On a Shallow Water Wave System

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Abstract.

In this paper we discuss symmetry reductions for the shallow water wave system

$$\begin{aligned} &u_{xxt} + \alpha u u_t + \beta v u_x - u_t - u_x = 0 \\ &v_x = u_t \end{aligned} \qquad (*)$$

where α and β are arbitrary, non-zero, constants. This arises in shallow water theory as a scalar equation in non-local form by setting $v = \partial_x^{-1} u_t$, where $(\partial_x^{-1} f)(x) = \int_x^\infty f(y)\,\mathrm{d}y$, using the so-called Boussinesq approximation; it has also been discussed in local form obtained by setting $u = U_x, v = U_t$. Two special cases of these scalar equations have been discussed in the literature, namely $\alpha = \beta$ and $\alpha = 2\beta$. The inverse scattering problem has been solved in both these cases, further they have both been studied using Hirota's bi-linear technique. A full catalogue of classical and nonclassical reductions have been given for the general local scalar equation.

We derive symmetry reductions of $(*)$ using the classical Lie method of infinitesimal transformations, the nonclassical method due to Bluman & Cole and the direct method due to Clarkson & Kruskal. We observe a large increase in the complexity of the calculation of the nonclassical method for the system $(*)$ as compared with the associated scalar equation and also the need for unusual techniques in both the direct and nonclassical method.

1. Introduction.

In this paper we discuss symmetry reductions for the shallow water wave system

$$\Delta_1 \equiv u_{xxt} + \alpha u u_t + \beta v u_x - u_t - u_x = 0 \qquad (1.1a)$$

$$\Delta_2 \equiv v_x = u_t \qquad (1.1b)$$

where α and β are arbitrary, non-zero, constants. This system arises in the non-local form,

$$u_{xxt} + \alpha u u_t - \beta u_x \partial_x^{-1} u_t - u_t - u_x = 0 \qquad (1.2)$$

where $(\partial_x^{-1} f)(x) = \int_x^\infty f(y)\,\mathrm{d}y$, from the classical shallow water theory using the so-called Boussinesq approximation (cf. [14]). It has also been discussed in the potential form

$$U_{xxxt} + \alpha U_x U_{xt} + \beta U_t U_{xx} - U_{xt} - U_{xx} = 0 \qquad (1.3)$$

by several authors [1, 12, 15, 17] (set $U_x = u$); a complete catalogue of classical and nonclassical symmetry reductions of (1.3) has been given by Clarkson and Mansfield [11] (see also [12, 13]). Two special cases of (1.1), or equivalently (1.3), have attracted some attention in the literature, namely $\alpha = 2\beta$ and $\alpha = \beta$.

- Both these special cases are solvable by inverse scattering. Their scattering problems are (i) a second order problem similar to that for the KdV equation if $\alpha = 2\beta$ [1], and (ii) a third order problem similar to that for the Boussinesq equation if $\alpha = \beta$.

- Ablowitz *et al.* [1] remark that when $\alpha = 2\beta$ (1.2) reduces in the long wave, small amplitude limit to the KdV equation, and also that it responds feebly to short waves (as does the regularized long wave equation).
- N-soliton solutions have been found for both these special cases. Further, (1.3) can be written in Hirota's bi-linear form [16] if and only if either $\alpha = 2\beta$ or $\alpha = \beta$ [15].
- Equation (1.3) satisfies the necessary conditions of the Painlevé tests due to Ablowitz *et al.* [2, 3] and Weiss *et al.* [26] to be completely integrable if and only if either $\alpha = 2\beta$ or $\alpha = \beta$ [11].

The classical method for finding symmetry reductions of PDEs is the Lie group method of infinitesimal transformations which was developed around the turn of the century (cf. [7, 22]). Though group-theoretic, the method depends on solving a system of *linear* differential equations, commonly known as the determining equations. The process by which these equations are found is entirely algorithmic, but is often both tedious and virtually unmanageable manually. Consequently, symbolic manipulation programs have been developed to aid the calculations; for this work we used the MACSYMA package `symmgrp.max` [8].

In 1969 Bluman and Cole [6] proposed a generalization of the classical method, the so-called "nonclassical method" (sometimes referred to as the 'method of partial symmetries of the first type' [25], or the 'method of conditional symmetries' [18]). In this method one is required to solve *nonlinear* determining equations, and possibly because it was first used to look at the *linear* heat equation, it didn't immediately catch on. The method used in this work to find solutions of the determining equations in the nonclassical method is that of differential Gröbner bases, implemented in the MAPLE package `diffgrob2` [20].

In 1989, motivated by the knowledge that symmetry reductions of PDEs to ODEs existed that could not be found using the classical method, Clarkson and Kruskal developed the so-called "direct method" of symmetry reduction [9]. Applying this completely direct and algorithmic method to the Boussinesq equation they were able to find both previously known and unknown reductions, outside the scope of the classical method. Levi and Winternitz [18] subsequently showed that all these new reductions could also be found using the nonclassical method of Bluman and Cole.

