MULTIPERIODIC PROCESSES IN FUNCTIONAL EQUATIONS

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Applying the formalism of dispersion equations there are discussed the solutions of functional equations having a close affinity with KdV and sG equations. There are considered functional and semifunctional (system of ode's) variants. Standard features typical for soliton type pde's appear in these cases also and particularly the multisoliton, multiphase quasi-periodic solutions and also solutions in form of solitons on periodic background exist.

Besides of a cognitive significance, the existence of soliton type solutions of functional equations is important also from an applicative point of view, since it makes a better approximation of numerical schemes for pde's.

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Introduction

Quite simultaneously with the beginning of soliton theory a discretized versions of partial differential equations (pde's) were considered exhibiting, as in case of Toda lattice, the multisoliton and quasiperiodic solutions. Strictly speaking it was a system of ordinary differential equations (ode) which can be considered as a particular case of pde's, when one independent variable was discretized. One can speak however about a more general case when this variable is continuous since a discrete variable will be a particular case of continuous one.

Thus considering nonlinear equations of two independent variables we can distinguish pde's, semi-differential equations (sd) and functional equations (fe), when a differentiation relates to both space and time variables (i.e. pde), next, when the differentiation relates to a single variable only (either space or time and we have ode) or the differentiation does not appear at all, respectively.

Below, we intend to discuss some equations of Korteweg de Vries (KdV) and sine-Gordon (sG) type and to show that sd-KdV, f-KdV sd- sG and f-sG admit not only multisoliton solutions but also quasiperiodic one as well as solutions in form of solitons on a quasiperiodic background. Solutions of this type we will denote later as solutions of a soliton-type (s-t).

A fact that besides of ds-equations there exists a class of fe's having s-t solutions seems interesting for two reasons: cognitive and applicative.

Quite recently there was shown that there exist quasiperiodic solutions of Calogero-Bruschi fe's and moreover, that a whole technique of "the finite gap integration" can be transformed to the matrix form and applied here [1]. This demonstrates how methods of the soliton theory are general. In the paper [2] there has been shown also that some sd-KdV and f-KdV equations have periodic although only single-phase solutions. Of course the multisoliton solutions were known in the past [3]. Enlarging a class of existing solutions there was shown in the paper [8] that sd-KdV and f-KdV equations have generally s-t solutions according to the meaning of this term adopted above.

Thus, the situation is quite analogous as for partial differential equations.

The second, applicative aspect relates to the numerical schemes. It is obvious that if pde of soliton type possesses a functional analogue, i.e. fe, having also soliton type solutions, one can expect that a numerical procedures based on this fe will represent a better approximation of pde since classes of solutions will be the same and the differences will have only a quantitative character. This will be seen clearly later on. In addition, such scheme automatically will conserve some constants of motion, since s-t fe, as a s-t pde should have also an infinite number of them. As it is known a numerical procedure for pde conserving at least some constants of motion (usually Hamiltonian) as a rule is much more effective than procedures without this property.

Discussing solutions of fe's we apply a technique of so-called dispersion equations (de's), e.g. [7], having a close relation with the bilinear operator technique used in the past effectively for various soliton equations. The method of de has some advantage over a bilinear technique since it is not restricted to the pure soliton solutions and can be applied without essential changes to all solutions of s-t.

In context of KdV-type equations discussion we will refer to paper [8] and the presented contribution can be considered as the generalization of results for pde's announced in [7]. Moreover, since the addition property, fundamental for next considerations, was originated by theory of Riemann theta functions, we precede general topics by a short and very elementary informations concerning Riemann theta functions.

