

Chapter 2: Profit Testing

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Profit testing can be described as the process of determining the appropriate premium for a life contract by examining its profit profile or profit signature. It is a very useful tool to price life contracts, such as unit-linked contracts, which are too complicated to for actuarial formulae. It also provides a very convenient method to perform sensitivity analysis for the profit signature to various “scenarios” corresponding to different actuarial assumptions. In practice, actuaries perform profit testing for real contracts with the aid of computer. In this chapter, I will give an introduction to some essential concepts in profit testing and illustrate the procedure to profit test life contracts. You will learn various important concepts and techniques in profit testing, such as the emergence of profit, allowance for reserves, profit measurement and actuarial bases, etc. The outline of this chapter is listed as follows:

Outline of Chapter 2

- Section 2.1: A review of the traditional approach
- Section 2.2: Emerging cashflows
- Section 2.3: Allowance for reserves
- Section 2.4: Profit measurement
- Section 2.5: Actuarial bases
- References: Course Note, Volume 4, Unit 4, Pages 7-50

Section 2.1: A review of the traditional approach

- Example: Consider a non-profit endowment assurance with the following contractual features
 - Term $n = 5$ years
 - Age at issue $x = 60$
 - Sum assured $SA = £10,000$ paid at the end of year of survival or earlier death
 - Mortality: Assured lives mortality table AM92 @ 4% (i.e. Interest Rate = 4%)
 - Initial expenses: £100
 - Renewal expenses: 5% of premium per year

- Problem: Evaluate the annual premium P paid at the beginning of each year such that the expected profit is £50 using the traditional approach

- Solution:

- Recall that for the traditional approach

1. Equation of value: $P\ddot{a}_{x,\overline{n}|} = e\ddot{a}_{x,\overline{n}|} + A_{x,\overline{n}|}$

2. Require the valuation of different elements in the equation of value separately

- Use actuarial functions:

$$0.95P\ddot{a}_{60,\overline{5}|} = 50 + 100 + 10000A_{60,\overline{5}|}$$

- $\ddot{a}_{60,\overline{5}|} = 4.550$ and $A_{60,\overline{5}|} = 0.82499$ (AM92 @ 4%, Yellow Tables, Page 101)
- $P = £1943.30$

Section 2.2: Emerging cashflows

- Convention: Estimate the expected cash-flows in **ANY** future year per policy in force at the start of the year
- Cash flow diagram:
- Cash flow in year t
 - CF_t : Cashflow expected $t - 1 \rightarrow t$ given that the policy is in force at $t - 1$
 - P_t : Premium at $t - 1$
 - E_t : Expenses at $t - 1$
 - i_t : Investment return $t - 1 \rightarrow t$
 - q_{x+t-1} : Probability of death age $x + t - 1 \rightarrow x + t$

- $p_{x+t-1} = 1 - q_{x+t-1}$
- D_t : Benefit on death $t - 1 \rightarrow t$
- S_t : Benefit on survival to t
- $CF_t = P_t - E_t + i_t(P_t - E_t) - q_{x+t-1}D_t - p_{x+t-1}S_t$

- Consider the last example again:

YR	Prem	E	i	D	Total
1	P	100	$0.038P$	80.22	$0.988P$
		$+0.05P$	-4		-184.22
2	P	$0.05P$	$0.038P$	90.09	$0.988P$
					-90.09
3	P	$0.05P$	$0.038P$	101.12	$0.988P$
					-101.12
4	P	$0.05P$	$0.038P$	113.44	$0.988P$
					-113.44
5	P	$0.05P$	$0.038P$	127.16	

- In year 5: Survival Benefit = $\pounds 10,000p_{64} = \pounds 10,000(1 - 0.012716) = \pounds 9872.84$ and
Total = $\pounds 0.988P - \pounds 10,000$
- $i_1 = (P - 100 - 0.05P) \times 0.04 = 0.038P - 4$
- $D_1 = 10,000q_{60} = 10,000 \times 0.008022 = 80.02$
- $D_2 = 10,000q_{61} = 90.09$
- CF_t = Cashflow at t given in force at $t - 1$
- Golden rule: EPV = Amount $\times$ Prob $\times$ Discount
- Fill in the missing items

