Two-Dimensional Solitary Waves in Arrays of Nonlinear Waveguides

T. Peschel and F. Lederer
Institut für Festkörpertheorie und Theoretische Optik
Friedrich-Schiller-Universität Jena
Max-Wien-Platz 1, 07743 Jena, Germany
email: ohp@physik.uni-jena.de

Abstract

Using both a variational and a fully numerical approach we derive solutions to the two-dimensional Discrete Nonlinear Schrödinger Equation which describes stationary optical beams in two-dimensional arrays of nonlinear waveguides. We compare our results to the respective solutions for the time-dependent case in a one-dimensional DNLS, which in contrast to the above fully discrete one represents a mixed discrete-continuous system. Possibilities for moving solitary waves are discussed on the basis of the Hamiltonian of the systems.

1 Introduction

There is a growing interest in arrays of nonlinear waveguides [1], because they may provide controllable point-to-point connections between different input and output channels [2, 3, 4]. Furthermore the propagation of light in arrays of nonlinear waveguides represents a fundamental problem, which has parallels in very different branches of natural sciences, like the description of excitations in macromolecules [5], the behaviour of electrical networks [6] or lattice vibrations in solids [7]. The common property of such systems is their discrete character.

The characteristic modes of excitation found in the system of waveguides, are so-called solitary waves, which may propagate across the array without changing their shape due to diffraction. From the point of view of applications they can be regarded as channels in which information is guided from the input to the output ports. Different possibilities to control their direction of propagation have been identified:

i) variation of the phase of the input beam [8, 9],

ii) variation of the coupling strength across the array, which acts like an effective potential [3, 4] or

iii) variation of the guided power which leads to a localization of the beam [1, 10].

There are two problems of major interest, which will be the topic of the present paper:

i) Three-dimensional integrated optics will use multible layers of waveguides, and hence involve two-dimensional arrays of waveguides [13, 12, 11].

ii) Any kind of information processing will require the usage of short pulses [14]. Thus the question arises, how the pulse duration will influence the propagation in arrays.

The latter problem seems similar to the first one, but exhibits also some pecularities due to its mixed, discrete-continuous character. Hence, the question remains, if it is possible to approximate the derivatives with respect to time simply by finite differences.

Furthermore we will discuss, how the additional degrees of freedom introduced by the higher dimensionality will influence the possibilities for controlling the propagation of solitary waves.

Throughout the paper we will use the variational approach as a tool to obtain equations, which determine the physical parameters of the solitary wave. This approach has been proven fruitful already for many different nonlinear systems [15], because it guarantees the best possible result for the trial function used. Of course the discrete character of the system will require a modification of the variation procedure [9].

2 Variational Model

We start our considerations with the coupled-mode equations in two dimensions, which determine the evolution of the amplitudes $u_{m,n}$ in the individual waveguides labelled by m, n. We assume a Kerr type nonlinearity.

$$0 = i\frac{du_{m,n}}{dz} + (u_{m-1,n}u_{m+1,n}) + C\,(u_{m,n-1}u_{m,n+1}) + 2\left|u_{m,n}\right|^2 u_{m,n} \qquad (1)$$

Without loss of generality we have expressed the coordinate of propagation z in units of the inverse coupling strength, which refers to the 1-direction. The equations of motion (1) may be derived from the Lagrangian of an infinite two-dimensional array.

$$\begin{aligned} L \;=\; & \sum_{m,n=-\infty}^{\infty}\left[\frac{i}{2}\left(u^*_{m,n}\frac{du_{m,n}}{dz} - u_{m,n}\frac{du^*_{m,n}}{dz}\right) + \left(u_{m+1,n}u^*_{m,n} + u^*_{m+1,n}u_{m,n}\right)\right. \\ & \left. +C\left(u_{m,n+1}u^*_{m,n} + u^*_{m,n+1}u_{m,n}\right) + \left|u_{m,n}\right|^4\right] \qquad (2) \end{aligned}$$

