The singular manifold method: a fresh approach

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Abstract

The singular manifold method of Weiss allows for many completely integrable partial differential equations the recovery of the Lax pair and Darboux transformation (DT), and so also the Bäcklund transformation, from a truncated Painlevé expansion. Here we present a natural extension of this singular manifold method whereby the DT is now identified with an infinite Painlevé expansion for a certain choice of the arbitrary coefficients: this then leads us to a new and more consistent definition of "singular manifold equation." This summation of an infinite Painlevé expansion, which involves of course only one singular manifold, is achieved by seeking a truncated expansion in a new Riccati variable Z. This new Riccati variable also allows us to place within the context of Painlevé analysis a larger class of exact solutions than was possible hitherto. Full details of this work are presented elsewhere.

1 Introduction: the singular manifold method

The Weiss-Tabor-Carnevale (WTC) Painlevé test [1] is now well-known as a test of partial differential equations (PDEs) for complete integrability. This test involves seeking a solution of the PDE, for example an equation in $U(x,t)$,

$$E[U] = 0, \tag{1}$$

as an expansion of the form

$$U = \varphi^p \sum_{j=0}^{\infty} U_j^{\varphi} \varphi^j, \tag{2}$$

where $\varphi(x,t) = 0$ defines an arbitrary non-characteristic [2] movable singular manifold. Here the superscript φ on U_j^{φ} denotes the expansion variable. For each family

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(choice of leading order behaviour) there is a set of indices $\{j_1, \ldots, j_N\}$ which give the values of j for which the coefficient $U_j^\varphi(x,t)$ in (2) remains arbitrary. For any family which represents either the general or a particular solution the Painlevé test requires [1, 3]: p to be integer; the indices to be distinct integers; all compatibility conditions for each index to be satisfied.

However, Painlevé analysis can also provide a constructive proof of integrability: it has been shown [1, 4] that from a truncated expansion U_T (so we have $p < 0$),

$$U_T = \varphi^p \sum_{j=0}^{-p} U_j^\varphi \varphi^j, \tag{3}$$

it is possible to obtain the Lax pairs of many integrable PDEs. This singular manifold method of Weiss [4], later phrased algorithmically by Musette and Conte [5], can also give the Darboux transformation (DT) — and so together with the Lax pair the Bäcklund transformation (BT) — for the PDE.

The singular manifold method involves seeking a constraint on φ, the so-called "singular manifold equation" (SME), for the constant level truncation (3) to exist. We later give a new definition of "singular manifold equation." For now we simply remark that (3) represents the trivial (because finite) summation of (2) for certain choices of the arbitrary coefficients, and with φ subject to the SME. In particular, all arbitrary coefficients of positive powers of φ in (2) have been set equal to zero.

2 A new expansion variable

2.1 The invariant Painlevé analysis

Conte [6] introduced an expansion variable χ given in terms of the WTC φ as

$$\chi = \left(\frac{\varphi_x}{\varphi} - \frac{\varphi_{xx}}{2\varphi_x} \right)^{-1}. \tag{4}$$

This variable χ satisfies the system of Riccati equations

$$\chi_x = 1 + \frac{1}{2} S \chi^2, \qquad \chi_t = -C + C_x \chi - \frac{1}{2}(C_{xx} + CS)\chi^2, \tag{5}$$

where S, the Schwarzian derivative of φ, and C are given by

$$S = \{\varphi; x\} = \left(\frac{\varphi_{xx}}{\varphi_x} \right)_x - \frac{1}{2} \left(\frac{\varphi_{xx}}{\varphi_x} \right)^2, \qquad C = -\frac{\varphi_t}{\varphi_x}. \tag{6}$$

The cross-derivative condition (identically satisfied in terms of φ) on (5) is

$$S_t + C_{xxx} + 2SC_x + S_x C = 0. \tag{7}$$

The resulting expansion in χ is a resummation of the WTC expansion (2),

$$U = \chi^p \sum_{j=0}^{\infty} U_j^\chi \chi^j. \tag{8}$$

Formulae relating the coefficients U_j^χ and U_j^φ can be found in [6].