In 1992 Nucci and Clarkson [21] showed for the Fitzhugh-Nagumo equation the nonclassical method is more general than the direct method, i.e. there exist reductions obtainable by the nonclassical method but not by the direct method. Subsequently Arrigo *et al.* [5] Olver [23], and Pucci [24] all addressed the relationship between these two methods for a single scalar equation in one dependent and two independent variables. The difference between the methods depends on whether the transformation (2.1) is so-called *fiber-preserving* i.e. if $(\xi/\tau)_u = 0$, or not. It has recently been shown by Ludlow [19] that the direct method does *not* find all classical reductions for some PDEs.

In sections 2, 3 and 4 we apply the classical, nonclassical and direct methods of symmetry reduction respectively, to the system of equations (1.1). In section 5 we discuss the applications of these methods to this system as compared with the scalar equation (1.3).

2. Classical symmetries

The theory behind the classical method is thus: consider the one-parameter Lie group of infinitesimal transformations in (x, t, u, v) given by

$$x^* = x + \varepsilon\xi(x,t,u,v) + O(\varepsilon^2) \tag{2.1i}$$
$$t^* = t + \varepsilon\tau(x,t,u,v) + O(\varepsilon^2) \tag{2.1ii}$$
$$u^* = u + \varepsilon\phi_1(x,t,u,v) + O(\varepsilon^2) \tag{2.1iii}$$
$$v^* = v + \varepsilon\phi_2(x,t,u,v) + O(\varepsilon^2) \tag{2.1iv}$$

where ε is the group parameter. Then one requires that this transformation leaves invariant the set of solutions of (1.1) given by

$$S_\Delta \equiv \{u(x,t), v(x,t) : \Delta_1 = 0, \Delta_2 = 0\}. \tag{2.2}$$

This yields an overdetermined, linear system of equations for the infinitesimals $\xi(x,t,u,v)$, $\tau(x,t,u,v)$, $\phi_1(x,t,u,v), \phi_2(x,t,u,v)$; the associated Lie algebra is realised by vector fields of the form

$$\mathbf{v} = \xi(x,t,u,v)\partial_x + \tau(x,t,u,v)\partial_t + \phi_1(x,t,u,v)\partial_u + \phi_2(x,t,u,v)\partial_v. \tag{2.3}$$

Having determined the infinitesimals, the symmetry variables are found by solving the characteristic equation

$$\frac{\mathrm{d}x}{\xi(x,t,u,v)} = \frac{\mathrm{d}t}{\tau(x,t,u,v)} = \frac{\mathrm{d}u}{\phi_1(x,t,u,v)} = \frac{\mathrm{d}v}{\phi_2(x,t,u,v)} \tag{2.4}$$

which is equivalent to solving the invariant surface conditions

$$\begin{aligned} \psi_1 &\equiv \xi(x,t,u,v)u_x + \tau(x,t,u,v)u_t - \phi_1(x,t,u,v) = 0 \\ \psi_2 &\equiv \xi(x,t,u,v)v_x + \tau(x,t,u,v)v_t - \phi_2(x,t,u,v) = 0 \end{aligned} \tag{2.5}$$

The set S_Δ is invariant under the transformation (2.1) provided that

$$\mathrm{pr}^{(3)}\mathbf{v}(\Delta_1, \Delta_2)|_{\Delta_1=0,\Delta_2=0} = 0 \tag{2.6}$$

where $\mathrm{pr}^{(3)}\mathbf{v}$ is the third prolongation of the vector field (2.3), which is given explicitly in terms of ξ, τ, ϕ_1 and ϕ_2 (cf. [22]). This yields the determining equations. The Kolchin-Ritt algorithm in `diffgrob2` simplifies the determining equations to the system

$$\xi_v = 0 \qquad \xi_u = 0 \qquad \xi_t = 0 \qquad (\alpha-\beta)\xi_{xx} = 0 \qquad \tau_v = 0 \tag{2.7i}$$
$$\tau_u = 0 \qquad \tau_x = 0 \qquad \beta\tau_t v + \beta\xi_x v + \beta\phi_2 - \tau_t - \xi_x = 0 \tag{2.7ii}$$
$$(\alpha-\beta)(2\alpha\xi_x u + \alpha\phi_1 - 2\xi_x) = 0. \tag{2.7iii}$$

This system is easily solved to yield

$$\xi = c_1 x + c_2 \qquad \tau = f(t) \qquad \phi_1 = -2c_1\left(u - \frac{1}{\alpha}\right) \qquad \phi_2 = -[f_t(t) + c_1]\left(v - \frac{1}{\beta}\right).$$

Note that lower case alphabetic subscripts have been used to denote both partial and total differentiation. The correct type of differentiation in any particular instance should be obvious. The case $\alpha = \beta$ must be considered separately, but gives the same result. We have two canonical reductions

