Chapter 1: Pensions

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A pension is an income paid to someone from the time he/she ceases full time employment for the rest of his/her life. The pension can be paid by an insurance company, the worker's employer and the government. In this chapter, I will provide an introduction to the concept of pension funds and the mathematics of pension funds. You will learn the basic underlying mechanisms of various pension funds and how to perform simple calculations involving pension benefits. More advanced topics and practical details about pension funds will be covered in the fourth year pensions module and professional examinations. The outline of this chapter is listed as follows:

Outline of Chapter 1

- Section 1.1: An orientation to pension funds and their variants
- Section 1.2: Valuation of the liabilities
- Section 1.3: Valuation by commutation functions
- Reference: Part III Pension Funds by M. R Hardy and M Willder

Section 1.1: An orientation to pension funds and their variants

- Question: What is a pension?

- A Pension is an income paid to someone from the time he/she ceases full time employment for the rest of his/her life

- Question: Why are pensions important?
  - About 40% actuaries work in pensions
  - Pension funds account for significant percentage in stock market
  - Pension benefits may represent an important part of an employee's remuneration package

- Question: Who pays the pension?
  - Insurance company: Premiums (e.g. Personal pensions)
  - Employer: Occupational pension schemes OPS
  - The government: Tax (e.g. Basic State Pension (BSP))

- Occupational Pension Scheme (OPS):
  - Both the employer and the employee pay into a fund during employee's working life
  - Fund pays pension at retirement
  - Deferred pay

- Question: What is the amount of pension people receive?

  - Same as salary?

  - Half of salary?

  - A fixed amount to employee per annum, say £12,000 per annum?

- The size or amount of the pension is related to salary (S). Why?

  - Higher wages can afford higher contributions to pension schemes

  - Maintain the standard of living before retirement

- Pensions are typically lower than pre-retirement wages

- Explanations:
  - Living costs are lower in retirement (e.g. The employee's mortgage has been paid off)
  - People can receive basic state pension (BSP) after retirement

- The size of the pension is also related to years of service (N). How?
  - The longer an employee's years of service is, the greater amount of the pension per annum is
  - Example: Pension per annum $= \frac{N}{60}S$ at retirement

- Most occuptional pension scheme (OPS) targets on two-third of salary

- Suppose the employee's years of service $N = 40$ years
- Pension per annum $= \frac{40}{60}S = \frac{2}{3}S$ at retirement

- Question: What are the main benefits from an OPS?
  - Age Retirement => Immediate Pension + Lump Sum
  - Ill-Heath Retirement => Immediate Pension + Lump Sum
  - Death in Service (DIS) => Lump Sum + Spouse's Pension
  - Withdrawal from Service => Deferred Pension or Return of Contributions

- Remarks:
  - Pensions are not only paid on age retirement
  - Pensions can be 'Deferred'

- Defined Benefit Schemes (DBS)
  - DBS is an OPS where the benefits are evaluated by refering to a formula
  - Example 1: Pension $= \pounds 10,000$ per annum
  - Example 2: Pension $= \frac{N}{80} \times$ Average Salary
  - Example 3: Lump Sum of DIS $= 4 \times$ Current Salary

- The exact amount of the benefit may not be known in advance (i.e. random), but we know how to evaluate it from a formula

- DBS is also called Final Salary Schemes since the benefits are often evaluated based on salary at retirement

- Defined Contribution Schemes

  - Varying benefits depending on the value of the fund at retirement

  - Contributions are fixed

  - Another name: Money Purchase

- Pensionable Salary
  - Salary used to calculate the **contributions** and **benefits**
  - Example: Pension salary = Basic salary received in the previous calendar year
  - Final Pensionable Salary: The average salary received in the **three** years prior to retirement

Section 1.2: Valuation of the liabilities

- Objectives:
  - Calculate the amount of money we need now to afford the benefits we promised (i.e. reserve)
  - Calculate the amount of money we will need to pay in the future to honor our promise on the benefits (i.e. premiums or contributions)

- Four causes for the payment of benefits
  - Active (a) $\rightarrow$ Age retirement (r)
  - Active (a) $\rightarrow$ Ill-health retirment (i)
  - Active (a) $\rightarrow$ Dealth (d)