YR	CF_t	${}_{t-1}p_x$	v^t	EPV
1	$0.988P$ -184.22	1	0.96154	-177.13 $+0.95P$
2	$0.988P$ -90.09	0.991978	0.92456	-82.63 $+0.9061P$
3				
4				
5				

- ${}_{t-1}p_x$ can be calculated based on AM92 table (Page 78) with $x = 60$
- v^t @ 4%
- Suppose the required expected profit is £50. Then, $P =$
- Note that in theory, P is the same as the result from the traditional method; Slightly different because of the round-off error

- Given that $P = £1943.30$

YR	PREM	EXP	INT	D	S	CF _t
1	1943.30	197.17	69.85	80.22	0	1735.76
2						
3						
4						
5						

- In year one, $EXP = 0.05 P + 100$
- $INT = 0.04 \times (PREM - EXP)$
- $D_t = 10,000q_{60+t-1}$ from AM92 table
- For $t = 1, 2, 3, 4$, $S_t = 0$; $S_5 = 9872.84$
- $CF_t = PREM - EXP + INT - D - S$

Section 2.3: Allowance for reserves

- Profit testing
 - Definition: The process of estimating future expected **cashflows** and the cost of setting up **reserves** in terms of a rate of return on capital
 - Application: Determine the appropriate premium for a life contract by examining its profit profile
- Profit vector PRO
 - PRO_t : The profit made in the year $t - 1$ to t per policy in force at the start of the year, after allowance for setting up reserves according to our basis

- $PRO = (PRO_1, PRO_2, \dots, PRO_n)$, for some year n

- PRO_t : Profit vector in year $t - 1 \rightarrow t$

- ${}_tV$: Reserve held per policy in-force at time t

- Then,

$$PRO_t = P_t - E_t + i_t(P_t - E_t) - q_{x+t-1} D_t - p_{x+t-1} S_t + {}_{t-1}V(1 + i_t) - p_{x+t-1} \times {}_tV$$

- Hence,

$$PRO_t = CF_t + i_t \times {}_{t-1}V - (p_{x+t-1} \times {}_tV - {}_{t-1}V)$$

- $i_t \times {}_{t-1}V$ = Interest on the reserve held at the start of the year t

- $p_{x+t-1} \times {}_tV - {}_{t-1}V = \text{Increase in reserves}$

- Profit signature σ_t

- Definition: The profit made in the year $t - 1 \rightarrow t$ **per policy sold at time zero**, after allowance for setting up reserves according to our basis

- $\sigma_t = {}_{t-1}p_x \times \text{PRO}_t$

- Expected profit $= \sum_t {}_{t-1}p_x \times \text{PRO}_t \times v^t = \sum_t \sigma_t \times v^t$

- Calculating reserves

- Use a **strong** basis $\Rightarrow$ **High** reserves

- Prudential or Cautious: A larger amount of money is available when an unexpected cost occurs

– Example: Suppose $x = 60$; $n = 5$; Mortality: AM92 @ $i = 4\%$; $SA = £10,000$; The interest rate on reserves is 4% . Calculate the net premium reserve

– Solution:

$$1. \text{ Net Premium NP} = £10,000 \times \left(\frac{A_{60:\overline{5}|}}{\ddot{a}_{60:\overline{5}|}} \right) =$$

$$£10,000 \times \frac{0.82499}{4.550} = £1,813.17 \text{ @ } i = 4\%$$

2. Calculate reserves recursively:

$${}_5V = 0$$

$$({}_tV + NP)(1.04) = 10,000q_{60+t} + {}_{t+1}V \times p_{60+t} + S_{t+1} \times p_{60+t}$$

$$3. \text{ Hence, } {}_4V = £7802.22$$

4. Fill in the missing items in the following tables:

Table 1.