Case 2.1 $c_1 \neq 0$. In this case we set $c_1 = 1, c_0 = 0$ and $1/f(t) = -g_t(t)/g(t)$, without loss of generality

REDUCTION 2.1.

$$u(x,t) = w_1(z)g^2(t) + 1/\alpha \qquad v(x,t) = w_2(z)g_t(t) + 1/\beta \qquad z = xg(t)$$

where w_1, w_2 satisfy the following system of ODEs

$$zw_{1,zzz} + 4w_{1,zz} + \alpha z w_1 w_{1,z} + 2\alpha w_1^2 + \beta w_2 w_{1,z} = 0 \tag{2.8i}$$

$$w_{2,z} = zw_{1,z} + 2w_1 \tag{2.8ii}$$

which may be solved in terms of the third Painlevé equation in the special cases $\alpha = \beta$ and $\alpha = 2\beta$.

Case 2.2 $c_1 = 0$. In this case we set $c_2 = 1$ and $1/f(t) = g_t(t)$, without loss of generality

REDUCTION 2.2.

$$u(x,t) = w_1(z) \qquad v(x,t) = w_2(z)g_t(t) + 1/\beta \qquad z = x - g(t)$$

Equation (1.1b) then becomes

$$w_{2,z} = -w_{1,z} \tag{2.9}$$

which can be integrated and used to simplify (1.1a) thus

$$w_{1,zzz} + (\alpha + \beta)w_1 w_{1,z} - (\beta c_3 + 1)w_{1,z} = 0 \tag{2.10}$$

If $\alpha + \beta \neq 0$ this is equivalent to the Weierstrass elliptic function equation (3.23), else (2.10) yields the exact solution

$$u(x,t) = A_1 \mathrm{e}^{c_4 z} + A_2 \mathrm{e}^{-c_4 z} + A_3$$

where the A_i are arbitrary constants and $c_4^2 = \beta c_3 + 1$.

3. Nonclassical symmetries

In the nonclassical method one requires only the subset of S_Δ given by

$$S_{\Delta,\psi} = \{u(x,t), v(x,t) : \Delta_1(u,v) = 0, \Delta_2(u,v) = 0, \psi_1(u,v) = 0, \psi_2(u,v) = 0\} \tag{3.1}$$

where S_Δ is defined in (2.2) and $\psi_{1,2} = 0$ are the invariant surface conditions (2.5), is invariant under the transformation (2.1). The usual method of applying the nonclassical method (e.g. as described in [18]) involves applying the prolongation $\mathrm{pr}^{(3)}\mathbf{v}$ to the system composed of (1.1) and the invariant surface conditions (2.5) and requiring that the resulting expressions vanish for $u, v \in S_{\Delta,\psi}$, i.e.

$$\mathrm{pr}^{(3)}\mathbf{v}(\Delta_1, \Delta_2)|_{\Delta_{1,2}=0, \psi_{1,2}=0} = 0 \qquad \mathrm{pr}^{(1)}\mathbf{v}(\psi_1, \psi_2)|_{\Delta_{1,2}=0, \psi_{1,2}=0} = 0. \tag{3.2}$$

It can easily be shown that the latter vanishes identically when $\psi = 0$ without imposing any further conditions upon ξ, τ, ϕ_1 and ϕ_2. To apply the method in practice we use the algorithm given in [10] which avoids difficulties arising from using differential constraints of the invariant surface conditions (2.5). We shall only consider the canonical case when $\tau \not\equiv 0$, in which we set $\tau = 1$, since when

$\tau \equiv 0$ the method produces a single equation which is considerably more complicated than the original system (1.1). We proceed by eliminating u_t and u_{xxt} in (1.1) using the invariant surface conditions (2.5) yielding

$$\begin{aligned}
&\phi_{1,v}v_{xx} - \xi_v u_x v_{xx} - \xi_{vv}u_x v_x^2 + \phi_{1,vv}v_x^2 - 2\xi_v u_{xx}v_x - 2\xi_{uv}u_x^2 v_x + 2\phi_{1,uv}u_x v_x \\
&\quad - 2\xi_{xv}u_x v_x + 2\phi_{1,xv}v_x - \xi u_{xxx} - 3\xi_u u_x u_{xx} + \phi_{1,u}u_{xx} - 2\xi_x u_{xx} \\
&\quad - \xi_{uu}u_x^3 + \phi_{1,uu}u_x^2 - 2\xi_{xu}u_x^2 + 2\phi_{1,xu}u_x - \xi_{xx}u_x + \alpha\phi_1 u \\
&\quad + \phi_{1,xx} - \phi_1 - \alpha\xi u u_x + \beta v u_x + \xi u_x - u_x = 0 \qquad (3.3a) \\
&v_x - \phi_1 + \xi u_x = 0 \qquad (3.3b)
\end{aligned}$$

which involves the infinitesimals ξ and ϕ_1 (though not ϕ_2) that are to be determined. Now we apply the classical Lie algorithm to this equation using the third prolongation $\mathrm{pr}^{(3)}\mathbf{v}$ and eliminate u_{xxx} using (3.3a) and v_x using (3.3b) in that order.