In what follows we will assume a spatial solitary wave with a sech shape in each direction as a trial function

$$\begin{aligned} u_n \;=\; & \tilde{A}\sqrt{b_1 b_2}\,\mathrm{sech}\,(b_1(m-x_1))\;\mathrm{sech}\,(b_2(n-x_2)) \\ & \times \exp\left(i\left(\phi + k_1(m-x_1) + k_2(n-x_2) + c_1(m-x_1)^2 c_2(n-x_2)^2\right)\right) \qquad (3) \end{aligned}$$

which is centered at the position (x_1, x_2) and may show phase differences k_1, k_2 between adjacent waveguides. To allow for a variable width $1/b$ of the beam we have added a phase front curvature c for each dimension, which is the conjugate variable to b. Then the maximum amplitude of the spatial solitary wave is given by $A = \tilde{A}\sqrt{b_1 b_2}$.

Using the trial function (3) we may reformulate the Lagrangian (2) in terms of soliton parameters

$$L = -E\left[\frac{d\phi}{dz} + L_1(1) + L_1(2) + h(1,2)\right] \qquad (4)$$

where the arguments 1 and 2 stand for all variables with indices 1 an 2, respectively,

$$L_1(j) = -k\frac{dx_j}{dz} + \mathrm{S}_1(b_j, x_j)\left(\frac{dk_j}{dz} - 2c_j\frac{dx_j}{dz}\right) + \mathrm{S}_2(b_j, x_j)\frac{dc_j}{dz} \tag{5}$$

Here we use the Hamiltonian h

$$h(1,2) = -2\mathrm{Z}(b_1, c_1, x_1, k_1) - 2C\mathrm{Z}(b_2, c_2, x_2, k_2) - E\mathrm{N}(b_1, x_1)\mathrm{N}(b_2, x_2) \tag{6}$$

where E denotes the total guided power. Both E and the Hamiltonian h are conserved quantities. The first terms in the Hamiltonian h refer to coupling, while the third one, which depends explicitly on the total guided power E, contains the effect of the nonlinearity.

The abbreviations used in (4-6) for the different sums over the trial function may be found in the appendix. The sums show a characteristic periodicity with the position of the solitary wave x because they are invariant when an arbitrary integer number is added to x and the summation index is redefined appropriately. Because of the smaller width of $\mathrm{sech}^4(x)$ compared to $\mathrm{sech}^2(x)$ this periodic behavior is most pronounced for the nonlinearity term $\mathrm{N}(b, x)$. Eventually its periodicity proves responsible for the localization of the solitary wave.

In fig.1 we have plotted the functions $\mathrm{Z}(b, 0, x, 0)$ and $\mathrm{N}(b, x)$, which determine the Hamiltonian, in dependence on the beam width and position. The role of the nonlinearity (N) is increasing for narrower beams, while coupling (Z) becomes weaker. Furthermore fig.1 demonstrates the increase in spatial modulation for narrow beams, which is the reason for localization.

3 Results

Stationary solitary waves always correspond to extrema of the Hamiltonian eq.(6). From the approximate form of the coupling term eq.(11) we may easily deduce the conditions for the phase front tilt $k_j = 0$ and curvature $c_j = 0$.

The optimization with respect to the beam position results in values of $x = 0, 1/2$, which correspond to excitations centered on one or between two waveguides, respectively. From the behaviour of the functions Z and N (see fig.1) we may deduce, that $x = 1/2$ represents a saddle point of the Hamiltonian. The respective solutions will be unstable, but they may be important as intermediate states of a moving solitary wave.

