How to predict, generate and observe strain solitons in solids

Alexander M. Samsonov

Theoretical Dept., A.F.Ioffe Physical Technical Institute,
the Russian Academy of Sciences, St.Petersburg, 194021 Russia
e-mail: samsonov@math.ioffe.rssi.ru

Abstract

Mathematical theory of long nonlinear strain waves in elastic wave guides was developed in order to initiate the successful generation of strain solitons in solids. New nonlinear quasi hyperbolic 'double dispersion equation' (DDE) was derived to describe long nonlinear wave evolution in a thin hyperelastic rod.
Various exact travelling wave solutions to the DDE were obtained, additional conditions were found, under which the long nonlinear solitary strain waves may exist in nonlinearly elastic materials.
Based on the theory, the experiment was proposed and longitudinal strain solitons in nonlinear elastic wave guides were generated successfully, using laser pulse source, and observed by means of holographic interferometry method. The compression strain solitary wave as a whole is shown on pictures. We demonstrate that the strain soliton can be reproduced and observed in a physical lab in ordinary conditions.

1 Introduction

Even recently one of the most surprising things in physics of condensed matter implied that there was no individual mention to experiments devoted to successful observation of solitary strain wave in solids. Indeed, the description of a long wave propagation in liquids and in solids is quite similar, providing a good forecast in soliton existence in solids. Even at the time of the pioneering soliton generation experiments made by J.Scott Russell the theory of elasticity was sufficiently developed, and the nonlinearity of elastic materials and the stress-strain relation was widely discussed, see, e.g., a review by J.F.Bell (1973).

When a powerful strain wave propagates in a nonlinearly elastic and bounded solid, the curvating of a wave front can increase rapidly right up to the irreversible deformations appearance. This phenomenon can be balanced with the dispersion of a wave inside a wave guide, having small, finite but not an infinitesimal cross section size, that is in close correspondence with an appropriate balance in a shallow water wave theory. However the velocity of a longitudinal (density) wave in solids is much greater then in liquids, [1] and it was

[1] I am pleased very much to refer to new successful experiment in solitons on shallow water surface, see *Nature* Editorial (1995) and the Soliton Home pages on this server

a reason of former explanation of unsuccessful soliton observation in experiments in solids.

The solitary strain wave was not observed in solids till the last decade. Furthermore, the availability of powerful and highly resolutive modern methods of wave detection in solids allowed to believe that a problem was in the theory also. Solitary wave solutions were found and thoroughly studied in multiple (and very useful) numerical simulations of nonlinear elasticity problems, nevertheless the difference between virtual and physical reality remained remarkable.

The mathematical problem of nonlinear elastic guided wave propagation can be reduced to coupled nonlinear p.d.e. In order to initiate a physical experiment a theoretician has to study their simplest reductions to the one p.d.e., to select types of self-similar (or travelling) waves as the simplest detectable signals, to estimate the influence of inhomogeneity, impurity and dissipation processes, as well as the results of numerical simulation as the cheapest experiment, being very keen about possible artefacts.

As a result of theoretical analysis the appropriate type of material should be chosen, as well as the suitable detection, generation and observation methods. After that an experimental setup will be determined. Our experience in this area led us to the conclusion that the more complex is a nonlinear phenomenon, the more simple has to be an observation approach.

Both physical and geometrical nonlinearities of solids have various sources, e.g., structural and kinematic nonlinearities, such as the Coulomb friction force, that discontinuously depends upon the body slip velocity, rotational twisting of elastic helix, motion of a beam with elastic constrains etc., and complex nonlinearities as thermomechanical stresses, prestressing of an elastic material and viscous (active) liquid—solid interactions.

Nevertheless, the intrinsic or *material nonlinearity* of elastic solids and the *nonlinearity of deformation* are of most importance for nonlinear elastic wave propagation problems.

The subdivision of elastic material's nonlinearity is carrying out, as a rule, into the 'geometrical' and 'physical' nonlinearities. The first one is governed by the finite deformation tensor (the Green tensor) :

$$\mathbf{C} = [\nabla\mathbf{U} + (\nabla\mathbf{U})^T + \nabla\mathbf{U}\cdot(\nabla\mathbf{U})^T]/2 \tag{1}$$

written in terms of a vector gradient $\nabla\mathbf{U}$ and its transpose $(\nabla\mathbf{U})^T$, both defined with respect to the displacement vector $\mathbf{U} = \{U_k\}$. The third term in (1) reflects small but finite, however *not infinitesimal* deformations (as it is in linear elasticity), and strains, resulted from them, respectively.