Riemann theta functions

Let B be a Riemann matrix, (i.e. $B \in C^{g\times g}$ is symmetric and with a positively definite imaginary part), $x \in C^g$. The Riemann theta functions with characteristics $[\alpha, \beta]$, $\alpha, \beta \in R^g$, are defined as [4], [5], [6]

$$\Theta[\alpha,\beta](z|B) := \sum_{n\in Z^g} \exp\left\{i\pi[2\langle(z+\beta),(n+a)\rangle + \langle(n+\alpha),B(n+\alpha)\rangle]\right\}, \quad (1)$$

$$\text{with}\quad \langle \mathrm{z}, \mathrm{n}\rangle := \sum_{\mathrm{i}=1}^{\mathrm{g}} \mathrm{z}_{\mathrm{i}} \mathrm{n}_{\mathrm{i}}. \tag{2}$$

The Riemann theta function (ordinary) is defined usually assuming zero characteristics i.e. as

$$\Theta(z|B) := \Theta[0,0](z|B) = \sum_{n \in Z^g} \exp\left\{ i\pi [2\langle z, n\rangle + \langle n, Bn\rangle]\right\}. \tag{3}$$

If n and m are integer, $(m, n \in Z^g)$ then

$$\Theta(z + n + Bm|B) = \exp[-i\pi(2\langle z, m\rangle + \langle m, Bm\rangle)]\Theta(z|B). \tag{4}$$

In the one-dimensional case, when $z \in C$ and $B \in C$, the multiple sums in (3) reduce to a single one and then to the Riemann functions of a single variable. The four functions $\Theta[0,0](z|B)$, $\Theta[1/2,0](z|B)$, $\Theta[0,1/2](z|B)$ and $\Theta[1/2,1/2](z|B)$ also known as Jacobi functions and are usually discussed in textbooks.

In general, for fixed g, we deal with 2^{g+1} independent functions but the most important for our purposes is the "addition relation"

$$\Theta(z + w|B)\Theta(z - w|B) = \sum_{\epsilon \in Z_2^g} W_\epsilon(w) Z_\epsilon(z), \tag{5}$$

where sum is over $\epsilon \in Z_2^g$, i.e. over g-dimensional hypercube (the set of 2^g vectors with components either 0 or 1).Functions $Z_\epsilon(z)$ and $W_\varepsilon(w)$ are given by

$$Z_\epsilon(z) = \Theta(z + \epsilon/2|B)^2, \tag{6}$$

$$W_\varepsilon(w) = 2^{-g} \sum_{\mu \in Z_2^g} (-1)^{\langle \epsilon, \mu\rangle} \exp(i2\pi\langle w, \epsilon\rangle)\Theta(2w + B\epsilon|2B)/\Theta(B\epsilon|2B). \tag{7}$$

There are other equivalent variants of the addition property, but because of the validity in limit, a choice (6) and (7) is the most convenient for our purposes.

KdV-type equations

We shall consider so-called sd-KdV equation

$$u(x,t)_{,t} = [1 + u(x,t)]^2 [u(x - 1/2, t) - u(x + 1/2, t)], \tag{8}$$

for $x, t \in R$, with substitution

$$u(x,t) := \tau(x-1/2,t)\tau(x+1/2,t)/\tau(x,t)^2 - 1, \tag{9}$$

and f-KdV equation with a suitable substitution, (δ -const.)

$$-[1/u(x,t+\delta/2) - 1/u(x,t-\delta/2)]/\delta = u(x-1/2,t) - u(x+1/2,t), \tag{10}$$

$$u(x,t) := \tau(x-1/2,t)\tau(x+1/2,t)/\tau(x,t+\delta/2)\tau(x,t-\delta/2) - 1. \tag{11}$$

We are using a shorthand notation $u_{,t} := \partial u/\partial t$ and we stress that both variables x and t are continuous.

It is obvious that as $\delta \to 0$, then (10)$\Rightarrow$ (8) and (11)$\Rightarrow$ (9). Less obvious is a reason that (9) is related to KdV equation. The details one can find in [8] and we repeat arguments here. By substitution $v = \ln(1+u)$, equation (8) can be transformed into

$$v(x,t)_{,t} = \exp[v(x,t) + v(x-1/2,t)] - \exp[v(x,t) + v(x+1/2,t)],$$

and by a standard procedure [9], substituting $v(x,t) = h^2 w(x+h/2,t)$, with a space step h, taking the truncated Taylor series up to h^5, one obtains $w_{,t} = h^3 w_{,x} + h^5(2ww_{,x} + w_{,xxx}/24)$, which in variables $X = 8^{1/2}(x + h^3 t)$, and $T := h^5 8^{1/2} t/3$ takes standard KdV form.