The remaining task is the minimization of the Hamiltonian with respect to the inverse beam widths b_j. In the next step the propagation constant may be found easily by a differentiation of the Lagrangian eq.(4) with respect to the guided power E to be

$$\frac{d\phi}{dz} = -h(1,2) + \mathrm{N}(b_1, x_1)\mathrm{N}(b_2, x_2) \tag{7}$$

Explicite stationary solutions for the total guided power as a function of the beam parameters may be obtained in two limiting cases only:

i) If the coupling in both directions is equal, and the soliton is centered either on a single waveguide $x_1 = x_2 = x = 0$ or in the center between four waveguides $x_1 = x_2 = x = 1/2$. Then also $b_1 = b_2 = b$ holds and the minimum of the Hamiltonian may be written as

$$E = -2\left.\frac{\dfrac{\partial \mathrm{Z}(b, 0, x, 0)}{\partial b}}{\mathrm{N}(b, x)\dfrac{\partial \mathrm{N}(b, x)}{\partial b}}\right|_{x=0,1/2} \tag{8}$$

ii) If the coupling in one direction is very strong ($C >> 1$). This case is of particular interest, if we regard eq.(1) as a discrete approximation to the time dependent one-dimensional system, where $C = 1/\Delta t^2$. Then our model describes the propagation of a light bullet through the array. In this case $b_2 << 1$ holds, and we may expand the Hamiltonian around $b_2 \approx 0$ and obtain

$$E = \sqrt{-12C \frac{\frac{\partial \mathrm{Z}(b_1, 0, x_1, 0)}{\partial b_1}}{\mathrm{N}(b_1, x_1) \frac{\partial \mathrm{N}(b_1, x_1)}{\partial b_1}} \Bigg|_{x_1=0,1/2}} \tag{9}$$

where $x_1 = 0, 1/2$, and $b_2 = E/(2C)\mathrm{N}(b_1, x_1)$ hold. If we are interested in the time-dependent problem we have furthermore to consider the pulse energy $\tilde{E} = \Delta t E$.

In the general case finding the stationary solitary wave involves solving an algebraic equation.

First we present results for both the dispersion relation and the Hamiltonian of solitary waves in an symmetric array in figures 2 and 3. Using the variational approach we obtain a power threshold for stationary solutions. The only stable solutions are highly localized and correspond to the upper branch of the dispersion relation. The lower branch, which has a negative slope, is unstable. This result is consistent with those obtained from the respective continuous model where no stable solitons can exist.

However, the fully numerical approach yields an additional stable branch, which exists also for arbitrarily small power. This apparent contradiction may be resolved by looking at the distribution of the amplitudes over the array, which is nearly sin-shaped. The confinement of the field is due to the boundaries of the array rather than by nonlinearity. Thus the lower stable branch may be identified as a result of the finite array size, which cannot be dealt with by the variational approach. Of course this "bistable" behaviour of the dispersion relation, which resembles that of the so-called nonlinear guided waves [16], may be very interesting for possible applications.

From the Hamiltonian displayed in figure.3 we find, that for solitary waves which correspont to the upper branch of the dispersion relation there is a rather high barrier between the stable state centered on a waveguide and the unstable intermediate one between two waveguides. This will make it very unlikely to find a moving solution. The reason is, that the phase front tilt needed to overcome the difference in the Hamiltonian would be of the order of $\pi/2$, which would destabilize the beam considerably [9].

In the next step we considered an asymmetric array, where the coupling in one direction is six times that in the other one ($C = 6$, see figures.4 and 5). In this case localization takes place in two steps, each of which is connected with one backbending branch in the dispersion relation.

The respective dependence of the Hamiltonian is depicted in figure.5. Again we find a rather high barrier for the motion in the first direction where the coupling is weak. In the direction of strong coupling the barrier is much lower. Thus motion in this direction is possible.

In the time-dependent (that is mixed discrete-continuous) case the variational approach again yields a threshold to the necessary pulse energy of a stable solitary wave (see figure.6). If we use, in contrast, the fully numerical approach we find an additional low-energy branch. From the respective amplitude distributions we find that the latter one corresponds to a well-defined temporal soliton on the finite one-dimensional array, which is confined spatially merely by the boundaries of the array.

From the behaviour of the Hamiltonian (see figure.7) we may again deduce strong restrictions for a motion of light bullets across the array.

4 Conclusions

We have shown that the behaviour of solitary waves in an two-dimensional array of waveguides may be described well by a discrete variational approach. A characteristic feature of both discrete and mixed discrete-continuous models of finite arrays are the S-shaped dispersion relations. Their stable branches correspond to quasi linear modes of the array and highly localized solitary waves, respectively. In all cases considered the motion of solitary waves across the array seems to be strongly limited by localization, if we start from the stable solutions.