The 'physical' nonlinearity is based on the representation of an internal energy U of a material as a partial sum of power series written with respect to $I_k(\mathbf{C}^n)$, $k = 1, 2, 3$, and with coefficients, depending on the elastic moduli (λ, μ) of second order, and of third order – ν_k, etc.,

$$\rho_0 U = (\lambda/2)I_1^2(\mathbf{C}) + \mu I_1(\mathbf{C}^2) + (\nu_1/6)I_1^3(\mathbf{C}) + \nu_2 I_1(\mathbf{C})I_1(\mathbf{C}^2) + (4\nu_3/3)I_1(\mathbf{C}^3) + \dots \tag{2}$$

Terms proportional to $\nabla\mathbf{U}\cdot(\nabla\mathbf{U})^T$ represent the *geometrical* nonlinearity, to ν_i – the *physical* nonlinearity; evidently this subdivision is colloquial one due to internal overlap in the invariants of $\mathbf{C}$:

$$I_1(\mathbf{C}) = tr\mathbf{C},\ I_2(\mathbf{C}) = [(tr\mathbf{C})^2 - tr\mathbf{C}^2]/2,\ I_3(\mathbf{C}) = \det\mathbf{C}, . \tag{3}$$

Figure 1: Stress-strain relations can be either linear (a), or nonlinear even for small elastic deformations (b).

Figure 2: Diagrams of potentials introduced by Toda *(a)* and by Morse *(b)*.

It was shown that both nonlinearities must be taken into consideration, even for metals under finite, but not infinitesimal deformation.

Surprisingly the concept of nonlinear elasticity is not a modern one. Various corrections of linear stress-strain relation begun shortly after a linear Hookean law publication in 1678, see Bell (1973), and we will cite only few of them proposed by Bernoulli (1694) as $\epsilon = a\sigma^m$, by Poncelet (1839) for brass in tension $\epsilon = \sigma[a + b\exp(m\sigma)]$Hodgkinson (1849) [2] as $\sigma = a\epsilon + b\epsilon^2$.

It is well understood now that various hyperelastic continuum models of real materials should be based on the corresponding limits of atomic lattice models. The interaction lattice potential and, consequently, the potential energy of deformation both will have to contain not only terms of the second order with respect to the distance between atoms but also the higher order terms. Only for sufficiently small and one-dimensional longitudinal lattice displacements, i.e., variations of interatomic distance, the energy can be approximated by means of a parabola, and strains will be linear in displacement gradients, respectively. Otherwise one should consider the often called "physical" nonlinearity of solids together with the nonlinearity in deformations.

For the atoms interaction in a crystalline lattice of a solid the Morse potential function (1929) is valid, introduced in a different problem [3] (quantum mechanics of an electron motion):

$$\mathcal{V}(x) = \mathcal{V}_0[\exp(-2\alpha x) - 2\exp(-\alpha x)], \tag{4}$$

with constant values of α and $(-\mathcal{V}_0)$.
It was shown by Valkering, (1978), that it governs precisely the nearest-neighbour interaction in an anharmonic lattice of the atoms in a diatomic molecule of solids in a continuum limit, see Fig.2. Still no exact solutions are found for elasticity problems governed by the Morse potential.

Toda & Wadati, (1973), introduced an exponential (Toda) potential function

$$\mathcal{V}(x) = (a/b)\exp(-bx) - ax,$$

and obtained some exact explicit solutions for the model. It can be reduced in the continuum limit to nonlinear Boussinesq equation, however a linear growth for $x \to \infty$ and a requirement for constants $ab > 0$ provides $V(x) > 0$, and the negative value of a potential for $x = 0$ requires to add a negative constant. These constraints seems to be very restrictive from physical viewpoint, too.

[2]the last one was proposed for considerable use by the British Iron Committee in 1849 for iron in tension, compression and flexure and is not changed up to date. Is it a genuine source of efficiency of british industry?

[3]A caution: the potential in atomic interaction theory differs from the potential energy by a factor, namely, by an electron (particle) charge only, and for this reason physicists use the term 'potential' for brevity. There is no real charge in continuum mechanics, therefore one should introduce the potential energy of interaction, appropriately normalized, instead of interaction potential itself, especially, when a model is subjected to generalize to the continuum limit.

We shall start with an approximation suitable for small (elastic) deformations due to sufficiently long wave propagation. If the deformation process is adiabatic, the internal energy U can be expressed as a partial sum of power series with respect to $I_k(\mathbf{C}^n),\ k=1,2,3,$ as it was defined in (2).