This procedure generates a system of eleven determining equations, but it is a large system, too large to be included here (the biggest equation has 166 summands!). The first equation is

$$\xi\xi_v - \xi_u = 0 \qquad (3.4)$$

and the analysis splits into two dependent upon whether ξ_u and ξ_v are zero or non-zero. The solving of these equations would be highly intractable should one attempt it manually, however using symbolic manipulation programs, in particular `diffgrob2` in MAPLE, the problem becomes more tractable though still non-trivial. We find both the classical reductions, and also two nonclassical reductions, which themselves have interesting special cases. These two new reductions arise from infinitesimals which give the following invariant surface conditions

$$f_t(t)u_x + u_t = f_t(t)w(z_1, z_2) \qquad (3.5i)$$

$$f_t(t)v_x + v_t = \frac{1}{\beta f_t(t)}[\alpha(f_t(t))^3 w(z_1, z_2) - (1 - \beta v)f_{tt}(t)] \qquad (3.5ii)$$

where

$$z_1 = \frac{1}{f_t(t)}\left(v + f_t(t)u - \frac{1}{\beta}\right) \qquad z_2 = \frac{\beta x}{\alpha} + f(t)$$

provided that $\alpha^2 - \beta^2 = 0$. If $\beta = \pm\alpha$ then w satisfies

$$\begin{aligned}
&\alpha z_1 w_{z_2} \pm \alpha z_1 w w_{z_1} + w_{z_2 z_2 z_2} \pm w_{z_1} w_{z_2 z_2} + 3w w_{z_2} w_{z_1 z_1} \pm 3w_{z_1 z_2} w_{z_2} + (w_{z_1})^2 w_{z_2} - w_{z_2} \pm w^3 w_{z_1 z_1 z_1} \\
&+ 3w^2 w_{z_1 z_1 z_2} \pm 4w^2 w_{z_1} w_{z_1 z_1} \pm 3w w_{z_1 z_2 z_2} + 5w w_{z_1} w_{z_1 z_2} \pm w(w_{z_1})^3 \mp w w_{z_1} + \alpha w^2 = 0 \qquad (3.6\pm)
\end{aligned}$$

As equations (3.6) stand they appear to be difficult to solve in full generality (indeed they are more complex than our original equation), however by using the associated invariant surface conditions we can make progress. Remember that in (3.5) the variables u, v are dependent rather than independent variables. If we define

$$P(x,t) = \frac{1}{f_t(t)}\left(v + f_t(t)u - \frac{1}{\beta}\right)$$

then (3.5) yield

$$f_t(t)P_x + P_t = \left(1 + \frac{\alpha}{\beta}\right) f_t(t)w(z_1, z_2) \qquad (3.7)$$

In the case $\alpha + \beta = 0$ this is easily solved to give

$$P(x,t) = p(z) \qquad z = x - f(t)$$

Our two earlier variables z_1 and z_2 are now equivalent to the new variable z defined above so we let $w(z_1, z_2) = R_z(z)$ (for convenience). The invariant surface conditions (3.5) are now in a form that can be solved and give the following reduction

REDUCTION 3.1. $\beta = -\alpha$

$$u(x,t) = f(t)R_z(z) + H(z), \qquad z = x - f(t)$$
$$v(x,t) = f_t(t)(K(z) - f(t)R_z(z)) - 1/\alpha$$

where $H(z), K(z)$ and $R(z)$ satisfy

$$K_z + H_z = R_z \tag{3.8i}$$
$$\alpha H R_{zz} + R_{zzzz} + \alpha K R_{zz} - R_{zz} - \alpha (R_z)^2 = 0 \tag{3.8ii}$$
$$H_{zzz} + \alpha H H_z + \alpha K H_z - H_z - \alpha R_z H - R_{zzz} + R_z = 0 \tag{3.8iii}$$

Whilst the reduction holds, $R(z)$ should be pre-determined by the infinitesimals or should in some sense satisfy (3.6–). To show that this is the case, notice that $P(x,t)$ has the same form as z_1, though the former is a dependent and the latter an independent variable. We make the hodograph transformation

$$w(z_1, z_2) = Q_s(s,y) \qquad z_1 = Q(s,y) \qquad z_2 = y \tag{3.9}$$

to equation (3.6–), which implements this role reversal to leave a (large) equation for $Q(s,y)$. Since $w(z_1, z_2)$ becomes a function of one variable only we let Q be the same, i.e. set $y = s$. Our large equation for Q then simplifies enormously to

$$Q_{ssss} + \alpha Q Q_{ss} - Q_{ss} - \alpha Q_s^2 = 0 \tag{3.10}$$

We note that $s = y = z_2 = x - f(t) = z$ and so it is not difficult to show that $R_z(z)$ is equivalent to $Q_s(s)$ as required.