YR	CF_t	$_{t-1}V$	Interest on Reserves	Increase in Reserves
1	1735.76	0	0	1805.21
2	1829.89			
3	1818.86			
4	1806.54			
5	-8080.02	7802.22		

Table 2.

YR	PRO_t	σ_t
1	-69.45	-69.45
2		
3		
4		
5		

- Interest on reserves = $0.04_{t-1}V$ (i.e. Not the interest rate in the valuation basis)

- $PRO_t = CF_t + \text{Interest on reserves} - \text{Increase in reserves}$
- $\sigma_t = {}_{t-1}p_x \times PRO_t$
- Expected profit $= \sum_{t=1}^5 \sigma_t v^t = 50.26 \approx 50 @ 4\%$, which is the same as the case without reserves
- Typical pattern of profits: $\sigma_1 < 0$ and $\sigma_t > 0$, for all $t > 1$

- New business strain:

- Equal to $-\sigma_1$
- The first premium is insufficient to pay for both the initial expenses and setting up the reserve at the end of the year
- The insurer needs a source of funds to pay for the initial reserve or loss

- The capital of the insurer
 - Additional assets above those needed for reserves
 - $\text{Total Assets} = \text{Reserves} + \text{Capital}$
 - **Reduced** by new business strain
 - **Increased** by later profits
 - Provided by the owners or shareholders of the insurance company
 - The owners will expect a good **rate of return** on the capital used

Section 2.4: Profit Measurement

- Traditional approach
 - Evaluate the expected profit discounted at the premium basis interest rate i_p using actuarial functions
 - **ONLY** method available using actuarial functions
 - The premium basis interest rate i_p is the **ONLY** appropriate discount rate using actuarial functions
 - In practice, i_p is **Unlikely** to be an appropriate discount rate
- Four more informative **profit criteria** can be introduced with profit testing

- Expected profit at RDR
 - Profit margin
 - Discounted payback period
 - Internal rate of return
- Expected profit at RDR
 - Risk Discount Rate (RDR): An investor's **Required** rate of return allowing for the risk of the investment
 - Then, expected profit = $\sum_t \sigma_t v^t$ @ RDR
 - The expected profit at RDR is the same as that in the traditional approach if $RDR = i_p$
 - Usually, $RDR > i_p$

- $i_p = \textbf{Expected}$ return on our assets (i.e. An estimate of the actual average return)
- Normally invest in safe assets (e.g. Gilts) which provides a low rate of return
- The insurer's capital is invested in a riskier asset - Insurance policies
- What are the major risks?
 1. Interest
 2. Mortality
 3. Expenses
- Use of the **Capital Asset Pricing Model (CAPM)** or **Arbitrage Pricing Theory (APT)** to evaluate the required discount rate given the risk

- Typically, $10\% < RDR < 15\%$
- Consider the last example:
 1. Expected Profit (EP) @ 10% = 33.56
 2. $RDR > i_p = 4\% \Rightarrow EP @ RDR < EP @ i_p = 50.26$
- The rewards for selling the policy
 1. Split between the salesperson and the insurer
 2. Saleperson receives commission
 3. Insurer's profit should be related to commission
- Example: Suppose we (i.e. the insurer) only require an expected profit equal to half of the initial commission. If the initial commission = $0.5 \times \text{Initial Expense} = 50$, then we must pass the test since $EPV @ 10\% = 33.56$

- Profit Margin

- Definition: The proportion of each premium taken as profit

- Profit Margin = $\frac{\text{EP @ RDR}}{\text{EPV of Prem @ RDR}}$

- Can be used to set our profit criteria (e.g. Profit criteria may be that the profit margin exceeds 1 %)

- Example: Consider EP @ 10% = 33.56.
EPV of premiums = $\sum_{t=0}^4 P(1/1.1)^t {}_t p_x = 7972.25$. Then,

$$\text{Profit Margin} = \frac{33.56}{7972.25} = 0.421\%$$

- Profit is a very small percentage of premiums

- Discounted payback period (DPP)
 - The time it takes to repay the new business strain with interest at the RDR
 - The first time t such that EPV of profits in the first t years

$$= \sum_{s \leq t} (1 + \text{RDR})^{-s} \sigma_s \geq 0$$

- The faster the strain is repaid (i.e. the smaller t), the sooner the insurer can invest in a new project