The technique of dispersion equations developed earlier for pde's, allows to determine the multiperiodic (multi-gap) solutions, not referring them however to the structure of Riemann surfaces being a natural scenery when IST is applied. As it was mentioned above, it is surprising that for the functional equations, the formalism of dispersion equations and IST also holds.

As it is known, the essential for further consideration are functions $F : C^g \to C$ having the following factorization property:

$$F(z+w)F(z-w) = \sum_{\epsilon} W_\epsilon(w) Z_\epsilon(z), \tag{12}$$

where the sum is over a finite set and the functions Z_ϵ and W_ϵ also map $C^g \to C$.

A few classes of functions fulfill (12). Among them, there are reported above the Riemann theta functions and exponential functions occurring in solutions of soliton type equations.

It is obvious that $W_\epsilon(w) = W_\epsilon(-w)$, but the most important consequence of (12) is that for the functions F having this property one can easy calculate subsequent derivatives.

The reader is referred to [7] or [8] for details and here we report only final relations. For example, if the property (12) holds, we have

$$\frac{\partial \ln[F(z+w)/F(z-w)]}{\partial z_j} = \frac{\sum_\epsilon \frac{\partial W_{\epsilon,}(w)}{\partial w_j} Z_\epsilon(z)}{\sum_\epsilon W_\epsilon(w) Z_\epsilon(z)}. \tag{13}$$

Denoting $L := \ln F(z)$ and by $\partial/\partial z_j := (.)_{,j}$ for the second and fourth derivatives we have

$$2L(z)_{,ij} = \frac{\sum_\epsilon W_{\epsilon,ij}(0) Z_\epsilon(z)}{\sum_\epsilon W_\epsilon(0) Z_\epsilon(z)}, \tag{14}$$

$$L_{,ijkl} + 2(L_{,ij}L_{,kl} + L_{,ik}L_{,jl} + L_{,il}L_{,jk}) = \frac{\sum_\epsilon W_{\epsilon,ijkl}(0) Z_\epsilon(z)}{\sum_\epsilon W_\epsilon(0) Z_\epsilon(z)}, \tag{15}$$

respectively. Note that variable z on the right hand side, appears only as an argument of functions $Z_\epsilon(z)$ which can form a basis if are independent. The sequence of higher order differential operations can be determined and differential properties of F functions satisfying (12) form a certain, known before, so-called second hierarchy of KdV equations [8].

Equations (12)-(15) allow to write the dispersion equation for standard KdV equation and the same procedure can be applied to other s-t equations.

In order to illustrate we consider as the first a standard KdV equation. Indeed, for KdV equation in the form $u_t + 6uu_x + u_{xxx} = 0$, using the substitution

$$u(x,t) = 2(\ln F)_{xx}, \quad \text{where} \quad \mathrm{F} = \mathrm{F}(\kappa \mathrm{x} + \omega \mathrm{t}), \tag{16}$$

and $\kappa, \omega \in C^g$,with a c being constant, we obtain

$$\sum_{i,j=1}^{g} \omega_i \kappa_j L_{,ij} + \sum_{i,j,k,l=1}^{g} \kappa_i \kappa_j \kappa_k \kappa_l (L_{,ijkl} + 2(L_{,ij}L_{,kl} + L_{,ik}L_{,jl} + L_{,il}L_{,jk})) = c. \tag{17}$$

Making use of (12)-(15) and considering $Z_\epsilon(z)$ as linearly independent, for any ϵ specified by (12), equation (17) reduces to the system of algebraic equations

$$\sum_{i,j=1}^{g} \omega_i \kappa_j W_{\epsilon,ij}(0) + \sum_{i,j,k,l=1}^{g} \kappa_i \kappa_j \kappa_k \kappa_l W_{\epsilon,ijkl}(0) - cW_\epsilon(0) = 0. \tag{18}$$

It is a system of 2^g equations for $2g$ unknown quantities $\kappa_j/c^{1/4}, \omega_j/c^{3/4}, (j = 1, \dots g)$ and thus for $g > 2$ it is overdetermined. Hence for $g > 2$ the system of de's supplies additional conditions for involved functions, which, when quasiperiodic solutions are considered, are of great importance for determination a relevant Riemann surface solving so called Schottky problem, [10].