- Active (a) $\rightarrow$ Withdrawal (w)

- Service Table
  - A multiple decrement table with the decrements for the four causes for the payment of benefits
  - Contruct the table for l_x, r_x, i_x, d_x and w_x, where l_α is the radix of the table associated with the entry age α

- Notation:
  - Differ from standard multiple decrement notation
  - Pr(A member is in service at age $x + t$ | In service at age x) $= {}_tp_x^{aa}$
  - ${}_tp_x^{aa} = \frac{l_{x+t}}{l_x} = {}_tp_x^{\overline{aa}}$

- Pr(A member retires $x+t \to x+t+1$ | In service at age x) $= {}_tp_x^{aa} \times {}_1p_{x+t}^{ar} = \frac{r_{x+t}}{l_x}$

- Pr(A member retires through ill-health $x+t \to x+t+1$ | In service at age x) $= {}_tp_x^{aa} \times {}_1p_{x+t}^{ai} = \frac{i_{x+t}}{l_x}$

- Pr(A member dies $x+t \to x+t+1$ | In service at age x) $= {}_tp_x^{aa} \times {}_1p_{x+t}^{ad} = \frac{d_{x+t}}{l_x}$

- Pr(A member withdraws $x+t \to x+t+1$ | In service at age x) $= {}_tp_x^{aa} \times {}_1p_{x+t}^{aw} = \frac{w_{x+t}}{l_x}$

- Example: **Yellow Tables (Page 142)**; Green Tables (Page 90)

- Remarks:
  - Lecture notes and tutorials: **Yellow Tables**
  - Past examination papers: **Green Tables**
  - Examination: **Yellow Tables**
  - In real situations, every scheme is different and has its own table

- Service Table (**Yellow Tables, Page 142**; Green Tables, Page 90): Qualitative Analysis
  - Withdrawal: Probability is initially high, but decreases with age
  - Death: Probability is initially low, but increases with age
  - Ill-health: Probability increases with age, but is very low

- Age retirement
  - Normal retirement age (NRA) = 65
  - Everyone must retire
  - Normally retire by age 65
  - Early retirement: Allowed from 60 to 65

- Salary Scale
  - OPS benefits are related to salary
  - Salaries increase over time due to
    1. Inflation
    2. Promotion
    3. Merit
    4. Experience
  - A promotional salary scale, s_x is set up

$$\frac{s_{x+t}}{s_x} = E\left(\frac{\text{Salary Earned } x+t \to x+t+1}{\text{Salary Earned } x \to x+1}\right)$$

provided that the employee is in service in the entire period

– Example: Salary earned between ages 18 and 19 is £18,000. Based on the salary scale in the Yellow Tables, calculate the expected salary earned between

1. 19 and 20

2. 40 and 41

– Solution:

$$
\begin{aligned}
s_{18} &= \\
s_{19} &= \\
s_{40} &=
\end{aligned}
$$

1. The expected salary between 19 and 20 =

2. The expected salary between 40 and 41 =

- Relationship between salary earned and salary rate
  - Salary Earned: The amount paid over a time interval
  - Salary Rate: The instantaneous rate of payment
  - Thus, the salary earned between ages x and $x + t$

    $$\int_0^t (\text{Salary rate at age } x + s)ds$$

  - The salary rate increases continuously through the year
  - The salary earned between ages x and $x + 1 \approx$ The salary rate at age $x + 1/2$
  - The equality holds when ?

- Example: Suppose the salary rate at age 30 is £20,000 and salary rate increases by £4,000 at age 30.75. Calculate the salary earned between ages 30 and 31.

- Solution:

  The salary earned
  $=$

- Final Pensionable Salary (FPS)

  - The scheme rules will specify the pension in terms of FPS

  - FPS can be any function of the current or past salaries

  - Common definition: The average salary earned over the last n years of service

- In the **Yellow Tables (Page 142)** Green Tables (Page 93), $n = 3$ $z_x = \frac{1}{3}(s_{x-3} + s_{x-2} + s_{x-1})$

