Collision of solitary waves in media with a second-order nonlinearity

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Abstract

We consider the collision behaviour of self-trapped optical beams in media with a quadratic nonlinearity. It is found that the solitary waves behave similarly to true solitons above a critical velocity. Below this velocity they merge and form oscillating states.

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Since the dawn of nonlinear physics solitary waves and solitons are interesting due to their particle-like behavior [1]. They represent fundamental excitations of many nonlinear systems with different kinds of nonlinearities [2]. Many phenomena which occur in nonlinear optics are related to solitary waves as well. Optical pulses propagating through a two-level system give rise to the so-called self-induced transparency solitons which are described by the Sine-Gordon equation [3]. In Kerr-like media, both the temporal and spatial evolution of the optical field can be approximated by the nonlinear Schrödinger equation. The soliton solutions of this equation, extensivly studied for many years, correspond to self-trapped beams or stable pulse propagation [4]. With the recent introduction of the concept of cascading into nonlinear optics it was found that spatial and temporal solitary waves can exist in media with a quadratic nonlinearity as well [5] - [12]. Thus a completely new and promising field has opened for investigations. Concerning the spatial solitary waves the first experiments were reported recently [12]. The observed self-trapped beams are a mixed state of the fundamental and the second harmonics. They represent an equilibrium state between up and down-conversion mediated by an instantaneous second order nonlinearity. During this process both waves suffer a phase modulation which can compensate either the diffraction or dispersion. The phase modulation is surprisingly large [13, 14] and corresponds to an effective cubic nonlinearity which may be considerable larger [15] than any truly cubic fast nonlinearity, known for example from semiconductors operated below half the band gap. Therefore the new solitary waves can exist at much lower power levels. They are promising candidates for signal processing in short integrated devices like nonlinear directional couplers and Mach-Zehnder interferometers [15–17]. As for the theoretical treatment some steps have been already carried out. Both temporal and spatial effects can be described by essentially the same equations [9, 10]. In this work we restrict ourselves to spatial effects having in mind that at least the principal results apply to the temporal case as well. The corresponding system of two complex nonlinear equations seems do be not integrable. In the limit of an infinite phase mismatch or beam width the equations can be reduced to the nonlinear Schrödinger equation [5]. This is integrable and thus soliton solutions exist [4]. But this limit corresponds to diverging beam powers required to get nonlinear effects. Therefore, a detailed analysis of the full equations is necessary.

It is known that the equations yield a family of stationary bright solitary wave solutions [9, 10]. They have to be determined numerically with the exception of one element of the family which is known analytically [6, 7]. Some of these solutions are stable during propagation, some are not [11]. Evidently, in what follows we are concentrating on the stable solutions. It is also interesting to know whether these solitary wave solutions are stable with respect to collisions. As it is known this is a basic criterion showing whether the stable bright solitary waves behave like true solitons, as suggested by the limiting case of the nonlinear Schrödinger equation.

We consider the propagation of two beams (fundamental and second harmonics) in a planar waveguide geometry with a quadratic nonlinearity. If we take into account spatial effects only (diffraction), the evolution equations for the slowly varying amplitudes a_1 and a_2 of the fundamental and second harmonics are [5,7]

$$\begin{aligned} i\frac{\partial a_1}{\partial z} + \frac{1}{2}\frac{\partial^2 a_1}{\partial y^2} + a_1^* a_2 &= 0\,, \\ i\frac{\partial a_2}{\partial z} + \frac{\sigma}{2}\frac{\partial^2 a_2}{\partial y^2} - \beta a_2 + a_1^2 &= 0\,. \end{aligned} \tag{1}$$

A possible walk-off in Eqs. (1) can be removed by means of a phase transformation [9,10] provided that $\sigma \neq 1/2$ [8]. The scaling of Eqs. (1) is as follows:

$$\begin{aligned} z &= \frac{1}{\beta_1 w^2} z'\,, \quad y = \frac{1}{w} y'\,, \quad \beta = w\beta_1(2\beta_1 - \beta_2)\,, \\ a_1 &= w\sqrt{\chi_1\chi_2}\, a_1'\,, \quad a_2 = w\chi_1 a_2'\,, \quad \sigma = \frac{\beta_1}{\beta_2}\,. \end{aligned} \tag{2}$$