Since (18) involves the propagation vectors κ_j and angular frequencies ω_j, it is natural to call this system of equations just as a system of dispersion equations.

For $g = 1$, all functions are of a single variable and de reduces then to the simple formula

$$\omega = -\kappa^3 [W_1(0)W_0^{(4)}(0) - W_0(0)W_1^{(4)}(0)]/[W_1(0)W_0^{(2)}(0) - W_0(0)W_1^{(2)}(0)]. \tag{19}$$

For $g = 2$, the relations are only slightly more complicated, but for $g \geq 3$ the situation becomes qualitatively different, since system is overdetermined.

Till now all our considerations are related to standard KdV equations without a specification of the solution class and remain valid for multisoliton, quasiperiodic and other, mixed solutions. It appears however, that the same procedure can be repeated for sd and fe's.

Let us consider the sd-KdV equation, (8) with the substitution (9), which in fact represents the discretized version of $(\ln f)_{,xx}$. With a shorthand notation

$$\tilde{\tau}(\alpha) := \tau(\kappa(x + \alpha) + \omega t)), \tag{20}$$

where $\kappa, \omega \in C^g$, sd-KdV can be written in the form

$$\sum_{i=1}^{g} \omega_i \left\{ \ln \left[\frac{\tilde{\tau}(1/4 + 1/4)}{\tilde{\tau}(1/4 - 1/4)} \cdot \frac{\tilde{\tau}(-1/4 - 1/4)}{\tilde{\tau}(-1/4 + 1/4)} \right] \right\}_{,z_i} =$$

$$= \frac{\tilde{\tau}(-1/4 + 3/4)\, \tilde{\tau}(-1/4 - 3/4)}{\tilde{\tau}(-1/4 + 1/4)\, \tilde{\tau}(-1/4 - 1/4)} - \frac{\tilde{\tau}(1/4 + 3/4)\, \tilde{\tau}(1/4 - 3/4)}{\tilde{\tau}(1/4 + 1/4)\, \tilde{\tau}(1/4 - 1/4)}, \tag{21}$$

which is convenient in order to make use of equations (12) and (13). A quite elementary procedure [8] leads to conclusion similar as for standard KdV:

if there exists a solution of the system of algebraic equations (for each ϵ specified by addition property)

$$\sum_{i=1}^{g} \omega_i W_{\epsilon,i}(\kappa/4) + C_0 W_\epsilon(\kappa/4) + W_\epsilon(3\kappa/4) = 0, \tag{22}$$

where C_0 is a constant and $W_{\epsilon,i}(\kappa/4) := \frac{\partial W_\epsilon}{\partial w_i}\Big|_{w=\kappa/4}$, then equation (21) will be satisfied.

Since in this equation, the quantities to be determined are $\kappa \in C^g, \omega \in C^g$ and C_0, it is natural to denote it as a system of dispersion equations, similarly as previously for standard KdV.

For $g = 1$, we have in particular case

$$\omega = -\frac{W_0(3\lambda)W_1(\lambda) - W_1(3\lambda)W_0(\lambda)}{W_0'(\lambda)W_1(\lambda) - W_1'(\lambda)W_0(\lambda)}, \tag{23}$$

where $\lambda = \kappa/4$, and W' denotes the derivative with respect to an argument.

The same trick repeated with respect to the f-KdV (3) with substitution (4) leads to

$$W_\epsilon({}^1w)/\delta + W_\epsilon({}^2w) + CW_\epsilon({}^3w) = 0, \tag{24}$$

i.e. to a system of de's for f-KdV equation. Here

$${}^1w := (3\omega\delta + \kappa)/4,\ {}^2w := (\omega\delta + 3\kappa)/4,\ {}^3w := (\omega\delta - \kappa a)/4;\ {}^1w, {}^2w, {}^3w \in C^g.$$

C is again a constant with respect to x and t, dependent however on $\delta\omega$ and κ. Observe now that in contrast to the case of standard KdV both quantities to be determined $\kappa \in C^g$, and $\omega \in C^g$ appear as the arguments of W functions.