Recalculating $I_k(\mathbf{C}^n)$ of power of $\mathbf{C}$ in terms of $I_k(\mathbf{C})$, the energy U can be expressed in terms of $I_k(\mathbf{C})$ only, and with coefficients, depending on the elastic moduli (λ,μ) of 2nd order, and those of 3d order (l,m,n), of 4th order (γ_i) etc., that introduces a model proposed by Murnaghan (1954) for a hyperelastic medium:

$$\Pi=(\lambda+2\mu)I_1/2-2mI_2+(l+2m)I_1/3-2mI_1I_2+nI_3+\gamma_1I^4+\gamma_2I_2^2+\gamma_3I_1I_3+\gamma_4I_2^2+\dots \quad (5)$$

Well before Murnaghan a general power series expansion was proposed by Landau and Rumer (1937), (see also Landau, Livshitz, 1987). They formally introduced two square (u_{ik}^2,u_{ll}^2) and three cubic $(u_{ll}^3,\ u_{ll}u_{ik}^2,\ u_{ik}u_{il}u_{kl})$ scalars defined by components of a symmetric second rank tensor $\mathbf{u}$. The most general scalar form, containing square and cubic terms with respect to u_{ik} will be defined as follows:

$$\mathcal{E}=\mu u_{ik}^2+(K/2-\mu/3)u_{ll}^2+(A/3)u_{ik}u_{il}u_{kl}+B\ u_{ll}u_{ik}^2+(C/3)u_{ll}^3 \quad (6)$$

where the bulk modulus K was introduced. Substituting here (1) will result in the elastic energy expression with terms up to the third order in $(\partial u_i/\partial x_k)$. All coefficients of (5) can be transformed into those of another set (6) and vice versa.

The Murnaghan approximation (5) for Π is not unique. The Blatz-Ko potential, (1962) was proposed for a rubber-like (compressible) material in a form:

$$\Pi=\mu\kappa[i_1-3+(1/i_3^\gamma-1)]/(2\gamma)+\mu(1-\kappa)[i_2/i_3-3+(i_3^\gamma-1)]/(2\gamma), \quad (7)$$

where $\gamma=2\nu/(1-2\nu),\ \kappa$ is the only modulus of third order $(0.5<\kappa<1)$, and all $i_k=i_k(\mathbf{G})$ are determined for the Green deformation measure tensor $\mathbf{G}=2\mathbf{C}+\mathbf{E}$, i.e.,

$$i_1=2I_1+3,\ i_2=4I_2+4I_1+3,\ i_3=2I_1+4I_2+8I_3+1,\ I_k=I_k(\mathbf{C}). \quad (8)$$

For incompressible materials, with a restriction $\det\{\mathbf{G}\}=1$ being taken into account, the Mooney potential, (1940) can be used in a form

$$\Pi=\mu[(3-\eta)I_1+2(1-\eta)I_2]/2,\ \mid\eta\mid\le 1, \quad (9)$$

where μ is the shear modulus and η is the third order elastic modulus. For Treloar's model, the simplest one, it is assumed in addition that $\eta=1$ and $\nu=1/2$:

$$\Pi=EI_1/3. \quad (10)$$

Each of these relations results in coupled nonlinear p.d.e. and the problem arises how to get the only equation for a nonlinear wave motion? For usage of variational principle it is assumed to be given the volume density U of internal energy as the function of entropy $\mathcal{S}$ and invariants I_k of any tensor of deformation. Another thermodynamic potential can be considered for thermoelastic deformations, e.g., the Helmholz free energy $\mathcal{F}$, see (Engelbrecht, 1983; Maugin, 1994).

Several points can be outlined here. Firstly, despite of a common experience with erasers and child's catapults, not only for rubber-like materials but even for metals the physical nonlinearity should be taken into account, see, e.g., Parker, (1984). For copper and iron, brass and lead the next order elasticity values are much larger then the Lamé constants, and therefore may provide the relative increase in potential energy up to 20% for a small second order terms in tensor **C**, as it was shown by Pleus and Sayir, (1983). Similar problems for granular and porous materials were considered also by Wright (1984) and Nesterenko (1992). One can conclude that the potential energy density must remain fully nonlinear, unless a correctness of any particular linear approximation stated. In other words the condition $\mid \partial U_i/\partial x_j \mid << 1$ does not necessarily lead to the relatively small addition in strains, $(u_{nonlin} - u_{lin})/u_{lin} << 1$, that is typical for linear theory of elasticity, and the reason of this difference consists, at least, in possible large intervals of space and time.