Here the primed quantities refer to the corresponding unscaled variables. The coordinates z and y denote the propagation direction and the transverse direction, respectively. The free scaling parameter w controls the slowness of the coordinates z and y. It has the meaning of a beam width. The constants χ_1 and χ_2 are the normalized integrals over the waveguide modes and the nonlinear coefficients, β_1 and β_2 are the propagation constants of the corresponding waveguide modes and β is the phase mismatch. Since we are interested in moving solitary wave solutions with arbitrary velocities, we look for the Galilean invariance of Eqs. (1). To this end we introduce the transformation

$$\begin{aligned} Y &= y - vz\,, \quad Z = z\,, \\ a_1 &= A_1 \cdot \mathrm{e}^{iqY}\,, \quad a_2 = A_2 \cdot \mathrm{e}^{2iqY} \end{aligned} \tag{3}$$

to obtain

$$i\left(\frac{\partial A_1}{\partial Z} - (v-q)\frac{\partial A_1}{\partial Y}\right) + \frac{1}{2}\frac{\partial^2 A_1}{\partial Y^2} + q\left(v - \frac{q}{2}\right) A_1 + A_1^* A_2 = 0\,,$$

$$i\left(\frac{\partial A_2}{\partial Z} - (v - 2\sigma q)\frac{\partial A_2}{\partial Y}\right) + \frac{\sigma}{2}\frac{\partial^2 A_2}{\partial Y^2} + (2q(v - \sigma q) - \beta)A_2 + A_1^2 \; = \; 0 \,. \tag{4}$$

Thus if $q = v$ and $\sigma = 1/2$ Eqs. (1) are Galileian invariant. Obviously Galilean invariance corresponds to a rotational symmetry in the plane of the guiding film. On the other hand, a possible walk-off breaks that symmetry. Therefore the walk-off cannot be eliminated by a simple phase transformation if $\sigma = 1/2$ holds exactly. We note that this condition was identified as a critical point of all transformations in [8]. In what follows we restrict ourselves to $\sigma = 1/2$ which is a good assumption for all realistic situations where spatial effects are involved and neglect any walk-off. In this case stationary solitary wave solutions are

$$\begin{aligned} A_1 &= u(Y) \cdot \mathrm{e}^{i\kappa Z} \,, \\ A_2 &= w(Y) \cdot \mathrm{e}^{2i\kappa Z} \,, \end{aligned} \tag{5}$$

where the field profiles u and w obey

$$\begin{aligned} \frac{1}{2}\frac{d^2 u}{dY^2} - \left(\kappa - \frac{v^2}{2}\right) u + u^* w &= 0 \,, \\ \frac{1}{4}\frac{d^2 w}{dY^2} - \left(2(\kappa - \frac{v^2}{2}) + \beta\right) w + u^2 &= 0 \,. \end{aligned} \tag{6}$$

The velocity v and the wavenumber κ are free parameters. Thus Eqs. (5) produce a two-parameter family of solutions. In this work we are interested in bright solitary waves, i. e., with $u, v \to 0$ if $Y \to \pm\infty$. For $\kappa = -2\beta/3 + v^2/2$ we have the well-known analytical solutions (see for example [6–8])

$$\begin{aligned} a_1 &= -\frac{\beta}{\sqrt{2}\cosh^2(\sqrt{-\beta/3}Y)} \cdot \mathrm{e}^{i\kappa Z + ivY} \,, \\ a_2 &= -\frac{\beta}{\cosh^2(\sqrt{-\beta/3}Y)} \cdot \mathrm{e}^{2(i\kappa Z + ivY)} \,. \end{aligned} \tag{7}$$

Otherwise u and w with the appropriate boundary conditions have to be calculated numerically. The solution given by Eqs. (7) exists only for $\beta < 0$.

In our numerical simulations we restrict ourselves to the analytical solutions of Eqs. (6) as initial conditions (with $\beta = -1$), except when the limit of large phase mismatch is explored. We examine the collision of two solitary wave solutions of Eqs. (7) separated sufficiently far from each other and with opposite but equal velocities. The phase difference between the waves was 0 or π. They are then propagated by means of a beam propagation method based on a split-step FFT algorithm.

It is observed that for sufficiently large velocities and zero phase difference the two solitary waves pass through each other unscathed (Fig. 1). The energy is conserved during the propagation, as far as numerical accuracy can tell. Thus the two solitary waves behave like true solitons. On the other side, for sufficiently small velocities the two solitary waves do not separate after they collided (Fig. 2). They merge and an oscillating single-solitary-wave state is created. A considerable amount of the initial energy decays into radiation. If the wave is propagated over a sufficiently large distance, the oscillations of this state become regular (periodic). Unlike the case of the nonlinear Schrödinger equation these oscillations are persistent and seem not to decay into radiation. This is similar to the numerical simulation of the excitation of solitary waves in this system [8]. Starting with an arbitrary beam profile one always gets oscillating solutions.