2.2 A new expansion variable

The linearisation of the Riccati system (5) via $\chi^{-1} = \psi_x/\psi$ is

$$\psi_{xx} = -\frac{1}{2}S\psi, \qquad \psi_t = -C\psi_x + \frac{1}{2}C_x\psi. \tag{9}$$

Thus a natural way to construct a more general Riccati system is to consider the nonlinearisation of the most general scalar linear system, given by

$$\eta_{xx} = A\eta_x + B\eta, \qquad \eta_t = -C\eta_x + \left(\int^x D\right)\eta. \tag{10}$$

The corresponding Riccati system [7], obtained via $Z^{-1} = \eta_x/\eta$, is

$$Z_x = 1 - AZ - BZ^2, \tag{11}$$
$$Z_t = -C + (AC + C_x)Z - (D - BC)Z^2, \tag{12}$$

which has as cross-derivative condition

$$(Z_x)_t - (Z_t)_x \equiv x_1 Z + x_2 Z^2 = 0, \tag{13}$$

where

$$-x_1 \equiv A_t + (AC)_x + C_{xx} - 2D = 0, \tag{14}$$
$$-x_2 \equiv B_t - D_x + 2BC_x + B_x C + AD = 0. \tag{15}$$

It might be objected that it is possible to set $A = 0$ in (11), (12) by using a simple gauge transformation

$$Z^{-1} = \chi^{-1} + \frac{1}{2}A, \tag{16}$$

which then gives a Riccati system of the same form as (5) with

$$-\frac{1}{2}S = B + \frac{1}{4}A^2 - \frac{1}{2}A_x, \qquad C = C, \tag{17}$$

and the same linearisation to (9). However this does not mean that consideration of the Riccati system (11), (12) is useless. If the truncation is such that the resulting expansion of $E[U]$ includes positive powers of Z, i.e. it is a higher order truncation

[8], then for $A \neq 0$ there corresponds to such a truncation in Z no *finite* series in χ. Further explanation can be found in [7].

The Riccati system (11), (12) has also the linearisation via $Z^{-1} = \psi_2/\psi_1$ onto

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} -\frac{1}{2}A & 1 \\ B & \frac{1}{2}A \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{18}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = \begin{pmatrix} \frac{1}{2}(AC + C_x) & -C \\ (D - BC) & -\frac{1}{2}(AC + C_x) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{19}$$

This linearisation lies behind the "two-singular manifold method" developed in [9, 10, 11]. We remark here that whilst there is clearly a close connection between the singularity structure and the DT (or indeed the Hirota form [12]) of the equations considered, the point of the present article is to show how the DT can be identified with a Painlevé expansion involving only one singular manifold.

2.3 Exact solutions

When seeking a higher order truncation in the case of constant coefficients $A = a$, $B = b$, $C = c$, and $D = 0$, we find that we are able to identify positive and negative powers of Z as

$$Z^{-1} = \frac{k}{2}\left(\tau - \sigma + \frac{a}{k}\right), \qquad Z = \frac{k}{2b}\left(\tau + \sigma - \frac{a}{k}\right), \tag{20}$$

where

$$\sigma = \frac{\sqrt{\mu^2 - 1}}{\cosh(k(x - ct + \gamma)) + \mu}, \qquad \tau = \frac{\sinh(k(x - ct + \gamma))}{\cosh(k(x - ct + \gamma)) + \mu}, \tag{21}$$

$b = (k^2 - a^2)/4$, $\mu = ia/\sqrt{k^2 - a^2}$, and γ is an arbitrary constant. In [13] the recovery of solutions polynomial in the two functions σ and τ was considered: we have now placed this entire class of solutions within the context of Painlevé analysis (a corresponding WTC expansion can be given). This therefore extends our previous work [8], which dealt with the case $a = \mu = 0$. Details can be found in [7]. The advantage of our approach is that we make an expansion in a single variable Z.