- Final Pensionable Salary Scale, z_x
  - $y =$ Age at retirement
  - SAL $=$ Salary earned $x - 1 \rightarrow x$
  - $x =$ Current age
  - $n = 3$
  - Then,

$$\begin{aligned} & \text{Expected FPS} \\ = \; & \frac{1}{3}\left(\text{SAL}\frac{s_{y-1}}{s_{x-1}} + \text{SAL}\frac{s_{y-2}}{s_{x-1}} + \text{SAL}\frac{s_{y-3}}{s_{x-1}}\right) \\ = \; & \text{SAL}\frac{z_y}{s_{x-1}}\ , \end{aligned}$$

    where $z_y := \frac{1}{3}(s_{y-1} + s_{y-2} + s_{y-3})$

Section 1.3: Valuation by commutation functions

- A quick review of commutation functions
  - Invented in the 18^{th} century
  - Simplify the calculation of numerical values for many actuarial functions
  - Popular tool for premium calculations in a deterministic model
  - lose their significance due to the development of powerful computers and insurance models based on probability theory
  - Life annuities
    * $D_x = v^x l_x =$ Discounted number of survivors

* $N_x = D_x + D_{x+1} + D_{x+2} \cdots$

* Recall that $\ddot{a}_x = \frac{N_x}{D_x}$

* $S_x = D_x + 2D_{x+1} + 3D_{x+2} \cdots = N_x + N_{x+1} + N_{t+2} + \ldots$

* Recall that $(I\ddot{a})_x = \frac{S_x}{D_x}$

- Life insurance

  * $C_x = v^{x+1} d_x =$ Discounted number of deaths

  * $M_x = C_x + C_{x+1} + C_{x+2} + \ldots$

  * $R_x = C_x + 2C_{x+1} + 3C_{x+2} + \cdots = M_x + M_{x+1} + M_{x+2} + \ldots$

  * Recall that $A_x = \frac{M_x}{D_x}$ and that $(IA)_x = \frac{R_x}{D_x}$

- Reference: Life Insurance Mathematics by Gerber H. U. (1986), 2^{nd} edition

- Valuation of past and future service
  - Accruals basis: Account for the benefits when they are earned, not when they are paid
  - Split the valuation into
    1. Past service accrual
    2. Future service accrual
  - Example: An OPS provides a pension of $\frac{1}{60}$ FPS for each year of service. Suppose a member joined this scheme aged x and he is now aged y. Assume the NRA is 65. What is the member's
    1. past service pension ?
    2. future service pension ?

- Solution:
  - Past Service $= y - x$
  - The member has already earned a pension of $\frac{y-x}{60} \times$ FPS, which is payable from age 65
  - Suppose the future service is F
  - Then, the member will earn a pension of $\frac{F}{60} \times$ FPS, where $y + F$ is the age when the member leaves the service or scheme
  - Note that $F \in [0, 65 - y]$

- Final Average Salary Schemes: Age Retirement
  - FPS is the average salary earned in the three years before retirement
  - Accrual $= \frac{1}{60}$ (i.e. Benefits of $\frac{1}{60} \times$ FPS is **earned** each year for each year of service)
  - Member

$$\begin{aligned} x &= \text{Current age} \\ n &= \text{Past service} \\ \text{SAL} &= \text{Salary earned} \\ \text{NRA} &= 65 \end{aligned}$$

  - General rule:

    The expected present value (EPV) of **past** service benefits **paid** in year $t \to t+$

$1\ (t < 65 - x) =$ Amount $\times$ Probability $\times$ Discount

- Suppose exits occur half-way through the year
- Past service pension $= \frac{n}{60} \times \mathsf{SAL} \times \frac{z_{x+t+0.5}}{s_{x-1}}$
- The pension is paid for the policyholder each year from age $x + t + 0.5$ onwards
- Let $\bar{a}^r_y =$ EPV of age retirement pension of 1 per annum from age y
- The total value of the pension = Past service pension $\times\ \bar{a}^r_{x+t+0.5}$
- Probability $= \frac{r_{x+t}}{l_x}$
- Discount factor $= v^{t+0.5} = \frac{v^{x+t+0.5}}{v^x}$