If the two initial solitary waves have the phase difference π they are repelled from each other, as expected (Fig. 3). The energy is conserved. Fig. 4 shows the transition between the quasi-elastic collision of the two solitary waves and the creation of a single-solitary-wave oscillating state, indicating that there may exist an unstable two-solitary-wave bound state. Although there is no bound state at $v = 0.348$ (Fig. 4(c)), the scattered solitary waves have a reduced velocity and there is some loss of energy. Also they oscillate after the collision.

For large phase mismatches Eqs. (1) can be reduced to the nonlinear Schrödinger equation, assuming $a_2 \simeq a_1^2/\beta$. To see whether the behavior is Schrödinger-like we performed some calculations solving Eqs. (1) with large phase mismatch ($\beta = 40$, $\kappa = 2/3$ if $v = 0$). The initial solitary waves were calculated by solving numerically Eqs. (6).

If we start with zero velocities of the initial solitary waves at a distance such that their overlap is sufficiently large, they attract each other forming a two-solitary-wave bound state similar to the case of the nonlinear Schrödinger equation (Fig. 5(a)). The collision length is in good agreement with the soliton half period found in this case [18]. Further, the soliton-like behavior is lost and an oscillating solitary-wave state as the one shown in Fig. 2 is created. There is still a critical velocity which is rather small (Fig. 5(b)). The behavior is similar to that described above with the initial velocity $v = 0$. In Fig. 5(c) we introduced a small relative phase shift between the two initial waves ($v = 0.05$). We observe that they are scattered asymmetrically. In contrast to the case of the nonlinear Schrödinger equation some energy is transferred from one solitary wave to the other. The collision behavior near the critical velocity seems to be quite sensitive to any changes of the relative phase of the waves.

In conclusion, we have observed a behavior different from what is found in integrable systems, for example in the case of the nonlinear Schrödinger equation. If the two initial solitary waves have a velocity below a critical one they merge and approach an oscillating one-solitary-wave state. Thus the solitary waves found in this system are not true solitons. Since it seems to be almost impossible to produce a stationary solitary wave from an arbitrary initial beam profile, it is noteworthy that for larger velocities the two initial solitary waves pass through each other unperturbed. For such velocities we have soliton-like behavior. The solitary waves considered were always somehow stable. We never observed that an initially stable solution decayed during the collision.

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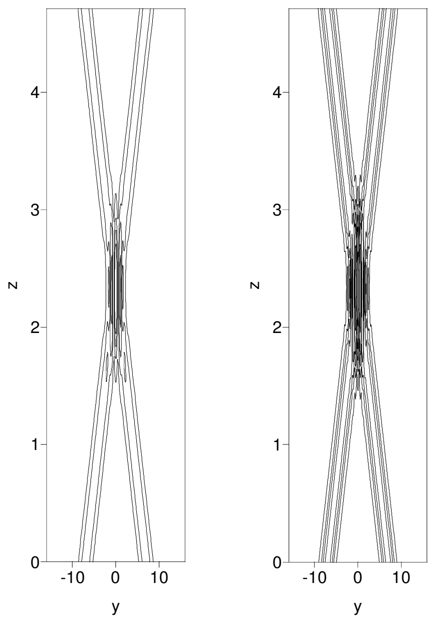

Figure 1: Intensity contour plots of the fundamental (a) and second harmonics (b) showing the propagation of two solitary waves with $v = \pm 3$ and relative phase difference zero for $\beta = -1$.

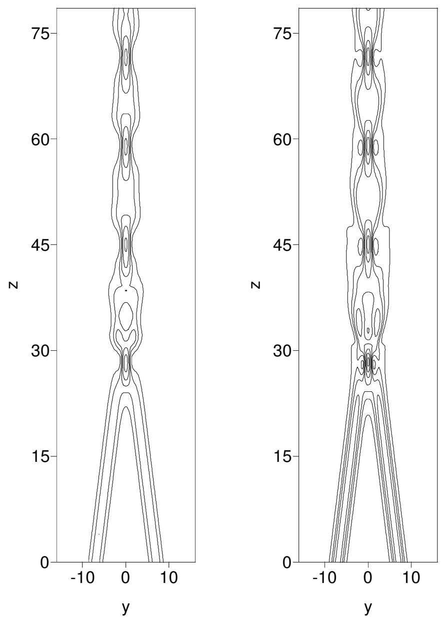

Figure 2: Intensity contour plots of the fundamental (a) and second harmonics (b) showing the propagation of two solitary waves with $v = \pm 0.2$ and relative phase difference zero for $\beta = -1$.