3 Example: mKdV5

We now give further details of an example presented in [7], i.e. the fifth order mKdV equation (mKdV5),

$$-U_t + \left(U_{xxxx} - \frac{10}{\kappa^2}U^2 U_{xx} - \frac{10}{\kappa^2}U U_x^2 + \frac{6}{\kappa^4}U^5\right)_x = 0, \tag{22}$$

which for simplicity we take in potential form (pmKdV5),

$$E[V] \equiv -V_t + V_{xxxxx} - \frac{10}{\kappa^2}V_x^2 V_{xxx} - \frac{10}{\kappa^2}V_x V_{xx}^2 + \frac{6}{\kappa^4}V_x^5 = 0, \qquad U = V_x. \tag{23}$$

This equation has the four (κ may take either sign) families $V = \kappa \log \chi + \cdots$ and $V = 2\kappa \log \chi + \cdots$, with indices $\{-1, 0, 2, 3, 6\}$ and $\{-3, -1, 0, 6, 8\}$ respectively. Here we consider the truncated expansion corresponding to either of the two principal families,

$$V_T = \kappa \log Z + v, \tag{24}$$

where for the full expansion v is the arbitrary coefficient corresponding to the index at 0. Substitution of (24) into (23) gives

$$E\left[V_T\right] \equiv Z^{-5} \sum_{j=0}^{10} E_j^Z Z^j = 0, \tag{25}$$

where $E_0^Z \equiv 0$ and $E_{10}^Z \equiv 0$ since the leading order coefficient is already set in (24). We then find

$$E_1^Z \equiv 10(\kappa A - 2v_x) = 0, \tag{26}$$

$$E_9^Z \equiv 10B^4(\kappa A - 2v_x) + 10\kappa B^3 B_x = 0, \tag{27}$$

which gives

$$A = \frac{2}{\kappa}v_x, \tag{28}$$

$$B = \lambda^2, \tag{29}$$

where $\lambda = \lambda(t)$ is an arbitrary function of integration. For this choice of A and B we find

$$E_2^Z \equiv 0, \qquad E_3^Z \equiv 0, \qquad E_7^Z \equiv 0, \qquad E_8^Z \equiv 0. \tag{30}$$

This then leaves us with the three equations:

$$\begin{aligned} E_4^Z \equiv\ & \kappa C + 16\kappa\lambda^4 - \frac{8}{\kappa}\lambda^2 v_x^2 + \frac{6}{\kappa^3}v_x^4 + 8\lambda^2 v_{xx} - \frac{12}{\kappa^2}v_x^2 v_{xx} + \frac{2}{\kappa}v_{xx}^2 \\ & - \frac{4}{\kappa}v_x v_{xxx} + 2v_{xxxx} = 0, \end{aligned} \tag{31}$$

$$\begin{aligned} E_5^Z \equiv\ & \left(-v_t + v_{xxxxx} - \frac{10}{\kappa^2}v_x^2 v_{xxx} - \frac{10}{\kappa^2}v_x v_{xx}^2 + \frac{6}{\kappa^4}v_x^5\right) - (E_4^Z)_x \\ & - \frac{2}{\kappa}v_x E_4^Z = 0, \end{aligned} \tag{32}$$

$$E_6^Z \equiv \left(\kappa D + 16\lambda^4 v_{xx} - \frac{24}{\kappa^2}\lambda^2 v_x^2 v_{xx} + 4\lambda^2 v_{xxxx}\right) - \lambda^2 E_4^Z = 0. \tag{33}$$