- Example: Suppose the profit signatures σ_t are given as before; $RDR = 10\%$. Fill in the missing items in the following table

YR	σ_t	$\sum_{s \leq t} (1 + RDR)^{-s} \sigma_s$
1	-69.45	-63.14
2	34.01	
3	33.70	
4	33.36	
5	32.99	

1. $DPP = 4$ years
2. Less than half of the new business strain has been repaid by year 2

- Internal rate of return (IRR)
 - The interest rate at which the present value of the expected profits is zero
 - $IRR = i$ such that $\sum_t \sigma_t(1 + i)^{-t} = 0$
 - Cannot always be used since there may be no solutions or more than one solutions to the above equation
 - Profit criteria: $IRR \geq RDR$ (i.e. Accept the contract with yield at least our required rate of return)
 - Example: Suppose RDR is 10% and $IRR = i$. Given σ_t in the following table:

YR	σ_t
1	−104.52
2	16.78
3	35.00
4	53.33
5	71.73

Then, i can be determined by the following equation of value:

$$0 = \frac{-104.52}{(1+i)} + \frac{16.78}{(1+i)^2} + \dots + \frac{71.73}{(1+i)^5}$$

– Hence, $IRR = i = 19.7\% > RDR$

- Note that DPP and IRR have widely been used for evaluating profits from investment projects in corporate finance
- Profit measurement in practice
 - Evaluate the four profit measures when designing a new product or setting premium rates
 - Adjust premium/benefits until the profit criteria are passed

- Marketing considerations: Very important to ensure that we are able to sell the contract (e.g. competitors' premiums)

Section 2.5: Actuarial bases

- Reference: Course Notes, Volume 4, Pages 30-40

- Recall that

$$PRO_t = P_t - E_t + i_t(P_t - E_t) - q_{x+t-1}D_t - p_{x+t-1}S_t + {}_{t-1}V(1 + i_t) - p_{x+t-1} \times {}_tV$$

$$\sigma_t = {}_{t-1}p_x \times PRO_t$$

- Calculate reserves, premium, PRO_t and σ_t
- Specify the **unknown** future levels of interest rate, mortality and expenses
- Actuarial Basis: A set of **assumptions** used to carry out one or more actuarial calculations

- Estimate reserves, premium, PRO_t and σ_t using an **Actuarial Basis** whose elements include assumptions on:
 - Interest rate
 - Mortality
 - Expenses
- Carry out the calculations based on these assumptions, but not the expectation of their actual future levels

- Experience Basis
 - A set of assumptions about the actual outcome
 - Suppose the premium and reserves are given. We can calculate PRO_t and σ_t under **ANY** set of assumptions of future experience
- Scenario Testing: Given the same premium and reserves, we perform profit test using several different sets of experience
- Sensitivity Analysis: Investigate the consequences of changing only one item of experience with other items of experience being fixed

- Example: Assume $i = 5\%$ (i.e. the experience basis interest rate); The interest rate on reserves is 5%; The RDR is 10%. Fix expenses and mortality; Premium and reserves are given as before.

YR	PREM	EXP	INT	D	S	CF _t
1	1943.30	197.16	87.31	80.22	0	1753.22
2	1943.30	97.16	92.31	90.09	0	1848.35
3	1943.30	97.16	92.31	101.12	0	1837.32
4	1943.30	97.16	92.31	113.44	0	1825.00
5	1943.30	97.16	92.31	127.16	9872.84	−8061.56

YR	${}_{t-1}V$ Given	Int on Res $0.05{}_{t-1}V$	Increase in Res	PRO _t	σ_t
1	0	0	1805.21	−51.99	−51.99
2	1819.81	90.09	1868.39	70.95	70.38
3	3721.73	186.09	1933.44	89.96	88.44
4	5712.94	285.65	2000.77	109.88	106.92
5	7802.22	390.11	−7802.22	130.77	125.81

$$- EP @ 10\% = 228.49$$

- Question: What will be the effect of a

change in the **Experience** basis on the expected profit?