– Therefore,

$$\begin{aligned}
EPV &= \frac{n}{60} \times \mathsf{SAL} \times \left(\frac{z_{x+t+0.5}}{s_{x-1}}\right) \\
&\quad \times \bar{a}^{r}_{x+t+0.5} \times \left(\frac{r_{x+t}}{l_x}\right) \times \\
&\quad \left(\frac{v^{x+t+0.5}}{v^x}\right) \\
&= \frac{n}{60} \times \mathsf{SAL} \times \left(\frac{{}^{z}C^{ra}_{x+t}}{s_{x-1}D_x}\right),
\end{aligned}$$

where ${}^{z}C^{ra}_{y} = z_{y+0.5} \times \bar{a}^{r}_{y+0.5} \times r_y \times v^{y+0.5}$ and $D_x = v^x l_x$ by the convention of commutation functions

- Summary: EPV of past service benefits paid on retirement at age $x+t$

  - When $x+t<65$,

$$
\begin{aligned}
EPV &= \frac{n}{60} \times \text{SAL} \times \left(\frac{z_{x+t+0.5}}{s_{x-1}}\right) \\
&\quad \times \bar{a}^{r}_{x+t+0.5} \times \left(\frac{r_{x+t}}{l_x}\right) \times \\
&\quad \left(\frac{v^{x+t+0.5}}{v^x}\right) \\
&= \frac{n}{60} \times \text{SAL} \times \left(\frac{{}^{z}C^{ra}_{x+t}}{s_{x-1}D_x}\right)
\end{aligned}
$$

  - When $x+t=65$,

$$
\begin{aligned}
EPV = &= \frac{n}{60} \times \text{SAL} \times \left(\frac{z_{65}}{s_{x-1}}\right) \times \bar{a}^{r}_{65} \\
&\quad \times \left(\frac{r_{65}}{l_x}\right) \times \left(\frac{v^{65}}{v^x}\right) \\
&= \frac{n}{60} \times \text{SAL} \times \left(\frac{{}^{z}C^{ra}_{65}}{s_{x-1}D_x}\right),
\end{aligned}
$$

  where ${}^{z}C^{ra}_{65} = z_{65} \times \bar{a}^{r}_{65} \times r_{65} \times v^{65}$

- Remarks:
  - Retirement at age 65 is at the exact age
  - Retirement before age 65 occurs at age $x+t+0.5$

- EPV of past service benefits paid on retirement at **ANY** future time
  - Equal to past service liability

$$\begin{aligned} & \text{Past service liability} \\ = \;& \frac{n}{60} \times \text{SAL} \times \left(\frac{\sum_{t=0}^{65-x} {}^{z}C_{x+t}^{ra}}{s_{x-1} D_x} \right) \\ = \;& \frac{n}{60} \times \text{SAL} \times \left(\frac{{}^{z}M_{x}^{ra}}{s_{x-1} D_x} \right), \end{aligned}$$

    where ${}^{z}M_{x}^{ra} = \sum_{t=0}^{65-x} {}^{z}C_{x+t}^{ra}$ by the convention of commutation functions

  - Example: Suppose the salary rate at age x, namely SAL', is given. What is the formula for the past service liability?

– Solution:

Past service liability

$=$

- Future service
  - Retirement can occur at any age y, where $x < y \leq 65$
  - Consider the benefits accrued this year for a member aged x
  - Note that if he retires this year at age $x + \frac{1}{2}$, he will only have accrued $\frac{1}{2}\times$ one year's benefit

– Then,

$$
\begin{aligned}
& \text{EPV of one year's accrual} \\
= \ & \frac{1}{60} \times \mathrm{SAL} \times \Bigg[\sum_{y=x}^{64} \left(\frac{z_{y+\frac{1}{2}}}{s_{x-1}} \right) \times \bar{a}^{r}_{y+\frac{1}{2}} \times \\
& \left(\frac{r_y}{l_x} \right) \times \left(\frac{v^{y+\frac{1}{2}}}{v^x} \right) - \frac{1}{2} \left(\frac{z_{x+\frac{1}{2}}}{s_{x-1}} \right) \times \bar{a}^{r}_{x+\frac{1}{2}} \\
& \times \left(\frac{r_x}{l_x} \right) \times v^{\frac{1}{2}} + \left(\frac{z_{65}}{s_{x-1}} \right) \times \bar{a}^{r}_{65} \times \left(\frac{r_{65}}{l_x} \right) \\
& \times v^{65-x} \Bigg] \\
= \ & \frac{\mathrm{SAL}}{60} \times \left(\frac{\sum_{y=x}^{64} {}^{z}C^{ra}_{y} - \frac{1}{2} {}^{z}C^{ra}_{x} + {}^{z}C^{ra}_{65}}{s_{x-1} D_x} \right) \\
= \ & \left(\frac{1}{60} \right) \times \left(\frac{\mathrm{SAL}}{s_{x-1} D_x} \right) \times \left({}^{z}M^{ra}_{x} - \frac{1}{2} {}^{z}C^{ra}_{x} \right)
\end{aligned}
$$