¿From these we easily obtain C and D,

$$C = -16\lambda^4 - 8\lambda^2\left(\frac{v_{xx}}{\kappa} - \frac{v_x^2}{\kappa^2}\right) - 6\left(\frac{v_{xx}}{\kappa} - \frac{v_x^2}{\kappa^2}\right)^2 - 2\left(\frac{v_{xx}}{\kappa} - \frac{v_x^2}{\kappa^2}\right)_{xx}, \tag{34}$$

$$D = -\frac{16}{\kappa}\lambda^4 v_{xx} + \frac{24}{\kappa^3}\lambda^2 v_x^2 v_{xx} - \frac{4}{\kappa}\lambda^2 v_{xxxx}, \tag{35}$$

and also the information from (32) that v is a second solution of pmKdV5. Substitution of these values of A, B, C, D into the cross-derivative conditions (14), (15) gives

$$-x_1 = -\frac{2}{\kappa}\left(-v_t + v_{xxxxx} - \frac{10}{\kappa^2}v_x^2 v_{xxx} - \frac{10}{\kappa^2}v_x v_{xx}^2 + \frac{6}{\kappa^4}v_x^5\right)_x = 0, \tag{36}$$

$$-x_2 = (\lambda^2)_t = 0, \tag{37}$$

which tell us that λ is a constant, and that (11), (12) form a Riccati pseudopotential for mKdV5 in $u = v_x$. Thus (18), (19) give the matrix spectral problem for mKdV5: the corresponding DT is obtained from (24) by setting $U_T = V_{T,x}$, $u = v_x$;

$$U_T = \kappa(\log Z)_x + u = \kappa(\log(\psi_1/\psi_2))_x + u. \tag{38}$$

The BT now follows easily.

We note that this calculation is very straight-forward, and certainly much less difficult than it would have been had we used the Riccati system previously employed [9, 10, 11]. This is due to the removal of a gauge freedom (see [7] for details). In addition, the DT is obtained from the analysis, so no attempt need be made to derive it beforehand (although of course the assumption of the form of the DT can prove useful [5]). This is true of all examples thus far considered [7].

The gauge transformation

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \lambda & 1 \\ -\lambda & 1 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{39}$$

yields the AKNS spectral problem for mKdV5 [14], and the identity

$$\log(\psi_1/\psi_2) = -2\tanh^{-1}(\zeta_1/\zeta_2) - \log(-\lambda) \tag{40}$$

gives the DT corresponding to this spectral problem as

$$U_T = -2\kappa\left(\tanh^{-1}(\zeta_1/\zeta_2)\right)_x + u, \tag{41}$$

equivalent to that given (for mKdV) by Chen [15], and Konno and Wadati [16].

We now rewrite the DT (38) in terms of χ using (16):

$$\begin{aligned} U_T &= \kappa\chi^{-1} - \kappa\lambda^2\chi\left(1+\frac{1}{\kappa}u\chi\right)^{-1} \\ &= \kappa\chi^{-1} - \kappa\lambda^2\chi + u\lambda^2\chi^2 - \frac{1}{\kappa}u^2\lambda^2\chi^3 + \frac{1}{\kappa^2}u^3\lambda^2\chi^4 - \frac{1}{\kappa^3}u^4\lambda^2\chi^5 + \cdots, \quad (42) \end{aligned}$$

where for this χ the coefficients S and C are obtained from (17) as

$$S = -2\lambda^2 - \frac{2}{\kappa^2}u^2 + \frac{2}{\kappa}u_x, \quad (43)$$

$$C = -16\lambda^4 - 8\lambda^2\left(\frac{u_x}{\kappa} - \frac{u^2}{\kappa^2}\right) - 6\left(\frac{u_x}{\kappa} - \frac{u^2}{\kappa^2}\right)^2 - 2\left(\frac{u_x}{\kappa} - \frac{u^2}{\kappa^2}\right)_{xx}. \quad (44)$$