Evidently we could not describe concisely main ideas of mathematical and physical treatment of nonlinear elasticity problems, and the best solution seems to refer to the well known books by Lurie, (1980); Maugin, (1988); Engelbrecht, (1983), and some others. Nevertheless, aiming to consider the simplest wave problem in a solid nonlinear wave guide, we will derive the main equation from the first principles in a form useful for further application to physical experiments, see Samsonov (1994), Dreiden et al (1988) for details.

2 Nonlinear wave in a rod

Let us consider the nonlinear wave propagation problem for an isotropic homogeneous nonlinearly elastic rod. Introducing the time variable t and the cylindrical coordinate system in space as (x, r, φ); where x is axis along the rod, $\varphi \in [0, 2\pi]$, $\quad 0 \le r \le R(x)$, we write the displacement vector as: $\mathbf{U}(u_i) \equiv \mathbf{U}(U, V, W)$, and assume each cross section to be circular after deformation.

Consider the nonlinearly elastic Murnaghan's material, having the elastic moduli of 2nd order (E, ν) and of 3rd order (l, m, n).

The Lagrangian density is written in a standard form:

$$\mathcal{L} = \frac{\rho}{2}\left(\frac{\partial \mathbf{U}}{\partial t}\right)^2 - \rho \Pi\left(I_k(\mathbf{C})\right) \tag{11}$$

that will lead to the coupled p.d.e after appropriate variation. To reduce them to the one equation we shall introduce the following assumptions:

1. torsions will be neglected: $\mathbf{U}_\varphi = 0$;

2. small, finite, but not infinitesimal strains: $\forall i, k \in [1, 3], \quad \partial u_i/\partial x_k << 1$ are considered;

3. long waves are considered only, having the length: $\lambda >> R$;

4. each cross sections remains plane after deformation: $U_r = 0$; then $u_i = u_i(x, t)$;

5. the following relationship (A.E.Love;1934) is valid between the main component of longitudinal strain U_x and transversal displacement V:

$$V = -\nu r U_x, \tag{12}$$

To avoid the possible lack of accuracy in asymptotics of stress tensor components on lateral surface of a wave guide the last assumption should be replaced with the refined relationship (Porubov, Samsonov; 1993):

$$U = U(x,t) + r^2\nu U_{xx}/2; \quad V = -r\nu U_x - r^3\nu^2 U_{xxx}/(2(3-2\nu)), \tag{13}$$

that leads to different coefficients in resulting equation, while its differential form will remain the same.

Taking the assumptions 1)-5) into consideration, we obtain from (11) the Lagrangian density for a long longitudinal wave in a rod made of any nonlinearly elastic material mentioned above:

$$\mathcal{L} = \frac{\pi R^2}{4}\left[(2\rho U_t^2 - 2EU_x^2) - \frac{2\beta U_x^3}{3} + \rho\nu^2 R^2 U_{xt}^2 - \mu\nu^2 R^2 U_{xx}^2\right] \tag{14}$$

where ρ is density per unit volume, E is the Young modulus, μ is the shear modulus, while the nonlinearity coefficient β depends upon the elastic moduli of 2nd and 3d orders, and for the Murnaghan model it is equal to $\beta = \beta(E, \mu; l, m, n)$.

The action functional A is defined using (11) as:

$$\mathcal{A} =_{t_0} \int^{t_1} \left(\int_W \mathcal{L}(\cdot)dW\right) dt; \tag{15}$$

that provides to use the Hamilton principle :

$$\delta\mathcal{A} = 0. \tag{16}$$

and to obtain from (14) the dimensional Double Dispersive Equation (DDE) in the form of the quasi hyperbolic nonlinear p.d.e of 4th order:

$$U_{tt} - \frac{\partial}{\partial x}\left[\frac{1}{\rho R^2}\left(ER^2 U\right)_x\right] = \frac{1}{2\rho R^2}\frac{\partial}{\partial x}\left[\beta R^2 U_x^2 + \rho\nu^2 R^4 U_{xtt} - (\mu\nu^2 R^4 U_{xx})_x\right], \tag{17}$$

as it was derived in 1982 by the author, see e.g., (Samsonov, 1988). When the wave guide is homogeneous and isotropic, the DDE can be reduced to the simplest form, written with respect to the strain component $u \equiv U_x$:

$$u_{tt} - c_0^2 u_{xx} = (1/2)\left[c_2 u^2 + \nu^2 R^2 (u_{tt} - c_1^2 u_{xx})\right]_{xx} \tag{18}$$

where all $c_0^2 = E/\rho$, $c_1^2 = \mu/\rho$, $c_2 = \beta/\rho$, $\mu = E/(2(1+\nu))$ and R, ν are constants now.