When $\delta \to 0$, as it was mentioned sd-KdV$\Rightarrow$f-KdV and this property concerns also the dispersion equations. Since for $\delta \to 0$, we have evaluation $C \sim -1/\delta + C_o + 0(\delta)$, also (24)$\Rightarrow$(22).

For $g = 1$ the dispersion relation reduces to

$$\delta = -\frac{W_0({}^1w)W_1({}^3w) - W_1({}^1w)W_0({}^3w)}{W_0({}^2w)W_1({}^3w) - W_1({}^2w)W_0({}^3w)}. \tag{25}$$

The class of functions F and then W and Z are not specified till now. There is only one requirement that the functions F have the addition property. If, as the functions F - the Riemann theta functions are chosen then (16) represent a g-phase quasiperiodic solution under condition that a suitable system of de's has nontrivial solution. Propagation vectors $\kappa \in C^g$, and angular frequencies $\omega \in C^g$, obtained from this system determine completely the quasi-periodic solution. As it was mentioned for $g \leq 2$ it can be done always, although for $g > 2$, the conditions for elements of B matrix have to be taken into account. In the language of the algebraic geometry g is a genus of the relevant Riemann surface and solvability of de's would have an interpretation that B-matrix is a period matrix of this Riemann surface.

As always, for $g = 1$ dispersion equations become relatively simply and we obtain

$$\pi\omega\Theta'[1/2, 1/2](0 \mid B) = \Theta[1/2, 1/2](\pi\kappa \mid B), \tag{26}$$

and

$$\Theta'[1/2,1/2](\pi\omega\delta \mid B) = \delta\Theta[1/2,1/2](\pi\kappa \mid B) \tag{27}$$

for sd-KdV and f-KdV, respectively. Up to the different definition of the Riemann theta, this result is equivalent with those given by in [2]. In the limit, we obtain the dispersion equations for a single soliton solutions

$$i\pi\omega = -\sinh(i\pi\kappa), \quad \text{and} \quad \sin(\pi\omega\delta) = -\delta\sin(\pi\kappa), \tag{28}$$

for sd-KdV and f-KdV, respectively.

For real δ the last equation has an imaginary solution $(\kappa, \omega \in iR)$ — then (16) represents the soliton solution. There also exists a real solution $(\kappa, \omega \in R)$ but then (16) represents a singular wavetrain solution. For the other s-t functional equations a situation looks similarly. Below we consider another example, namely f-sG equation.

Sine-Gordon equation

As before it would be convenient to use a shorthand notation

$$f_{m,n,\varepsilon} = F\left(\kappa m + \omega n + \varepsilon/2\right) \quad \textit{and} \quad f_{m,n} = F\left(\kappa m + \omega n\right), \tag{29}$$

where $\kappa, \omega \in C^g$, $\varepsilon \in Z_2^g$. Then we define the operators $\frac{\Delta^2}{\Delta h^2}$ and $\frac{\Delta^2}{\Delta s^2}$ by

$$\frac{\Delta^2 \ln f_{m,n}}{\Delta h^2} = \frac{1}{h^2}\left[\frac{f_{m+h/2,n} f_{m-h/2,n}}{f_{m,n}^2} - 1\right], \tag{30}$$

$$\frac{\Delta^2 \ln f_{m,n}}{\Delta s^2} = \frac{1}{h^2}\left[\frac{f_{m,n+s/2} f_{m,n-s/2}}{f_{m,n}^2} - 1\right], \tag{31}$$

which represents a discrete analogue of the second derivative of $\ln f$ with respect to m, or n , respectively, (since $(\ln F(x))'' = F''/F - (F'/F)^2$).

A functional version of sGe we assume in form

$$4\mathrm{Im}\left[\frac{\Delta^2 \ln \mathrm{f_{m,n}}}{\Delta \mathrm{h}^2} - \frac{\Delta^2 \ln \mathrm{f_{m,n}}}{\Delta \mathrm{s}^2}\right] = \sin\psi_{m,n}\ , \tag{32}$$

$$\psi_{m,n} = 2i\ln\frac{f^*_{m,n}}{f_{m,n}}\ , \tag{33}$$

Parameters h and s play a role of space and time steps, respectively. Due to definitions (30) and (31) a doubly discrete version of sGe one can write as