5 Appendix

The abbreviations used for the different sums over the trial function are defined as follows:

$$\mathrm{S}_j(b,x) = \frac{\sum\limits_{n=-\infty}^{\infty} (n-x)^j \mathrm{sech}^2\left(b(n-x)\right)}{\sum\limits_{n=-\infty}^{\infty} \mathrm{sech}^2\left(b(n-x)\right)} \tag{10}$$

$$\begin{aligned}\mathrm{Z}(b,c,x,k) &= \sum_{n=-\infty}^{\infty}\Bigg[\mathrm{sech}\left(b(n-x+1)\right)\mathrm{sech}\left(b(n-x)\right) \\ &\quad \times \cos\left(k+2c(n-x+\frac{1}{2})\right)\Bigg]\left[\sum_{n=-\infty}^{\infty}\mathrm{sech}^2\left(b(n-x)\right)\right]^{-1} \\ &\approx 2\pi\frac{\sin(c)}{\sinh(\pi c/b)\sinh(b)}\cos(k)\left[\sum_{n=-\infty}^{\infty}\mathrm{sech}^2\left(b(n-x)\right)\right]^{-1} \end{aligned} \tag{11}$$

$$\mathrm{N}(b,x) = \frac{\sum\limits_{n=-\infty}^{\infty}\mathrm{sech}^4\left(b(n-x)\right)}{\left[\sum\limits_{n=-\infty}^{\infty}\mathrm{sech}^2\left(b(n-x)\right)\right]^2} \tag{12}$$

where all sums have been normalized in order to factor the total guided power out of the Lagrangian.

Acknowledgement

The authors want to acknowledge financial support by the Deutsche Forschungsgemeinschaft in the framework of their initiative for optical signal processing.

References

[1] D. N. Christodoulides and R. I. Joseph, "Discrete self-focusing in nonlinear arrays of coupled waveguides", Opt. Lett. **13**, 794 (1988).

[2] H. A. Haus and L. Molter-Orr, "Coupled multiple waveguide systems", IEEE J. Quantum Electron. **QE-19**, 840 (1983).

[3] R. Muschall, C. Schmidt-Hattenberger, and F. Lederer, "Spatially solitary waves in arrays of nonlinear waveguides", Opt. Lett. **19**, 323 (1994).

[4] T. Peschel, R. Muschall, and F. Lederer, "Localization and beam steering with spatial solitons in arrays of nonlinear waveguides", Vol. 6, 1995 OSA Technical Digest Series (Opt. Soc. Am., Washington DC, 1995), pp. 20-22.

[5] A. S. Davydov and N. I. Kislukha, "Solitary excitons in one-dimensional molecular chains", Phys. Status Solidi (B) **59**, 465 (1973).

[6] P. Marqui, J. M. Bilbault, and M. Remoissenet, "Observation of nonlinear localized modes in an electrical lattice", Phys. Rev. E **51**, 6127 (1995).

[7] T. Holstein, "Studies of polaron motion; part I: the molecular-crystal model" Ann. Phys. **8**, 325 (1959).

[8] W. Krolikowski, U. Trutschel, M. Cronin-Golomb, and C. Schmidt-Hattenberger, "Solitonlike optical switching in a circular fiber array", Opt. Lett. **19**, 320 (1994).

[9] A. B. Aceves, C. De Angelis, T. Peschel, R. Muschall, F. Lederer, S. Trillo, and S. Wabnitz "Discrete Self-Trapping, Soliton Interactions, and Beam-Steering in Nonlinear Waveguide Arrays", to be published in Phys. Rev. E

[10] J. C. Eilbeck, P. S. Lomdahl, and A. C. Scott, "The discrete self-trapping equation", Physica D **16**, 318 (1985).

[11] W. K. Mesenzev, S. L. Musher, I. W. Ryzhenkova, S. K. Turitsyn, "Two-dimensional solitons in discrete systems" JETP Letters **60**, 815 (1994) (in russian).