The Painlevé indices for a principal family of mKdV5 (22) are $\{-1, 2, 3, 5, 6\}$ and so we see that the DT corresponds to an infinite expansion in χ of the form (8), with S and C as above and the arbitrary coefficients at 2, 3, 5 and 6 chosen as:

$$U_2^\chi = -\kappa\lambda^2, \quad U_3^\chi = u\lambda^2, \quad U_5^\chi = \frac{1}{\kappa^2}u^3\lambda^2, \quad U_6^\chi = -\frac{1}{\kappa^3}u^4\lambda^2. \quad (45)$$

Corresponding to this expansion we have an infinite WTC expansion of the form (2), where φ is subject to the constraints given by (43) and (44), and we have again a certain choice of arbitrary coefficients U_2^φ, U_3^φ, U_5^φ and U_6^φ.

The values of S and C (43), (44) are a parameterisation of the relation

$$C + S_{xx} + \frac{3}{2}S^2 + 10\lambda^2 S + 30\lambda^4 = 0. \quad (46)$$

This prompts us to introduce the following new and more consistent definition of "singular manifold equation:"

> *The singular manifold equation of a PDE is the constraint on the WTC expansion variable φ such that, for a certain choice of arbitrary coefficients, the Painlevé expansion corresponds to the Darboux transformation of the PDE.*

We therefore claim that (46) is the correct SME for mKdV5. That this is identical to that for KdV5 [4] should not be surprising since the spectral problems of mKdV5 and KdV5 are related by the gauge transformation (16) (see for example [17]). By using the Miura map we can obtain from the DT for mKdV5 the corresponding DT for KdV5 (a Painlevé expansion truncated at constant level).

We note that the SME (46), a constraint on a single singular manifold φ, can be obtained from the Painlevé expansion corresponding to *either* one of the two principal families: we do not no need to try to use these two families in conjunction. However, we *cannot* obtain this SME if we insist on truncation at constant level.

Indeed, we can see from (42) that constant level truncation in χ will only give this SME for a zero value of the spectral parameter,

$$C + S_{xx} + \frac{3}{2}S^2 = 0. \tag{47}$$

It is easy to verify that the constant level truncation in χ for a principal branch of mKdV5 is $U_T = \kappa\chi^{-1}$ together with the constraint (47). However, the constraint (47) should not be considered to be the SME for mKdV5.

We can mirror iterations of the DT for mKdV5 using the Painlevé expansion (42). Setting $u = 0$ gives

$$U_T = \kappa(\chi^{-1} - \lambda^2\chi), \qquad \text{with} \qquad S = -2\lambda^2,\ C = -16\lambda^4, \tag{48}$$

and the corresponding solution is the one-soliton of mKdV5:

$$U_T = -(ik\kappa)\mathrm{sech}\left(k(x + k^4 t + \beta)\right). \tag{49}$$

This corresponds to equation (3.1.9) in [8] for mKdV. The expression (48), although now finite in χ, still corresponds to an infinite WTC expansion in φ; it was in [8] that we first argued as to the significance of summing such expansions.

Finally, we remark that a corollary of our analysis is a direct proof of the convergence of the Painlevé expansion for a principal family of mKdV5 for

$$0 < |\chi| < \left|\kappa u^{-1}\right|, \tag{50}$$

provided that equations (43) and (44) hold and the arbitrary coefficients U_2^χ, U_3^χ, U_5^χ and U_6^χ are chosen as in (45).

4 Conclusions

We have introduced a new and more consistent definition of SME, as the constraint on φ such that for certain choices of arbitrary data the Painlevé expansion corresponds to the DT. This approach, whereby we use only one singular manifold but no longer expect the DT to correspond to a WTC expansion truncated at constant level, provides a natural extension of the Weiss singular manifold method. It was not necessary to derive the DT in advance. We have seen that the SME for mKdV5 can be obtained from either one of the two families: we do not need to use these two families in conjunction. This new understanding of the relationship between the WTC expansion and the DT arose from the introduction of a new Riccati variable which in itself greatly simplifies the process of obtaining the Lax pair and DT from Painlevé analysis. Further examples can be found in [7].

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