where ${}^{z}M^{ra}_{x} = \sum_{y=x}^{65} {}^{z}C^{ra}_{y}$

– Write ${}^{z}\bar{M}_x^{ra} = {}^{z}M_x^{ra} - \frac{1}{2}{}^{z}C_x^{ra}$

– Then,

$$\begin{aligned} & \text{EPV of one year's accrual} \\ = \ & \left(\frac{1}{60}\right) \times \left(\frac{\text{SAL}}{s_{x-1}D_x}\right) \times {}^{z}\bar{M}_x^{ra} \end{aligned}$$

– We have only considered this year's accrual for someone now aged x

– Need to sum for all future years accrual (i.e. Age $x, x+1, \ldots, 64$)

– Hence,

$$\begin{aligned} \text{FSL} &= \frac{1}{60} \times \left(\frac{\text{SAL}}{s_{x-1}D_x}\right) \times \sum_{t=0}^{64-x} {}^{z}\bar{M}_{x+t}^{ra} \\ &= \frac{1}{60} \times \left(\frac{\text{SAL}}{s_{x-1}D_x}\right) \times {}^{z}\bar{R}_x^{ra} \ , \end{aligned}$$

where ${}^{z}\bar{R}_x^{ra} = \sum_{t=0}^{64-x} {}^{z}\bar{M}_{x+t}^{ra}$

- R_x is the sum of $M_x, M_{x+1}, \ldots, M_{64}$ since the final year of accrual is the year from age 64 to 65

- Example: Suppose $x = 40$; SAL $= 30,000$; $n = 10$. Calculate the past service liability (PSL) and the future service liability (FSL) for age retirement using the Yellow Tables with 4% rate of interest

- Solution:

  PSL =

  FSL =

- Ill-health retirement

  - The EPV of past service benefits paid in year $t \to t+1$

$$
\begin{aligned}
& \left(\frac{n}{60}\right) \times \mathsf{SAL} \times \left(\frac{z_{x+t+0.5}}{s_{x-1}}\right) \times \bar{a}^{i}_{x+t+0.5} \\
& \times \left(\frac{i_{x+t}}{l_x}\right) \times v^{t+0.5} \\
= \; & \left(\frac{n}{60}\right) \times \mathsf{SAL} \times \left(\frac{{}^{z}C^{ia}_{x+t}}{s_{x-1}D_x}\right)
\end{aligned}
$$

    where $\bar{a}^{i}_{y}$ = EPV of ill-health pension of £1 per annum from age y

  - Then,

$$
\begin{aligned}
PSL \; &= \; \left(\frac{n}{60}\right) \times \mathsf{SAL} \times \left(\frac{\sum_{t=0}^{64-x} {}^{z}C^{ia}_{x+t}}{s_{x-1}D_x}\right) \\
&= \; \left(\frac{n}{60}\right) \times \mathsf{SAL} \times \left(\frac{{}^{z}M^{ia}_{x}}{s_{x-1}D_x}\right),
\end{aligned}
$$

    where $i_{65} = 0$

– Also,

$$
\begin{aligned}
& FSL \\
= \; & \frac{1}{60} \times \mathsf{SAL} \times \sum_{t=0}^{64-x} \left(\frac{{}^{z}M_{x+t}^{ia} - \frac{1}{2}{}^{z}C_{x+t}^{ia}}{s_{x-1}D_x} \right) \\
= \; & \frac{1}{60} \times \mathsf{SAL} \times \left(\frac{\sum_{t=0}^{64-x} {}^{z}\bar{M}_{x+t}^{ia}}{s_{x-1}D_x} \right) \\
= \; & \frac{1}{60} \times \mathsf{SAL} \times \left(\frac{{}^{z}\bar{R}_x^{ia}}{s_{x-1}D_x} \right)
\end{aligned}
$$