[12] W. J. Firth and B. A. Samson, "Clusterization and contrast enhancement at light propagation in a cable of nonlinear coupled fibers", *Digest of Nonlinear Guided-Wave Phenomena (NLGW) Conference 1993, September 20-22, Cambridge, UK* (Optical Society of America, Washington, DC, 1993), p. 82.

[13] D. B. Mortimore and J. V. Arkwright, "Monolithic wavelength-flattened 1x7 single-mode fused fiber couplers: theory, fabrication and analysis", Appl. Opt. **30**, 650 (1991).

[14] A. B. Aceves, C. De Angelis, A. M. Rubenchik, and S. K. Turitsyn, "Multidimensional solitons in fiber arrays", Opt. Lett. **19**, 329 (1994).

[15] D. Anderson, "Variational approach to nonlinear pulse propagation in optical fibers", Phys. Rev. A **27**, 3135 (1983).

[16] A. D. Boardman, P. Egan, F. Lederer, U. Langbein, and D. Mihalache, "Third-order nonlinear electromagnetic TE and TM guided waves", in *Nonlinear surface electromagnetic phenomena* eds. H. E. Ponath and G. I. Stegeman, Amsterdam, North Holland 1991.

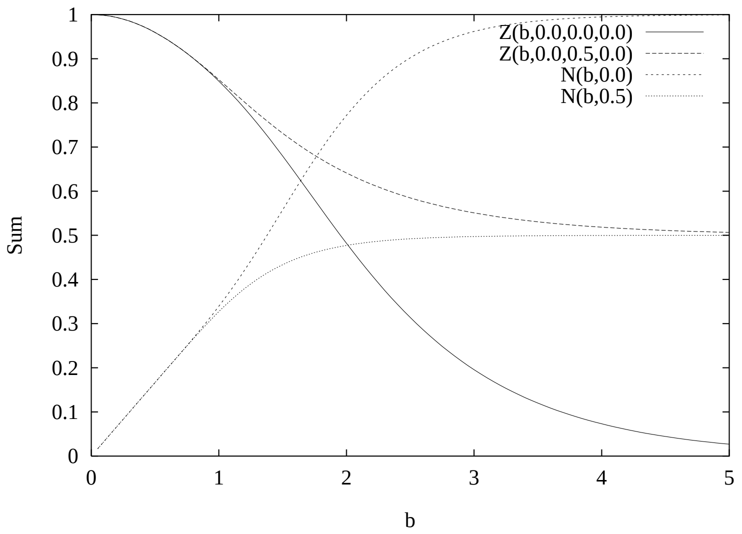

Figure 1: Functions $\mathrm{Z}(b,0,x,0)$ and $\mathrm{N}(b,x)$, which describe coupling and nonlinearity, respectively, in dependence on the inverse beam width b and the beam position x. $x = 0, 1/2$ stand for an excitation centered on a waveguide or between two waveguides, respectively.

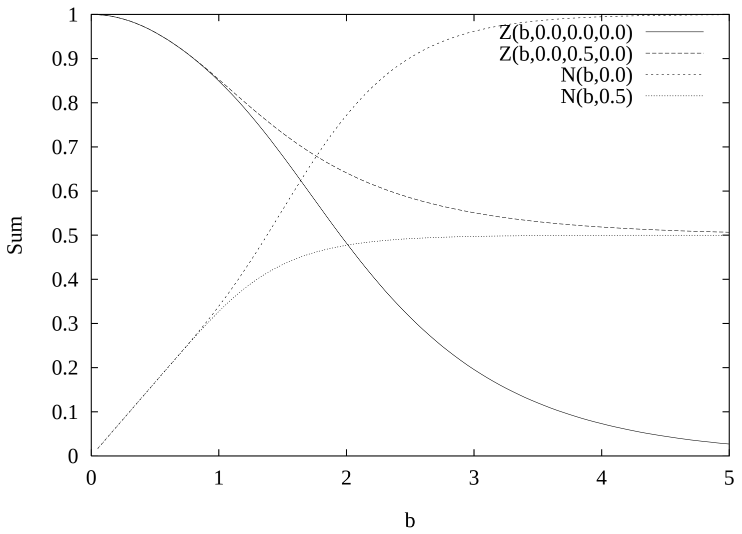

Figure 2: Dispersion relation as a function of the guided power for spatial solitary waves on a symmetric ($C = 1$) two-dimensional array. Variational results are given by lines, while the points represent numerical data. $x_2 = 0$ has been assumed.

