1 Revision

This course assumes knowledge of statistical and actuarial functions associated with a single life, as far as the calculation of premium rates using the **equivalence principle**. Some of these concepts are reviewed here.

When a life effects a policy, like a life insurance policy which pays out a lump sum on death, the actual future lifetime is **unknown.**

We represent this future lifetime of a life aged x by a random variable T_x. Some of the important characteristics of T_x are as follows.

(a) It is assumed to have a **continuous**distribution with cumulative distribution function (c.d.f.):

$$\mathbf{P}\{T_x \leq t\} = F_x(t).$$

Associated with this is the survival function:

$$S_x(t) = 1 - F_x(t)$$

.

In actuarial notation:

$$ {}_tq_x = F_x(t) \qquad {}_tp_x = S_x(t) $$

$$ \text{and} \quad {}_tq_x + {}_tp_x = 1. $$

(b) T_x has probability density function (p.d.f.):

$$ \begin{aligned} f_x(t) &= \frac{d}{dt}F_x(t) = \frac{d}{dt}(1 - S_x(t)) \\ &= -\frac{d}{dt}{}_tp_x. \end{aligned} $$

(c) The force of mortality denoted μ_{x+t} is:

$$ \mu_{x+t} = \lim_{h \to 0} \frac{\mathsf{P}\{T_x \le t + h \mid T_x > t\}}{h} $$

$$
\begin{aligned}
&= \lim_{h \to 0} \frac{F_x(t+h) - F_x(t)}{h\,(1 - F_x(t))} \\
&= \lim_{h \to 0} \frac{{}_h q_{x+t}}{h} \\
&= \frac{-\frac{d}{dt}\,{}_t p_x}{{}_t p_x}.
\end{aligned}
$$

Therefore:

$$
\frac{d}{dt}\,{}_t p_x = -f_x(t) = -{}_t p_x \mu_{x+t}
$$

and since:

$$
\frac{d}{dt} \log\left({}_t p_x\right) = \frac{\frac{d}{dt}\,{}_t p_x}{{}_t p_x}
$$

we get:

$$
{}_t p_x = \exp\left(- \int_0^t \mu_{x+r}\, dr \right).
$$

(d) If we know the p.d.f. of T_x we can calculate moments of T_x:

$$\begin{aligned} \mathsf{E}[T_x] &= \mathring{e}_x = \int_{t=0}^{\infty} {}_tp_x \, dt \\ \mathsf{Var}[T_x] &= \mathsf{E}[T_x^2] - (\mathsf{E}[T_x])^2. \end{aligned}$$

The probabilities ${}_tp_x$ are usually computed with the aid of **a life table.**

By considering a starting age α and a radix l_α which represents **the number of people alive at age** α we define the life table function l_x at all ages $x \geq \alpha$ by:

$$l_x = {}_{x-\alpha}p_\alpha \, l_\alpha.$$

which represents the **expected number**of these same people alive at age x. Therefore:

$$_tp_x = \frac{l_{x+t}}{l_x}.$$

The random variable $K_x = \text{int}[T_x]$, the integer part of T_x, defines the start of the year of death (measured from age x). The random variable $K_x + 1$defines **the end of the year of death.**The same life table probabilities $_tp_x$, with integer values of t, serve to define the (discrete-valued) distribution of K_x and $K_x + 1$. In fact, the term ‘life table’ usually refers to the tabulation of l_x at integer ages x.

Payments made on death, or as long as someone still lives, are made at **random times**, because T_x is a random variable. Given a force of interest δ, the present values of such payments are therefore also random variables. For example the present value of $1 payable immediately on death is:

$$\exp(-\delta T_x) = v^{T_x}$$

and the present value of $1 payable at the end of the year of death is:

$$\exp(-\delta(K_x + 1)).$$

The expectations of such present values **(EPVs)**are important in pricing life insurance contracts. Many are given special symbols in the international actuarial notation. For example:

$$\begin{aligned} \bar{A}_x &= \mathsf{E}[\exp(-\delta T_x)] \\ A_x &= \mathsf{E}[\exp(-\delta(K_x + 1))]. \end{aligned}$$

Similar EPVs ($\bar{a}_x$, a_x, $\ddot{a}_x$, etc., may be defined in respect of level annuity payments. There are many variants of such symbols to deal with **limited terms, deferred payment, frequency of payment**and other simple variants.

The **Principle of Equivalence**is often used to calculate premiums. It states that:

EPV of income = EPV of outgo

so that, for example, the equation:

$$A_x = P_x \ddot{a}_x$$

is solved to find P_x, the premium for a whole-life assurance of \$1, payable at the end of the year of death to a life age x, with premiums payable **annually in advance for life.**

Standard EPVs can be calculated **exactly** for benefits payable at the end of the year of life, or annuities payable annually, using a standard life table, for example:

$$A_x \quad = \quad \sum_{t=0}^{t=\infty} e^{-\delta(t+1)} \, {}_tp_x \, q_{x+t}$$

$$\ddot{a}_x = \sum_{t=0}^{t=\infty} e^{-\delta t} {}_t p_x.$$

When benefits are payable at the moment of death, or annuities are payable continuously, EPVs can be expressed **exactly** as integrals, for example:

$$\bar{A}_x = \int_{t=0}^{t=\infty} e^{-\delta t} {}_t p_x \, \mu_{x+t} \, dt$$

$$\bar{a}_x = \int_{t=0}^{t=\infty} e^{-\delta t} {}_t p_x \, dt$$

but these integrals may have to be evaluated approximately using **numerical methods**. Such numerical methods (generally needing a computer) were not covered in Intro to LIM but will be **very** important in LIM1.