If the refined relations (13) between U and V mentioned above are assumed, then we obtain the equation with different coefficients:

$$u_{tt} - c_0^2 u_{xx} = (1/2)\left[c_2 u^2 + \nu R^2\left((\nu-1)u_{tt} + c_0^2 u_{xx}\right)\right]_{xx} \tag{19}$$

In dimensionless form both versions of the DDE are written as:

$$u_{tt} - u_{xx} = \varepsilon(u^2 + au_{tt} - bu_{xx})_{xx}; \tag{20}$$

where a, b are constant. Similarly to it, the 1st Boussinesq equation can be written as:

$$u_{tt} - u_{xx} = \varepsilon(u^2 + au_{tt})_{xx}; \tag{21}$$

and the 2nd Boussinesq equation is:

$$u_{tt} - u_{xx} = \varepsilon(u^2 - bu_{xx})_{xx}; \tag{22}$$

The corresponding dispersion relations determined as $\omega = \omega(k)$ for $u \propto \exp i(kx - \omega t)$ are different for these equations. Indeed for the DDE one has:

$$\omega^2_{DDE} = k^2(1 + bk^2\varepsilon)/(1 + ak^2\varepsilon), \tag{23}$$

whereas for the 1st and the 2nd Boussinesq equations, respectively¡

$$\omega^2_{1B} = k^2/(1 + ak^2\varepsilon), \qquad \omega^2_{2B} = k^2(1 + bk^2\varepsilon). \tag{24}$$

Note the finite and non-zero value of the phase velocity $c_{ph} \neq 0$ for $k \to \infty$ of the wave described by the DDE. Moreover, the DDE wave can propagate without energy transfer:

$$c_G = \partial\omega/\partial k = 0 \quad \text{for} \quad c_{ph} = \omega/k = \sqrt{\varepsilon(a - b)/(1 + a\varepsilon k^2)} \neq 0. \tag{25}$$

Therefore energy could be locked or probably go back in a hyperelastic solid while the nonlinear wave would propagate in positive direction of x.

The DDE and both the Boussinesq eqs. are the nonlinear hyperbolic p.d.e. of 4th order. Two approaches can be used to obtain the travelling wave solution, depending upon the phase variable $z = x \pm Vt$, namely, the direct integration of the NHE, and the standard reduction to the nonlinear evolution equation (NEE), as follows:

$$\tau = t\varepsilon^{k+1}, \; \xi = \varepsilon^k(x - Vt); \tag{26}$$

that provides the NEE:

$$u_\tau + (u^k)_\xi/k + cu_{\xi\xi\xi} = O(\varepsilon). \tag{27}$$

For example, when $k = 2$ it leads to the KdV equation, while when $k = 3$ - to the mKdV equation. However, the problem arising on this way is how to satisfy the initial conditions prescribed for the initial problem: $u(t = 0) = u_0(x)$?

Indeed, if:

$$\partial^m u/\partial z^m \to 0, \quad |z| \to \infty, \quad m = 0...3, \tag{28}$$

both approaches are equal, while if

$$u \to u_1, z \to +\infty; \quad u \to u_2, z \to -\infty; \tag{29}$$

togheter with :

$$\partial^{(m+1)}u/\partial z^{(m+1)} \to 0, \quad z \to \infty, \quad m = 0...2 \tag{30}$$

there is no equivalence. In fact, we deal with the singular perturbation boundary value problem, arising when $\varepsilon \to 0$ in the NEE limit, this fact is important also for the problem under consideration.

Respectfully, the comparison between the DDE and the corresponding KdV equation, both written in dimensional form leads to the following different results. The DDE (18) has an exact solitary wave solution:

$$u = \frac{3(V^2 - c_0^2)}{c_2} \cosh^{-2} \frac{x \pm Vt}{\lambda} \equiv A \cosh^{-2} \frac{x \pm Vt}{\lambda} \tag{31}$$

where the wave length is

$$\lambda = \nu R \sqrt{\frac{2(V^2 - c_1^2)}{V^2 - c_0^2}} = \nu R \sqrt{2\left(1 + \frac{3(c_0^2 - c_1^2)}{A c_2}\right)} \tag{32}$$

Therefore the solitary wave velocity is estimated as

$$V^2 = \frac{A\beta}{3\rho} + c_0^2 = c_0^2(1 + A + AFf(l, m, n)) \tag{33}$$

and the strain soliton exists iff

$$V > c_0 \quad \text{or} \quad V < c_1 < c_0 \tag{34}$$

being either compressive or tensile one, in dependence of the sign of $\beta = 3E + f(l, .m, n)$.