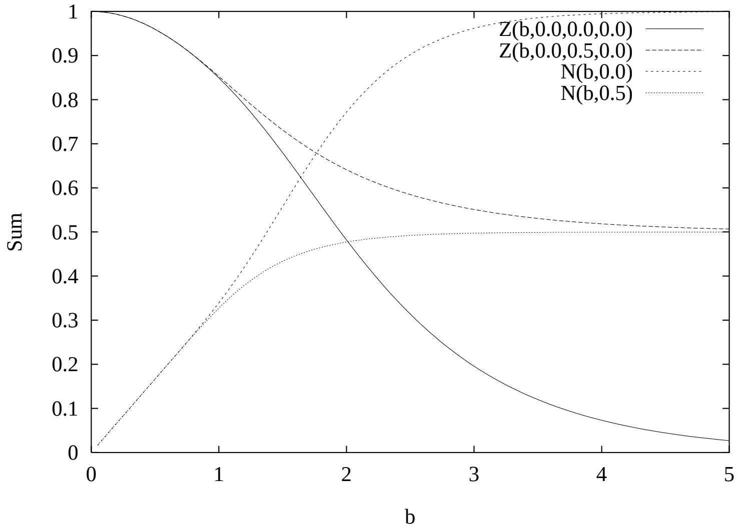

Figure 3: Hamiltonian as a function of the guided power for spatial solitary waves on a symmetric ($C = 1$) two-dimensional array.

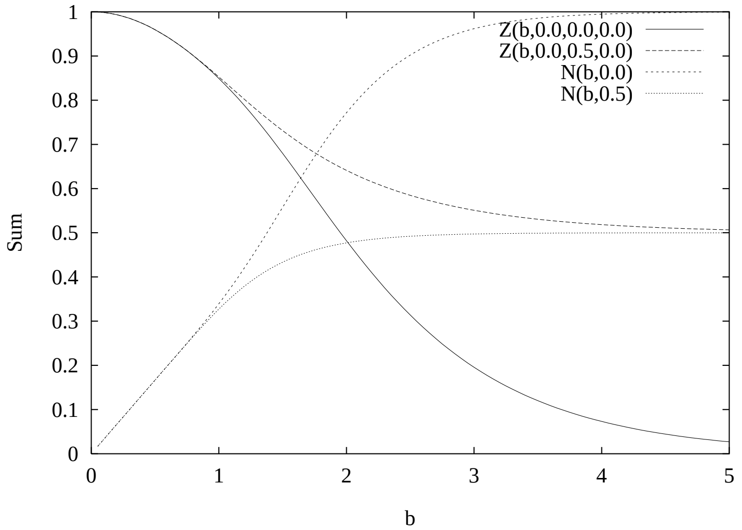

Figure 4: Variational results for the dispersion relation of an asymmetric array with ($C = 6$) demonstrate the occurance of gradual localization.