$$
\begin{aligned}
&\frac{1}{h^2}\left[\frac{f_{m+h/2,n,e}f_{m-h/2,n,e}}{f_{m,n,e}^2}-\frac{f_{m+h/2,n,0}f_{m-h/2,n,0}}{f_{m,n,0}^2}\right]\\
&-\frac{1}{s^2}\left[\frac{f_{m,n+s/2,e}f_{m,n-s/2,e}}{f_{m,n,e}^2}-\frac{f_{m,n+s/2,0}f_{m,n-s/2,0}}{f_{m,n,0}^2}\right]\\
=\ &\frac{1}{4}\left[\left(\frac{f_{m,n,e}}{f_{m,n,0}}\right)^2-\left(\frac{f_{m,,n,0}}{f_{m,n,e}}\right)^2\right]
\end{aligned}
\tag{34}
$$

which can be transformed to

$$
\begin{aligned}
&\frac{\left(f_{n+h/2,m,e}f_{n-h/2,m,e}\right)/h^2-\left(f_{n,m+s/2,e}f_{n,m-s/2,e}\right)/s^2+f_{n,m,0}^2/4}{f_{n,m,e}^2}\\
=\ &\frac{\left(f_{n+h/2,m,0}f_{n-h/2,m,0}\right)/h^2-\left(f_{n,m+s/2,0}f_{n,m-s/2,0}\right)/s^2+f_{n,m,e}^2/4}{f_{n,m,0}^2}
\end{aligned}
\tag{35}
$$

Next, since each side of equation should be constant, if the suitable functions are independent we obtain

$$\left(f_{n+h/2,m,e}f_{n-h/2,m,e}\right)/h^2-\left(f_{n,m+s/2,e}f_{n,m-s/2,e}\right)/s^2+f_{n,m,0}^2/4=Cf_{n,m,e}^2 \tag{36}$$

$$\left(f_{n+h/2,m,0}f_{n-h/2,m,0}\right)/h^2-\left(f_{n,m+s/2,0}f_{n,m-s/2,0}\right)/s^2+f_{n,m,e}^2/4=Cf_{n,m,0}^2 \tag{37}$$

Using an addition property (12) we obtain two sets of equations which are equivalent to the set ($\delta\in Z_2^g$) of 2^g algebraic equations

$$W_\delta\left(\kappa h/2\right)/h^2-W_\delta\left(\omega s/2\right)/s^2-CW_\delta\left(0\right)+W_{e-\delta}(0)/4=0. \tag{38}$$

It can be rewritten in a more legible form

$$\frac{\Delta^2 W_\delta\left(\kappa h/2\right)}{\Delta h^2}-\frac{\Delta^2 W_\delta\left(\omega s/2\right)}{\Delta h^2}+\tilde{C}\delta_{\delta,0}+\delta_{e,\delta}/2=0, \tag{39}$$

where $\tilde{C}=2\left(C-\frac{1}{h^2}-\frac{1}{s^2}\right)$ is a new constant and

$$\frac{\Delta^2 W_\delta\left(\kappa h/2\right)}{\Delta h^2}:=\left[W_\delta\left(\kappa h/2\right)+W_\delta\left(-\kappa h/2\right)-2W_\delta\left(0\right)\right]/h^2 \tag{40}$$

Equation (39) represents a system of 2^g algebraic equations for determination of 2g+1 quantities $\kappa_1 ... \kappa_g, \omega_1 ... \omega_g$ and $\tilde{C}$. In the limit, when $h\to 0$ and $s\to 0$ we obtain well known de's for standard sGe [7]

$$\sum_{0\le i\le g}\sum_{0\le j\le g}\left(\kappa_i\kappa_j-\omega_i\omega_j\right)W_{\delta,ij}\left(0\right)-\tilde{C}\delta_{\delta,0}+\delta_{e,\delta}/2=0. \tag{41}$$

Thus for fixed and finite h and s the functional sGe has a solution provided there exist the solutions of suitable algebraic equations (39).