Therefore the "dead zone" of velocities exists, in which an initial pulse does not provide the soliton generation. The KdV equation provides different and incorrect estimations for both the velocity V and the wave length λ :

$$c_0 < V < \infty, \; \lambda \sim \sqrt{1/A} \tag{35}$$

Another solitary wave on pedestal may be considered for (18) also:

$$u = u_0 + \left[\frac{3(V^2 - c_0^2)}{c_2} - 3u_0\right] \cosh^{-2} \frac{x \pm Vt}{\lambda} \equiv A \cosh^{-2} \frac{x \pm Vt}{\lambda} \tag{36}$$

with

$$\lambda = \nu R \sqrt{\frac{2(V^2 - c_1^2)}{V^2 - c_0^2 - c_2 u_0}} \tag{37}$$

Formally, this wave may exist iff $c_1^2 < V^2 < c_0^2 - (\beta/\rho)u_0$, however a pedestal is equvalent to the pre-stressed wave guide, that should be taken into account during the statement of the problem. Also in experiments it may be generated in a pre-stressed wave guide with an initial strain proportional to the 'pedestal'; this experiment is rather difficult.

The soliton propagation in a rod with varying cross section was studied also, see (Samsonov, 1988; 1994); theory was proved in experiments in soliton propagation along this geometrically inhomogeneous rod, that will be published elsewhere.

Figure 3: Sharp a and massive b initial pulses result in different soliton trains

3 Soliton observation

Numerical simulation of various initial pulses generation and propagation subjected to the DDE model (i.e., of a non-stationary problem) shown the dependence of both mass and a tail type behind a solitary wave upon the mass (duration) of initial pulse. It was found that a sharp pulse provides a solitary wave with a decaying oscillating tail behind, while a massive one leads to the solitary wave train followed by the oscillating wave trains, having an almost sinusoidal envelope wave, see Fig.3 . That seems to be a sign of non-integrability of the DDE by means of the IST method however several conserved densities were found for the equation, see Samsonov (1988).

In order to avoid any plastic flow of a specimen and to keep the elastic type of deformations the parameters of a pulse should satisfy the following condition for the finite strain tensor component:

$$|\sqrt{1+2\mathbf{C}_{xx}}-1| < Y_0, \tag{38}$$

where Y_0 is the yield point of a material. The amplitude of an initial pulse should be given in order to balance nonlinear and dispersive features of a wave guide. The parameters of tentative initial pulse and the corresponding solitary wave in a rod are summarized in the Table below, (see also Samsonov & Sokurinskaya (1988)), where 'C' denotes possible compressive strain, 'T' - the tensile strain, W is a transversal displacement of lateral surface and the velocity difference is $\delta V = (V - c_0)/c_0$.

The apparatus used for generation and observation of strain solitons (see Fig. 4) consisted of a channel to produce the wave in a solid, a synchronizer, a holographic interferometer and a device for measuring the laser pulse energy.

Table

Initial and solitary wave parameters in rod made of various nonlinearly elastic materials

	INITIAL	PULSE	DATA		SOLITARY	WAVE	DATA	
	Strain type	Initial pressure P_0, atm	Duration τ_0, μsec	Starting distance l, cm	Amplitude modulus A	Length λ, mm	Relative velocity δV	W, μm
Steel, Rex535	C	400	8	600	$2 \cdot 10^{-4}$	61	10^{-4}	0.2
Brass, L62	C	1000	6	61	10^{-3}	28	10^{-3}	1.2
Polystyrene	C	200	5	1.2	$6 \cdot 10^{-3}$	45	10^{-2}	8.0
NaCl	C	80	5	110	$2 \cdot 10^{-4}$	32	10^{-3}	1.2

Figure 4: Setup for nonlinear wave motion study in a rod

Figure 5: Wave propagation stages in a water cell and in a rod: 1) - weak shock wave initiation; 2) - soliton observation

Quartz	T	- 900	2	44	$1.2 \cdot 10^{-3}$	22	10^{-3}	0.2
PMMA	T	- 200	13	300	$4 \cdot 10^{-4}$	130	10^{-4}	0.6
Glass	T	- 100	6	110	$1.3 \cdot 10^{-4}$	66	10^{-4}	0.1

Solitons in a transparent solid rod were formed in our experiments from an initial shock wave, which was initiated in a liquid, surrounding the rod, by means of the laser explosive evaporation of a metal from a foil target surface, that was placed nearby the input rod's cross section.