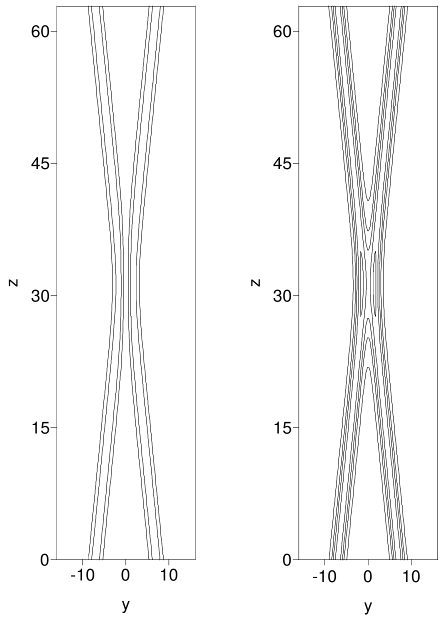

Figure 3: Intensity contour plots of the fundamental (a) and second harmonics (b) showing the propagation of two solitary waves with $v = \pm 0.2$ and relative phase difference π for $\beta = -1$.

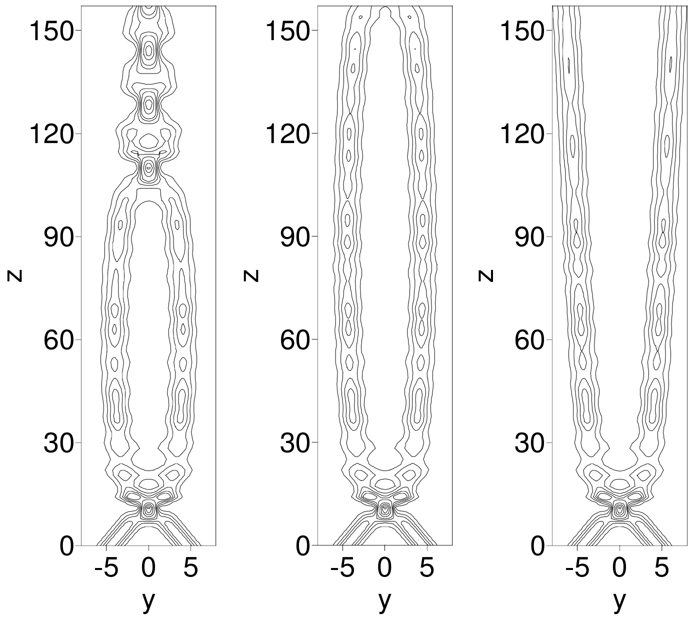

Figure 4: Intensity contour plots of the fundamental showing the propagation of two solitary waves with $v = \pm 0.346$ (a), $v = \pm 0.347$ (b) and $v = \pm 0.348$ (c) for $\beta = -1$. The waves have relative phase difference zero.

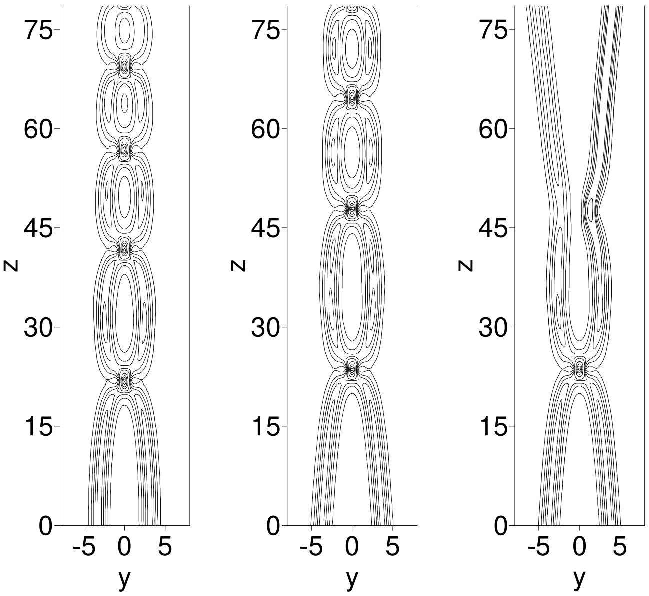

Figure 5: Intensity contour plots of the fundamental showing the propagation of two solitary waves with $v = 0$ (a), $v = \pm 0.05$ (b) and $v = \pm 0.05$ (c) for $\beta = 40$. The waves have relative phase difference zero ((a) and (b)) and 0.0625 (c).