There are three special cases of Reduction 3.1.

(a) If $R(z) = c_1$, a constant, then we have the classical reduction 2.2.

(b) If $R(z) = H(z)$, and we integrate (3.8i) with respect to z to give $K(z) = R(z) - H(z) + c_2 = c_2$, but let $c_2 = 0$ then Reduction 3.1 simplifies to

$$u(x,t) = H(z) + f(t)H_z(z) \qquad v(x,t) = -f(t)f_t(t)H_z(z) - 1/\alpha$$

where $H(z)$ satisfies

$$H_{zzzz} + \alpha H H_{zz} - \alpha H_z^2 - H_{zz} = 0 \tag{3.11}$$

which is not of Painlevé type.

(c) If $R(z) = c_3 \exp(c_4 z) - c_4^2/\alpha$, where we may assume that c_3, c_4 are non-zero constants, then again integrate (3.8i) with respect to z to give $K(z) = R(z) - H(z) + c_5$. If we now let $c_5 = 1/\alpha$ then (3.8ii) is identically zero and (3.8iii) becomes

$$H_{zzz} + [\alpha c_3 \exp(c_4 z) - c_4^2] H_z - \alpha c_3 c_4 \exp(c_4 z) H = c_3 c_4 (c_4^2 - 1) \exp(c_4 z).$$

Since $H = \exp(c_4 z)$ is a solution of the homogeneous equation we look for solutions of this in the form $H(z) = g(z)\exp(c_4 z)$. Making the transformation $m(y) = g_z(z)$ with $y = \exp(c_4 z)$ yields the Bessel-like equation

$$y^2 m_{yy} + 4y m_y + \left(2 + \frac{\alpha c_3}{c_4^2} y\right) m = \frac{c_3}{c_4}(c_4^2 - 1). \tag{3.12}$$

Then making the transformation $m(y) = l(y)/y^2$ yields

$$l_{yy} + \frac{\alpha c_3}{c_4^2}\frac{l}{y} = \frac{c_3}{c_4}(c_4^2 - 1) \tag{3.13}$$

which has general solution (cf. [4])

$$l(y) = c_6\sqrt{y}J_1\left(\frac{2\sqrt{\alpha c_3}}{c_4}\sqrt{y}\right) + c_7\sqrt{y}\,Y_1\left(\frac{2\sqrt{\alpha c_3}}{c_4}\sqrt{y}\right) + \frac{c_4}{\alpha}(c_4^2 - 1)y$$

where $J_1(z)$ and $Y_1(z)$ are the usual Bessel functions. Hence

$$\begin{aligned} u(x,t) =& A_1\exp(c_4 z)\int^z \exp(-3c_4 s/2)J_1\left(\frac{2\sqrt{\alpha c_3}}{c_4}\exp(c_4 s/2)\right)\,\mathrm{d}s + [A_3 + c_3 c_4 f(t)]\exp(c_4 z) \\ &+ A_2\exp(c_4 z)\int^z \exp(-3c_4 s/2)Y_1\left(\frac{2\sqrt{\alpha c_3}}{c_4}\exp(c_4 s/2)\right)\,\mathrm{d}s - \frac{(c_4^2 - 1)}{\alpha} \end{aligned}$$

where the A_i are arbitrary constants.

If $\beta = \alpha$ then the right hand side of (3.7) is no longer zero but yields

$$f_t(t)P_x + P_t = 2f_t(t)w(z_1, z_2) \tag{3.14}$$

First we consider the simplified case $w_{z_2} = 0$. Again we make a hodograph-type transformation

$$w(z_1, z_2) = 2Q_s(s), \qquad z_1 = 2Q(s) + c_6. \tag{3.15}$$

Thus (3.6+) becomes

$$Q_{ssss} + 2\alpha Q Q_{ss} + (\alpha c_6 - 1)Q_{ss} + 2\alpha Q_s^2 = 0 \tag{3.16}$$

which can be integrated twice to yield either the first Painlevé equation or the Weierstrass elliptic functon equation. However knowing this doesn't make (3.14) much easier to solve. To progress we note, as before, that $P(x,t)$ has the same form as z_1. Therefore by the chain rule and using (3.15) we have

$$Q_s(s)[f_t(t)s_x + s_t - 2f_t(t)] = 0 \tag{3.17}$$

If $Q_s = 0$ we obtain a classical reduction, hence assuming $Q_s \neq 0$ gives

$$s = 2f(t) + G(z), \qquad z = x - f(t) \tag{3.18}$$

Now we are able to solve the invariant surface conditions (3.5), since our z in (3.18) is a characteristic direction in both of equations (3.5). These yield

$$\begin{aligned} u(x,t) &= Q(s) + K(z), \qquad z = x - f(t) \\ v(x,t) &= f_t(t)[Q(s) + H(z)] + 1/\alpha \end{aligned} \tag{3.19}$$

