\mathrm{d}}{\mathrm{dt}} V(t)$$

$$= \frac{\mathrm{d}}{\mathrm{dt}}\left(1 - \frac{\bar{a}_{x+t}}{\bar{a}_x}\right) \quad \text{using (2.1)}$$

$$= -\frac{1}{\bar{a}_x}\frac{\mathrm{d}}{\mathrm{dt}}\,\bar{a}_{x+t}$$

$$= -\frac{1}{\bar{a}_x}\left(\mu_{x+t}\,\bar{a}_{x+t} - \bar{A}_{x+t}\right) \quad \text{using (2.2)}$$

$$= -\mu_{x+t}\frac{\bar{a}_{x+t}}{\bar{a}_x} + \frac{1 - \delta\,\bar{a}_{x+t}}{\bar{a}_x}$$

$$
\begin{aligned}
&= -\mu_{x+t}\,(1 - V(t)) + \frac{1}{\bar{a}_x} - \delta\,(1 - V(t)) \\
&\qquad\qquad\qquad\qquad \text{using (2.1)} \\
&= -\,\mu_{x+t}\,(1 - V(t)) + \frac{1 - \delta\,\bar{a}_x}{\bar{a}_x} + \delta\,V(t) \\
&= -\,\mu_{x+t}\,(1 - V(t)) + \frac{\bar{A}_x}{\bar{a}_x} + \delta\,V(t) \\
&= -\,\mu_{x+t}\,(1 - V(t)) + \bar{P}_x + \delta\,V(t).
\end{aligned}
$$