- Non-Salary Related Schemes: Ill-health Retirement

  – Pension of £P per year of service per annum

  – EPV of past service benefits paid in year $t \rightarrow t+1$

$$
\begin{aligned}
= \; & n \times P \times \bar{a}_{x+t+0.5}^{i} \times \left(\frac{i_{x+t}}{l_x} \right) \times v^{t+0.5} \\
= \; & n \times P \times \left(\frac{C_{x+t}^{ia}}{D_x} \right)
\end{aligned}
$$

- Past Service Liability:

$$\begin{aligned} PSL &= n \times P \times \left(\sum_{t=0}^{64-x} \frac{C_{x+t}^{ia}}{D_x} \right) \\ &= n \times P \times \left(\frac{M_x^{ia}}{D_x} \right) \end{aligned}$$

- Future Service Liability:

$$\begin{aligned} FSL &= P \times \sum_{t=0}^{64-x} \left(\frac{M_{x+t}^{ia} - \frac{1}{2} C_{x+t}^{ia}}{D_x} \right) \\ &= P \times \left(\frac{\bar{R}_x^{ia}}{D_x} \right) \end{aligned}$$

- Lump Sum of L per year of service
  - $C_y^i = i_y \times v^{y+0.5}$
  - $PSL = n \times L \times (\frac{M_x^i}{D_x})$ (Why?)
  - $FSL = L \times (\frac{\bar{R}_x^i}{D_x})$ (Why?)

- Age retirement
  - If $y < 65$, $C_y^{ra} = v^{y+\frac{1}{2}} \times r_y \times \bar{a}^r_{y+\frac{1}{2}}$
  - If $y = 65$, $C_{65}^{ra} = v^{65} \times r_{65} \times \bar{a}^r_{65}$
  - Past Service Liability:

$$\begin{aligned} PSL &= n \times \sum_{t=0}^{65-x} \left(\frac{PC_{x+t}^{ra} + LC_{x+t}^{r}}{D_x} \right) \\ &= n \left(\frac{PM_x^{ra} + LM_x^{r}}{D_x} \right) \end{aligned}$$

  - Future Service Liability:

$$\begin{aligned} FSL &= \sum_{t=0}^{64-x} \left(\frac{P\bar{M}_{x+t}^{ra} + L\bar{M}_{x+t}^{r}}{D_x} \right) \\ &= \frac{P\bar{R}_x^{ra} + L\bar{R}_x^{r}}{D_x} \end{aligned}$$

- Career Average Salary Schemes

- Pension per annum

$$= \frac{1}{60} \times \text{Total Salaries in Service}$$
$$= \frac{1}{60} \times \text{Career Average Salary} \times \text{Service}$$

- $x_0 =$ Age at entry to scheme

- $PSAL =$ Salary earned from $x_0 \to x$

- Past Service Liability:

$$PSL = \frac{PSAL}{60}\left(\frac{M_x^{ia} + M_x^{ra}}{D_x}\right)$$

- Future Service:
  - Ill-health
  - Age retirement

- Ill-health

  – EPV of future service benefits accrued in **ANY** future year

$$
\begin{aligned}
&= \frac{1}{60}\sum_{t=0}^{64-x}\frac{\mathsf{SAL}}{s_{x-1}}\Big(s_x + s_{x+1} + \cdots + \\
&\quad s_{x+t-1} + \frac{1}{2}s_{x+t}\Big) \times \bar{a}^{i}_{x+t+0.5} \times \left(\frac{i_{x+t}}{l_x}\right) \\
&\quad \times v^{t+0.5} \\
&= \frac{\mathsf{SAL}}{60 s_{x-1} D_x}\sum_{t=0}^{64-x} C^{ia}_{x+t}\Big(s_x + s_{x+1} + \\
&\quad \cdots + s_{x+t-1} + \frac{1}{2}s_{x+t}\Big)
\end{aligned}
$$