Substituting this into equation (1.1a) gives

$$H_z(z) + K_z(z) = 2[1 - G_z(z)]Q_s(s) \tag{3.20}$$

We have two possibilities. First if $G_z(z) \neq 1$ we divide by $1 - G_z(z)$ to yield

$$\frac{H_z(z) + K_z(z)}{1 - G_z(z)} = 2Q_s(s) = \lambda$$

where λ is a separation constant, since now s and z are independent. Putting $Q_s(s) = \lambda/2$ into (3.16) gives $Q(s) = 0$, a classical reduction. Second if $G_z(z) = 1$ then $s = x + f(t)$ and $H_z(z) + K_z(z) = 0$. Integrating this last expression and substituting our values of u, v into (1.1b) gives the following reduction

REDUCTION 3.2. $\beta = \alpha$

$$u(x,t) = Q(s) + K(z) \qquad z = x - f(t) \qquad s = x + f(t)$$
$$v(x,t) = f_t(t)[Q(s) + H(z)] + 1/\alpha$$

where $H(z), K(z), Q(s)$ satisfy

$$H(z) + K(z) = c_6 \tag{3.21i}$$
$$Q_{sss} + 2\alpha Q Q_s + (\alpha c_6 - 1)Q_s = \lambda \tag{3.21ii}$$
$$K_{zzz} + 2\alpha K K_z - (\alpha c_6 + 1)K_z = \lambda \tag{3.21iii}$$

where λ is a separation constant and the integration constant in (3.21i) can be chosen to be c_6 without loss of generality. If $\lambda \neq 0$ equations (3.21ii,iii) can both be solved in terms of the first Painlevé equation

$$y_{xx}(x) = 6y^2(x) + x \tag{3.22}$$

and if $\lambda = 0$ the Weierstrass elliptic function equation

$$\wp_x^2(x; g_2, g_3) = 4\wp^3(x; g_2, g_3) - g_2\wp(x; g_2, g_3) - g_3. \tag{3.23}$$

The general case $w = w(z_1, z_2)$ is more difficult to solve and we shall not pursue this further here.

4. Direct Method

In this section we assume that the shallow water wave system is in the simplified form

$$\tilde{u}_{xxt} + \alpha\tilde{u}\tilde{u}_t + \beta\tilde{v}\tilde{u}_x = 0 \tag{4.1a}$$
$$\tilde{v}_x = \tilde{u}_t \tag{4.1b}$$

which is equivalent to (1.1) via the transformation $u = \tilde{u} + 1/\alpha, v = \tilde{v} + 1/\beta$ (α and β are assumed non-zero). We now seek a solution of (4.1) in the form

$$\tilde{u}(x,t) = U(x,t,w(z(x,t))) \qquad \tilde{v}(x,t) = V(x,t,\eta(z(x,t)), w(z(x,t))) \tag{4.2a,b}$$

and require that the result is a system of ODEs for $w(z)$ and $\eta(z)$. In fact it is easily shown that we may consider, without loss of generality, the ansätze

$$\tilde{u}(x,t) = A(x,t)w(z) + B(x,t) \qquad \tilde{v}(x,t) = C(x,t)\eta(z) + D(x,t) \tag{4.3a,b}$$

Substituting these ansätze into (4.1a,b) yields

$$\begin{aligned} Az_x^2z_tw_{zzz} &+ [2A_xz_xz_t + A_tz_x^2 + 2Az_xz_{xt} + Az_{xx}z_t]w_{zz} + [A_{xx}z_t \\ &+ 2A_{xt}z_x + 2A_xz_{xt} + A_tz_{xx} + Az_{xxt} + \alpha ABz_t + \beta ADz_x]w_z \\ &+ [A_{xxt} + \alpha BA_t + \alpha AB_t + \beta DA_x]w + \alpha AA_tw^2 + \alpha A^2z_twz_z \\ &+ \beta CAz_x\eta w_z + \beta CA_x\eta w + \beta CB_x\eta + [B_{xxt} + \alpha BB_t + \beta DB_x] = 0 \qquad (4.4\text{i}) \\ Az_tw_z - Cz_x\eta_z &+ A_tw - C_x\eta + [B_t - D_x] = 0 \qquad (4.4\text{ii}) \end{aligned}$$

Here we make the following remarks:

(i) In (4.4i) we use the coefficient of w_{zzz} to be the normalizing coefficient, so all other coefficients should look like $Az_x^2z_t\Gamma(z)$ and in (4.4ii) we use w_z so all others look like $Az_t\Gamma(z)$, where $\Gamma(z)$ is a function to be determined.

(ii) We use the uppercase Greek alphabet to represent undetermined functions, and such that under any operation the result is given the same letter.