For the quasi periodic functions, since Riemann theta functions have the property (5)-(7), similarly as for ds-KdV or f-KdV, one can write de's for quasiperiodic solutions substituting (7) into (39), which will involve the finite second differences of Riemann theta functions. Since relations are rather long, although the procedure is elementary, we confine ourselves only reporting a limiting case of dispersion equation for a single soliton solution of f-sGe

$$\left[\frac{\sinh\left(\kappa h/2\right)}{h/2}\right]^2-\left[\frac{\sinh\left(\omega s/2\right)}{s/2}\right]^2=1,$$

which in the limit, when $h\to 0$ and $s\to 0$ takes a standard form $\kappa^2-\omega^2=1$.

The mixed solutions.

It is natural to consider the soliton processes as a limiting form of quasi-periodic ones and multisoliton process as a limit of a multiphase quasi-periodic process. One can also discuss a class of "intermediate" solutions, where the limiting procedure does not relate to the all phases of quasi-periodic solution, but only to a few of them. The solution in a form of multisoliton on a background of multiphase periodic process are obtained then immediately, [7], [8], although the cases of higher gena require additional restrictions.

Let the Riemannian matrix $B\in C^{g\times g}$ be decomposed into blocks $B^{ss}\in C^{s\times s}, B^{pp}\in C^{p\times p}, B^{ps}\in C^{p\times s}$ and $B^{sp}=(B^{ps})^t$ where superscript t denotes transposed matrix. Superscript s relates to the soliton subspace and p - a periodic subspace. Of course $s+p=g$ and matrices B^{ss} and B^{pp} are also Riemannian. Let the argument $z\in C^g$ be similarly decomposed into $z_s\in C^s$ and $z_p\in C^p$. The matrix $D^{ss}:=\mathrm{Im[Diag(B^{ss})]}$ has s diagonal elements d_i. We define the S-lim operation (S-oliton limit) as an s-fold limit

$$\mathrm{S-}\lim\Theta(\mathrm{z|B}):=\lim_{\mathrm{d_i}\to\infty}\Theta\left[\begin{array}{c|cc} z^s-iD^{ss}e^s/2 & B^{ss} & B^{sp}\\ z^p & B^{ps} & B^{pp}\end{array}\right]=:\mathrm{T(z)}, \tag{42}$$

where all elements of diagonal matrix d_i tend to infinity and $e^s=(1,\ldots,1)^t\in Z^s$. This limit we denote as $T(z)$ .It turns out that the limit for the fixed Riemannian matrix B exists always, and $T(z)$ functions satisfy have the addition property.

The explicit form of $T(z)$ functions following from (42) is

$$T(z) = \sum_{n\in(Z_2)^s} \exp\left[i\pi(2\langle n, z^s\rangle + \langle n, \widetilde{B}^{ss} n\rangle)\right] \Theta(z^p + B^{ps} n | B^{pp}), \tag{43}$$

where $\widetilde{B}^{ss} := B^{ss} - iD^{ss}$, i.e. having only real diagonal elements.

In application to the f-KdV (or sG) equation, the solution in form of soliton on a periodic background ($s = p = 1$) is discussed in paper [8], and the relevant T function has then a form

$$T(z) = \Theta(z_2|B_{22}) + \exp[i\pi(2z_1 + \widetilde{B}_{11})]\Theta(z_2 + B_{12}|B_{22}), \tag{44}$$

where index 1 is ascribed to soliton process and index 2 - to the periodic one. $\widetilde{B}_{11}$ is real and without losing generality can be chosen as zero. It is seen that the solution $(\ln T)_{xx}$ asymptotically ($z_1 \to \pm i\infty$) has a form of a periodic process, but there exists a shift between the left and right hand side asymptotics imposing by the off-diagonal element of the starting B matrix i.e. B_{12}. Thus the situation is quite identical as for standard KdV equation. The solution of f-KdV equation in form of single soliton on periodic background was discussed just in paper [8].

Fof sGe standard or functional one we have quite analogous situation. Since the solution is given always as $\psi = 2i \ln(T^*/T)$, substituting (44), in limit, when $z_1 \pm i\infty$ the solution has asymptotic evaluation $\psi \sim 2i \ln(\Theta^*/\Theta)$ with arguments either z_2 or $z_2 + B_{12}$, respectively. This means that asymptotically we deal with periodic processes but shifted in phase one with respect to other, as before for KdV equation.

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