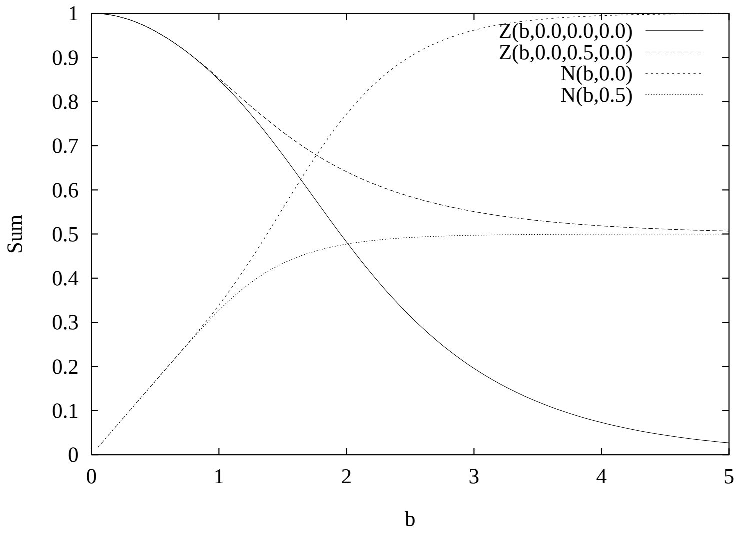

Figure 5: Hamiltonian as a function of the guided power for spatial solitary waves on an asymmetric ($C = 6$) two-dimensional array.

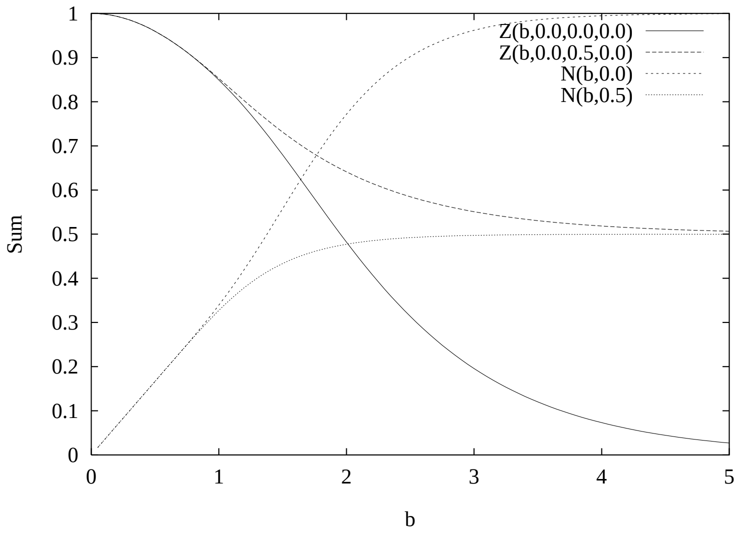

Figure 6: Dispersion relation as a function of the pulse energy for the time-dependent model (see eq. (9)). Lines represent variational and points numerical results, respectively. The pulse energy is defined as $\tilde{E} = \Delta t E$. The numerical data use a discretization $\Delta t = 0.2$.

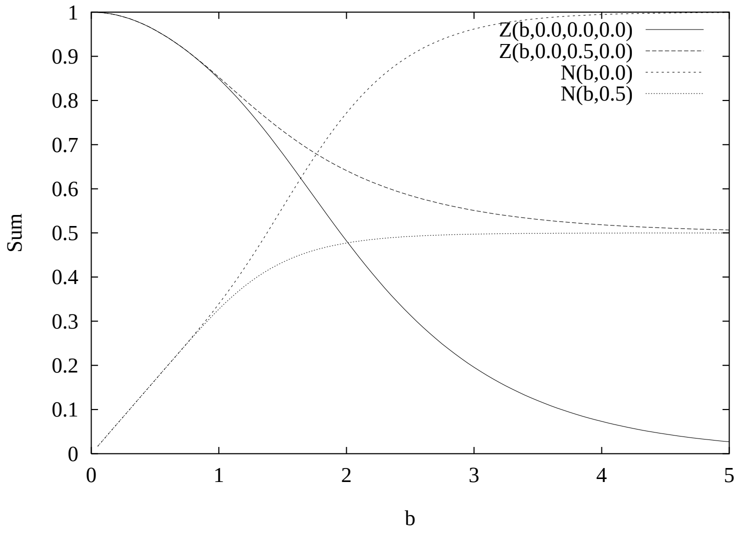

Figure 7: Hamiltonian as a function of the pulse energy for the time-dependent model. The parameters are the same as in fig. 6. The line $k = \pi/2$ has been added to demonstrate the hight of the localization potential.