The power density of laser radiation acting on the target was measured by the energy meter and was kept constant (equal to $2.3 \cdot 10^8 W/cm^2$) during the experiment. Observations were made in the perpendicular direction by means of the pulsed laser used for holographic interferometry, and two parallel cut-offs were made along a specimen in order to make transparent the central part of a rod. To study the soliton evolution we used a polysterene rod of 1 cm thick, that was probed by a plane light beam (50 mm in diameter) at different regions, the holograms of focused images were recorded. The setup allowed us to change precisely the delay time between the pulses from the laser, inducing a shock wave and from the recording one.

The wave picture was visualised due to the interferometry fringe shift Δk , which amplitude was used for direct calculations. The wave patterns were recorded at the various distances from the input edge of the rod. The wave propagation pictures obtained in experiments are shown on Figures 6, 7 and 8. The left and right edges of the frame are located, respectively, on 40 mm and 90 mm from the input cross section of the rod on Fig.6, on 60 and 110 mm on Fig.7 and on 75 and 125 mm on Fig.8. The sharp kink on the right side of each photograph corresponds to a weak shock wave (A), that has produced the strain soliton. The fringe shift, representing a soliton, is extracted and plotted separately in each graph below the observation area image for convenience. Let us note here that the strain soliton is the density wave, but not at all the lateral surface elevation wave as it could be on a water surface; the extraction is made for convenient measurements.

The recorded interferograms allowed us to follow the evolution of the wave pattern in the rod and to visualize the process of soliton formation and its succeeding propagation along the waveguide. We observed that shock waves (which are residuals of the initial shock wave used for soliton generation) decay, moving along the rod, however, their velocities remain higher than both the sound (rod's) velocity in polystyrene and the soliton velocity (from frame to frame the soliton lags behind them). The soliton exhibits a rather extended water-basin—shaped longitudinal wave, which is not followed by any tensile wave. Moving along the rod, the soliton does not undergo any noticeable change of its shape and amplitude; that is the well known major property of a nonlinear solitary wave, detected now in a nonlinear elastic wave propagation. Lack of attenuation is a distinctive soliton feature from any linear pulse, that is expected to disappear in PS at 10 radii distance from the input.

Figure 6: Soliton detection on distance 40-90 mm from the input tip.

Figure 7: Soliton detection on distance 60-110 mm from the input tip.

The following quantitative estimations of the soliton parameters were reached. The soliton length λ was measured directly from the interferogram, and the soliton amplitude A was calculated from the carrier fringe shift obtained by reconstruction of doubly exposed holograms and shown on interferogram.

It was found by means of the known formula (see, e.g., Dreiden et al., (1988)):

$$A = -\frac{\lambda \Delta k}{2h[(n_1 - 1)(1 - 2\nu) + \nu(n_1 - n_0)]} \tag{39}$$

where n_0, and n_1 are transparency indices of water and deformed material resp., λ is the light wave length, $2h$ is rod's thickness along the detecting laser light direction.

Figure 8: Soliton detection on distance 75-125 mm from the input tip.

We obtained the following parameters for a typical recorded soliton: $A = 2.82 \cdot 10^{-4}$; $\lambda = 32.5$ mm, that are in good agreement with our theoretical estimations.

Thus, using the laser weak shocks generator and the holographic interferometry method, we have generated, detected and recorded the strain solitary wave (a soliton) inside the nonlinearly elastic solid waveguide (a rod).

Soliton detection may allow to determine intrinsic physical properties of nonlinearly elastic materials and the wave guide parameters. It can be used for explanation of possible damage of optical fibers due to the light backscattering, as mentioned by Coste & Montes (1986) and Montes (1987), for energy concentration in a prescribed wave guide cross section, for energy transfer for quite a long distance in elastic structures made of materials, which are not transparent for linear waves due to viscosity etc.

Acknowledgement
It is a pleasure to thank G.Dreiden, A.Porubov and I.Semenova for close collaboration and illuminating discussions. The support of this study from the International Science Foundation and the Russian Government (Grants R56000 and R56300) is gratefully acknowledged.

References

[1] Bell J.F. (1973), Experimental foundation of solid mechanics. In: Encyclopaedia of Physics, YIa/1, S.Flügge ed., Springer, Berlin et al.

[2] Blatz, P.J., (1969). Application of large deformation theory to the thermo-mechanical behavior of rubberlike polymers-porous, unfilled and filled. Rheology - Theory and Applications, vol.5, (ed. F.R.Eirich), Academic Press, pp.1- 55.

[3] Chree C. (1889), Trans. Phil. Soc. Cambridge, v. 14.

[4] Clarkson P.A., LeVeque R.J., Saxton R. (1986) Solitary wave interactions in elastic rods. Stud. Appl. Math., v.75, 2, 95-122.