- Summation

$$
\begin{aligned}
&= s_x\left(\frac{1}{2}C_x^{ia} + C_{x+1}^{ia} + \cdots + C_{64}^{ia}\right) \\
&+ s_{x+1}\left(\frac{1}{2}C_{x+1}^{ia} + C_{x+2}^{ia} + \cdots + C_{64}^{ia}\right) \\
&+ \cdots + s_{64}\left(\frac{1}{2}C_{64}^{ia}\right) \\
&= \sum_{t=0}^{64-x} s_{x+t}\bar{M}_{x+t}^{ia} \\
&= \sum_{t=0}^{64-x} {}^s\bar{M}_{x+t}^{ia} = {}^s\bar{R}_x^{ia}
\end{aligned}
$$

- Hence,

$$FSL = \frac{\mathsf{SAL}}{60}\left(\frac{{}^s\bar{R}_x^{ia}}{s_{x-1}D_x}\right)$$

- Similarly for age retirement

- Age retirement:

$$FSL = \frac{\mathsf{SAL}}{60}\left(\frac{{}^{s}\bar{R}_x^{ra}}{s_{x-1}D_x}\right)$$

- Contributions:
  - The employer (and often the employee) will pay contributions (or premiums) to the pension fund
  - Salary-Related Contributions:

    $k =$ Contribution as a percentage of salary

– EPV of future contributions

$$
\begin{aligned}
&= \left(\frac{k}{100}\right) \times \mathsf{SAL} \times \sum_{t=0}^{64-x} \left(\frac{s_{x+t}}{s_{x-1}}\right) \\
&\quad \times \left(\frac{l_{x+t+0.5}}{l_x}\right) \times \left(\frac{v^{x+t+0.5}}{v^x}\right) \\
&= \left(\frac{k}{100}\right) \times \mathsf{SAL} \times \frac{\sum_{t=0}^{64-x} s_{x+t} D_{x+t+0.5}}{s_{x-1} D_x} \\
&= \left(\frac{k}{100}\right) \times \mathsf{SAL} \times \frac{\sum_{t=0}^{64-x} s_{x+t} \bar{D}_{x+t}}{s_{x-1} D_x} \\
&= \left(\frac{k}{100}\right) \times \mathsf{SAL} \times \frac{\sum_{t=0}^{64-x} {}^{s}\bar{D}_{x+t}}{s_{x-1} D_x} \\
&= \left(\frac{k}{100}\right) \times \mathsf{SAL} \times \frac{{}^{s}\bar{N}_x}{s_{x-1} D_x}
\end{aligned}
$$

- Fixed Contributions

  F = Contribution payable continuously (£ per annum)

- EPV of future contributions

$$= F \sum_{t=0}^{64-x} \left(\frac{l_{x+t+0.5}}{l_x} \right) \times \left(\frac{v^{x+t+0.5}}{v^x} \right)$$

$$= F \left(\frac{\bar{N}_x}{D_x} \right)$$

- $\bar{N}_x$ can be found from Yellow Tables, Page 143

- Return of Contributions

  - On death or withdrawal, a return of **employee** contributions accumulated with interest at j% per annum

  - ASAL = Salary earned $x_0 \to x$ accumulated at j% per annum

  - Past Service Liability:

$$PSL = \frac{k}{100} \times \text{ASAL} \times \left(\frac{{}^jM_x^w + {}^j M_x^d}{(1+j)^x D_x} \right)$$

- Future Service Liability:

$$FSL = \frac{k}{100} \times \text{SAL} \times \left(\frac{{}^{sj}\bar{R}_x^w + {}^{sj}\bar{R}_x^d}{(1+j)^x D_x} \right)$$

- Check **Yellow Tables, Pages 148 - 149, Page 148 for death and Page 149 for withdrawal**, accumulated interest rate $j = 2\%$ p.a. and discount interest rate $i = 4\%$ p.a.

- Green Table, Page 94, $j = 3\%$ p.a. and $i = 4\%$ p.a.

• Class Work: Suppose x_0 is the entry age; x is the current age; PSAL is the salary earned from x_0 to x; RATE is the salary rate at x. Derive suitable commutation functions to value an ill-health pension of $\frac{1}{60}$ average salary over member's career, per year of service

- Hints:
  - Construct a service table
  - Define a promotional salary scale
  - Evaluate the EPV of past service benefits paid in year from t to $t+1$
  - Evaluate the EPV of past service benefits paid in years from 0 to $65-x$

End of Chapter 1