- High interest $\Rightarrow$ High profit
- High expense $\Rightarrow$ Low profit
- High mortality: Depends on what kind of products

1. Term assurance: Low profit

2. Annuity: High profit

- Premium Basis: Calculate premiums using either

1. The traditional approach

2. A profit test

- Example: The premium basis can be 4% AM92

- Once the premium is calculated according to a given premium basis, we can test the profit under **different** experience bases with the premium being kept **fixed**
- Question: What will be the effect of a change in the premium basis on the expected profit (with the experience basis being held constant)?
 - High interest $\Rightarrow$ Low premiums $\Rightarrow$ Low profit
 - High expenses $\Rightarrow$ High premiums $\Rightarrow$ High profit

- High mortality

1. Term assurance $\Rightarrow$ High premium $\Rightarrow$ High profit

2. Annuity $\Rightarrow$ Low premium $\Rightarrow$ Low profit

- High premiums $\Rightarrow$ Higher profits $\Rightarrow$ Strong premium basis

- Marketability $\Rightarrow$ Premiums cannot be too high in order to sell the contracts

- Valuation basis

- Need to know the size of the reserves to calculate the profit vector PRO_t

- The valuation basis only specifies the size of the reserve ${}_tV$

- All other cashflows are given by the experience or premium basis
- The valuation basis is specified by legislation and has to be prudent or strong
=> Highly safe reserves
- Question: How are the reserves affected by a change in the valuation basis?
 - High interest => Low reserves
 - High expenses => High reserves
 - High mortality
 1. Term assurance => High reserves
 2. Annuity => Low reserves
- Question: How does a change in reserves affect the profit signature?

- Example:

- Experience basis: 4% AM92

- Valuation bases

1. 4% AM92 $\Rightarrow$ Net premium (NP) = $10,000 \frac{A_{60:\overline{5}|}}{\ddot{a}_{60:\overline{5}|}} = 1,813.17$

2. 6% AM92 $\Rightarrow NP = 10,000 \times (\frac{0.75152}{4.39}) = 1,711.89$

- Fill in the missing items in the following table

Valuation Basis 4% AM92

YR	CF_t	${}_{t-1}V$ Given	Int on Res $0.04 {}_{t-1}V$	Increase in Res	PRO_t	σ_t
1	1735.76	0	0	1805.21	-69.45	
2	1829.89					
3	1818.86					
4	1806.54					
5	-8080.02					

Valuation Basis 6% AM92

YR	CF_t	${}_{t-1}V$ Given	Int or Res $0.05 {}_{t-1}V$	Increase in Res	PRO_t	σ_t
1	1735.76	0	0	1734.73	1.02	
2	1829.89					
3	1818.86					
4	1806.53					
5	-8080.02					

- Remarks:

- High valuation basis interest rate $\Rightarrow$ Lower reserves
- Setting up larger reserves leads to **smaller** profits (larger losses) in the **early** years since the profit vector includes the increase in reserves as an item of loss
- Setting up larger reserves leads to **larger** profits in the **later** years since larger reserves early on

1. More interest on reserves
 2. Smaller increase in (or larger release of) reserves in the final year
 - The 6% valuation basis is **too weak**
 $\Rightarrow$ Reserve is too small to pay the benefits in the final year and $\sigma_5 = -47.2 < 0$ (a loss)
- Question: How does a change in reserves (i.e. choice of valuation basis) affect the EPV of profits?
 - Theorem: Suppose the RDR is the experience basis interest rate (i.e. $\text{RDR} = i$). Then, the EP is **NOT** affected by the choice of valuation basis

- Example:

Consider the last example. Suppose RDR = 4%. Then, the expected profit is 50.26. For valuation bases 4% and 6%, calculate the expected profits and check them by yourself

- Proof of the theorem:

- EPV of Profits

$$= \sum_{t=1}^n (1 + \text{RDR})^{-t} \times {}_{t-1}p_x \times \text{PRO}_t$$

- But, RDR = i , the experience basis interest rate

- EPV

$$= \sum_{t=1}^n (1 + i)^{-t} \times {}_{t-1}p_x \times [(P_t - E_t) \times (1 + i) - q_{x+t-1} \times D_t - p_{x+t-1} \times S_t + {}_{t-1}V(1 + i) - p_{x+t-1} \times {}_tV]$$

- Note that the premiums and reserves are given and that all other items are estimated on the experience basis

- EPV

$$\begin{aligned}
 &= \sum_{t=1}^n (1+i)^{-t} \times {}_{t-1}p_x \times CF_t + \\
 &\quad \sum_{t=1}^n (1+i)^{-t+1} \times {}_{t-1}p_x \times {}_{t-1}V \\
 &\quad - \sum_{t=1}^n (1+i)^{-t} \times {}_tp_x \times {}_tV
 \end{aligned}$$

- Let $s = t - 1$

- EPV

$$\begin{aligned}
 &= \sum_{t=1}^n (1+i)^{-t} \times {}_{t-1}p_x \times CF_t + \\
 &\quad \sum_{s=0}^{n-1} (1+i)^{-s} \times {}_sp_x \times {}_sV \\
 &\quad - \sum_{t=1}^n (1+i)^{-t} \times {}_tp_x \times {}_tV
 \end{aligned}$$

- However, ${}_nV = {}_0V = 0$
- Therefore, $EPV = \sum_{t=1}^n (1+i)^{-t} \times {}_{t-1}p_x \times CF_t$, which does not depend on the reserves
- Require a **high** rate of return (RDR) on capital for setting up large reserves
- Why?
 - The insurer needs to borrow a large amount of **capital** to finance large reserves
 - There is a risk that **actual** experience is **worse** than the premium basis, and so, the capital may never be returned
 - Require a high rate of return (RDR) on capital as compensation for the risk

- Example: Expected profits @ RDR = 10% for different valuation bases are evaluated as follows:

VALN BASIS	4%	6%
EPV	33.56	49.45

- Hence, the EP is **reduced** by setting up **large** reserves
- Question: What will the effect of a change in the valuation basis be on the expected profit?
 - High interest $\Rightarrow$ Low reserves $\Rightarrow$ High profit
 - High mortality
 1. Term assurance $\Rightarrow$ High reserves $\Rightarrow$ Low profit

2. Annuity $\Rightarrow$ Low reserves $\Rightarrow$ High profit

- In practice, we need to strike a balance of the tradeoff between
 - Low reserves $\Rightarrow$ More profit
 - High reserves $\Rightarrow$ More prudent
- Actual Experience
 - The valuation, premium and experience bases are all **Assumptions** of the **Unknown Future** experience
 - Once the contract is finished, the actual experience is the **Known** past interest, mortality and expenses

- Ideal situation: Our best estimates or assumptions are as close as the actual experience as possible
- In practice, **Estimation Error** does exist
- Emerging Cashflows v.s. Actuarial Functions
 - Advantage of actuarial functions
 1. Quick and easy to calculate by hand
 - Disadvantages of actuarial functions
 1. Ignore the cost of setting up reserves
 2. Provide no information on the timing of cashflows
 3. Difficult to use to price complex benefits (e.g Unit-linked policies)

- Advantages of emerging cashflows
 1. Easy to implement using a computer, in particular, spreadsheet
 2. Allow for the cost of reserves
 3. Provide the expected cashflows for each year => Convenient for financial planning, especially choice of assets
 4. Can be used for valuing even the most complex contracts or benefits

– Disadvantages of emerging cashflows

1. Very slow to calculate by hand
2. Even quicker to calculate commutation functions by computers

End of Chapter 2