[5] Coste J., Montes C. (1986). Asymptotic evolution of stimulated Brillouin backscaterring: implication for optical fibers. Phys.Rev.A., 34, 5, 3940.

[6] Crandall S.H. (1974), Nonlinearities in structural dynamics. Shock and Vibration Dig., 6, 8, 1-13.

[7] Dreiden, G.V. et al., (1988). Formation and propagation of strain solitons in non-linearly elastic solids. Sov.Phys.-Techn.Phys., v.33, no.10, p.1237-1241.

[8] Dreiden, G.V. et al., (1995). Experiments in the propagation of longitudinal strain solitons in a nonlinearly elastic rod. Techn.Phys. Letters, v.21, no.6, pp.415-417.

[9] Editorial: Soliton wave receives crowd of admirers. Nature, v.376, 3 Aug 1995, p.373.

[10] Engelbrecht J. (1983) Nonlinear wave processes of deformation in solids. London, Pitman.

[11] Love, A.E.H., (1952). A treatise on the mathematical theory of elasticity (4th ed). Cambridge: Cambridge University Press.

[12] Lurie A.I., (1980). Nonlinear theory of elasticity. Moscow, Nauka, 512 pp. *in Russian* (English edition: Amsterdam, Elsevier, 1990).

[13] Maugin G.A. (1994) Physical and mathematical models of nonlinear waves in solids. In: Jeffrey, Engelbrecht eds. (1994). Amsterdam et al., Elsevier.

[14] Mooney, M. (1940). A theory of large elastic deformation. J.Appl.Phys., 11, 582-592.

[15] Montes C. (1987). Inertial response to nonstationary stimulated Brillouin backscatering: damage of optical and plasma fibers. Phys.Rev.A., 36, 6, 2976.

[16] Morse P.M. (1929). Phys.Rev., v.34, p.57.

[17] Nesterenko V.N. (1992), Pulse loading of heterogeneous materials, Nauka, Novosibirsk, *in Russian.*

[18] Newell A. (1985), Solitons in Mathematics and physics. SIAM, Philadelphia.

[19] Parker D.F. (1984). On the derivation of nonlinear rod theories from three-dimensional elasticity. J.Appl.Math.and Phys. (ZAMP), 35, 833-847.

[20] Pleus, P., Sayir, M, (1983). A second order theory for large deflections of slender beams. J.Appl.Math and Phys. (ZAMP), v. 34, 192-217.

[21] Pochhammer L. (1876) J. f. Mathematik (Crelle), v. 81, p.324.

[22] Porubov A.V., Samsonov A.M., (1993). Refinement of longitudinal strain wave propagation in non-linearly elastic rod. Sov.Tech.Phys.Lett., 19, no.12, 365-366.

[23] Shield, R.T., (1983). Equilibrium solutions for finite elasticity. Trans. ASME, J. Appl. Math., v.50, 1171-1180.

[24] Samsonov A.M., (1982). Structural optimization in nonlinear elastic wave propagation problems. In: "Structural optimization under dynamical loading. Seminar and Workshop of junior scientists, Tartu, Estonia", U.Lepik ed., Tartu University Press, Tartu, Estonia, 75-76.

[25] Samsonov, A.M., (1984). Soliton evolution in a rod with variable cross section. Sov.Physics–Doklady, 29, 7, 586-587.

[26] Samsonov, A.M., (1988). On existence of longitudinal strain solitons in an infinite nonlinearly elastic rod. Sov.Physics– Doklady, v.33, pp.298-300.

[27] Samsonov A.M., (1994) Nonlinear strain waves in elastic waveguides. In: "Nonlinear waves in solids", A.Jeffrey, J.Engelbrecht (1994).

[28] Samsonov A.M., Sokurinskaya E.V.,(1988).On the excitation of a longitudinal deformation soliton in a nonlinear elastic rod. Sov.Phys.- Techn.Phys., v.33, no.8, 989-991

[29] Toda M., Wadati M., (1973), A soliton and two solitons in an exponential lattice and related eqs. J.Phys.Soc Japan, 34, 18-25.

[30] Treloar, L.R.G., (1958). Physics of rubber elasticity. Oxford, the University Press.

[31] Valkering T.P., (1978), Periodic permanent waves in an anharmonic chain with nearest-neighbour interaction. J.Phys. A, v. 11, 10, 1885-1897.

[32] Wright T., (1984). Weak shocks and steady waves.., Int.J.Solids & Struct., v.20, 911-919.

[33] Wu, J. et al. (1987). Observation of envelope solitons in solids. Phys.Rev. Lett., 59, 24, 2744-2747.