(iii) There are three kinds of freedoms that can, and should, be employed to reduce the complexity of the calculations, but which can only be used once. (a) and (b) are described here for the ansatz (4.3a), but apply equally well to (4.3b): (a) if $B(x,t)$ is of the form $B = B_0(x,t) + A(x,t)\Omega(z)$, then we can take $\Omega \equiv 0$ (by substituting $w(z) \to w(z) - \Omega(z)$); (b) if $A(x,t)$ has the form $A = A_0(x,t)\Omega(z)$, then we can take $\Omega \equiv 1$ (by substituting $w(z) \to w(z)/\Omega(z)$); and (c) if $z(x,t)$ is determined by an equation of the form $\Omega(z) = z_0(x,t)$, where $\Omega(z)$ is any invertible function, then we can take $\Omega(z) = z$ (by substituting $z \to \Omega^{-1}(z)$).

The coefficient of η_z in (4.4ii) gives the constraint $Az_t\Gamma(z) = Cz_x$ then by freedom (b), $C = Az_t/z_x$. From the coefficient of ww_z in (4.4i) and using freedom (b) again (though in the context of (4.3a)) we have $A = z_x^2$. From the coefficient of w^2 we find, after using freedom (c), that $z = \theta(x) + \sigma(t)$. The coefficient of $w\eta$ then gives us either $z = x + \sigma(t)$ or, by using a different scaling of z, $z = x\sigma(t)$. The remaining analysis then leads us to the two classical reductions found in section 2.

However, we are not finished yet. We can write (4.4ii) in the form

$$\Gamma_1(z)w_z + \Gamma_{1,z}(z)w + \Gamma_2(z)\eta_z + \Gamma_{2,z}(z)\eta + \Gamma_{3,z}(z) = 0 \qquad (4.5)$$

and now it can be integrated. Comparing coefficients with (4.4ii), and using the previously found form of $C(x,t)$, namely $C = Az_t/z_x$, we find $\Gamma_2 = -\Gamma_1$, and we require

$$C_x = A_t \qquad B_t - D_x = \frac{Az_t\Gamma_{3,z}(z)}{\Gamma_1(z)} \qquad \Gamma_1(z)A_t = Az_t\Gamma_{1,z}(z) \qquad (4.6\text{i,ii,iii})$$

Hence integrating (4.4ii) yields

$$\eta(z) = w(z) + \Gamma_4(z) \qquad \text{where } \Gamma_4(z) = \frac{\Gamma_3(z) + \gamma_4}{\Gamma_1(z)}. \qquad (4.7)$$

Substituting this into (4.4i), and requiring the result to be an ODE in $w - \Gamma_4(z)$ and its derivatives are treated like coefficients of the ODE – yields

$$\begin{aligned} Az_x^2z_tw_{zzz} &+ [2A_xz_xz_t + A_tz_x^2 + 2Az_xz_{xt} + Az_{xx}z_t]w_{zz} + [A_{xx}z_t \\ &+ 2A_{xt}z_x + 2A_xz_{xt} + A_tz_{xx} + Az_{xxt} + \alpha ABz_t + \beta ADz_x \\ &+ \beta CAz_x\Gamma_4(z)]w_z + [A_{xxt} + \alpha BA_t + \alpha AB_t + \beta DA_x + \beta CB_x \\ &+ \beta CA_x\Gamma_4(z)]w + [\alpha AA_t + \beta CA_x]w^2 + [\alpha A^2z_t + \beta CAz_x]ww_z \\ &+ [B_{xxt} + \alpha BB_t + \beta DB_x + \beta CB_x\Gamma_4(z)] = 0. \qquad (4.8) \end{aligned}$$

Again we use the coefficient of w_{zzz} to be the normalizing coefficient, so all other coefficients look like $Az_x^2 z_t \Gamma(z)$. We have used freedom (b) for ansatz (4.3b), and we can still use freedom (a) here since if $D = D_0(x,t) + C(x,t)\Omega(z)$ we can choose $\Omega \equiv 0$ since we can make the transformation $\Gamma_4(z) \to \Gamma_4(z) - \Omega(z)$. All the other freedoms apply since they are unaffected by (4.7). With $C = Az_t/z_x$ the coefficient of ww_z yields

$$Az_x^2 z_t \Gamma(z) = (\alpha + \beta) A^2 z_t \tag{4.9}$$

which splits the analysis into whether $\alpha + \beta$ is zero or non-zero.

Case 4.1 $\alpha + \beta \neq 0$. This case admits the classical Reduction 2.2 for all allowed α and β, but also admits another reduction namely

REDUCTION 4.1. $\alpha - \beta = 0$.

$$\tilde{u}(x,t) = w(z) + \nu(y) \qquad \tilde{v}(x,t) = \sigma_t(t)[w(z) + c_1 - \nu(y)]$$
$$z = x + \sigma(t) \qquad y = x - \sigma(t)$$

where c_1 is an arbitrary constant and $w(z), \nu(y)$ satisfy

$$w_{zzz}(z) + 2\alpha w(z) w_z(z) + \alpha c_1 w_z(z) = \lambda$$
$$\nu_{yyy}(y) + 2\alpha \nu(y) \nu_y(y) - \alpha c_1 \nu_y(y) = \lambda$$

where λ is effectively a separation constant. This reduction is equivalent to the nonclassical Reduction 3.2.

Case 4.2 $\alpha + \beta = 0$. This case is slightly more difficult to solve and so we used `diffgrob2`. It also admits a reduction that is equivalent to a nonclassical reduction, namely Reduction 3.1,

REDUCTION 4.2. $\alpha + \beta = 0$.

$$\tilde{u}(x,t) = w(z) + \sigma(t)\eta_z(z) \qquad \tilde{v}(x,t) = \sigma_t(t)(w(z) + \eta(z) + \sigma(t)\eta_z(z)) \qquad z = x + \sigma(t)$$

where $w(z), \eta(z)$ satisfy

$$w_{zzz} + \eta_{zzz} + \alpha w \eta_z - \alpha \eta w_z = 0$$
$$\eta_{zzzz} + \alpha \eta_z^2 - \alpha \eta \eta_{zz} = 0$$

5. Discussion

In this paper we have discussed symmetry reductions for the shallow water wave system (1.1) using the classical, nonclassical and direct methods. All three methods give the same reductions when applied to this system as when applied to its counterpart, the scalar equation (1.3). What is unusual about the calculation is the large increase in complexity in moving from a scalar equation (see Clarkson and Mansfield [11]) to this system. Whilst for both the system and the scalar equation the determining equations for the classical method are of similar complexity (and are all linear) the nonclassical method paints quite a different picture. For the system there are 11 determining equations, all nonlinear, which constitute 1254 lines of computer generated output. Even when we look at the subcase $\xi_u = \xi_v = 0$ this reduces to 8 nonlinear equations and 360 lines of output. However the scalar equation has only 8 determining equations, 3 linear and 5 nonlinear, which produce 162 lines of output, greatly simplifying the problem at hand.

Another difficulty we observed was that the reductions obtained using the nonclassical method arise in a very unusual manner, and one has to use a hodograph transformation. Indeed we can establish a set of infinitesimals which would give Reduction 3.2 more naturally by working the method of characteristics backwards, namely

$$\xi = f_t(t) \qquad \phi_1 = 2f_t(t)P_y(y) \qquad \phi_2 = \frac{f_{tt}(t)}{f_t(t)}\left(v - \frac{1}{\alpha}\right) + 2(f_t(t))^2 P_y(y) \qquad y = x + f(t)$$

where $f(t)$ is an arbitrary function, and $P(y)$ satisfies

$$P_{yyy} + 2\alpha P P_y - P_y = c_1 \tag{5.1}$$

with c_1 an arbitrary constant. If we look at section 3, equation (3.6) this is equivalent to $w(z_1, z_2) = Q(z_2) = P_y(y)$ and satisfying (5.1). The first term of (3.6), as the only $O(z_1)$ term, must be zero then we find $w = 0$ and no such solutions are possible. Thus our nonclassical determining equations do not admit these infinitesimals as solutions.

The direct method is not straightforward either. Unless we integrate the smaller equation (4.4ii) and use the information gained in the main equation (4.4i), we only extract the classical reductions. It appears that we must have interaction between the equations to get non-classical reductions. Despite this the direct method is much easier to implement.

Open Questions:

(i) How should we deal with non-local terms like that in (1.2), by creating systems or higher order PDEs? The following suggests that this may not be as clear cut as one may think.

(ii) *Potential symmetries.* In the notation of Bluman and Kumei [7], for a system of PDEs $R\{\boldsymbol{x}, \boldsymbol{u}\}$ and an auxiliary system $S\{\boldsymbol{x}, \boldsymbol{u}, v\}$ where $R\{\boldsymbol{x}, \boldsymbol{u}\}$ is embedded in $S\{\boldsymbol{x}, \boldsymbol{u}, v\}$, a local (classical) symmetry of $S\{\boldsymbol{x}, \boldsymbol{u}, v\}$ will induce a nonlocal (not classical) symmetry admitted by $R\{\boldsymbol{x}, \boldsymbol{u}\}$ if the infinitesimals of variables $(\boldsymbol{x}, \boldsymbol{u})$ of $S\{\boldsymbol{x}, \boldsymbol{u}, v\}$ depend explicitly on the potential variable v. Such a nonlocal symmetry is called a "potential symmetry" of $R\{\boldsymbol{x}, \boldsymbol{u}\}$. Whilst the theory only concerns the classical method, does it have any bearing on the nonclassical method? For both the nonclassical symmetries we have found, the infinitesimal of u, ϕ_1, depends on v. Despite this both reductions are obtainable using the nonclassical method on (1.3). Therefore we ask "do *nonclassical* potential symmetries exist?"

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