Solitary ion-acoustic waves in a plasma with negative ions and non-isothermal electrons *

E. J. Parkes †

Abstract

The Sagdeev potential formalism is used to investigate the propagation characteristics of a finite amplitude compressive solitary ion-acoustic wave in a collisionless plasma comprising cold positive and negative ions and non-isothermal electrons. Attention is focused mainly on the effects of the degree of non-isothermality.

For strong non-isothermality and to lowest order, small amplitude waves have a sech^4 form. Higher order corrections to this are obtained via the Sagdeev potential rather than using the more usual method of reductive perturbation with renormalization. Existing results in the literature derived using the latter approach are corrected and extended.

1 Introduction

The propagation properties of solitary ion-acoustic waves in a collisionless, unmagnetized plasma have been studied by many authors. Two main approaches have been adopted in these investigations: finite amplitude waves have been investigated by using the Sagdeev potential formalism and small amplitude waves by using reductive perturbation theory with renormalization.

Finite amplitude waves in a plasma comprising cold positive ions and hot electrons were studied by Sagdeev [1]. His work has been extended to include, for example, the effects of warm ions [2], weak relativistic effects [3], the addition of negative ions [4] and two electron fluids at different temperatures [5]. The formative work of Washimi & Taniuti [6] on small amplitude waves was again in the context of a plasma comprising cold positive ions and hot electrons. They derived a KdV equation from the governing equations by using the reductive perturbation method and discussed its solitary wave solution. This work has

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†Department of Mathematics, University of Strathclyde, Livingstone Tower, Richmond Street, Glasgow G1 1XH, Scotland. (e.j.parkes@strath.ac.uk)

been extended by many authors. In particular the effect of the addition of negative ions has been investigated (see [7, 8, 9] for example). A second order correction to Washimi & Taniuti's first order solution was derived by Kodama & Taniuti [10] using the reductive perturbation method with renormalization.

In all the papers cited above, the hot electrons are assumed to have a Boltzmann distribution for the electron number density, namely

$$n_e = \exp(\phi), \tag{1.1}$$

where n_e and ϕ are the non-dimensional electron number density and electrostatic potential respectively. To derive (1.1) it is assumed that thermal electrons move freely at a speed much higher than an ion-acoustic wave. Schamel [11, 12] pointed out that the Boltzmann distribution does not take into account the resonant electrons which interact with the wave during its evolution. Accordingly he suggested that two different distribution functions should be used to describe the electrons, one for the free electrons and one for the trapped ones. The need for this is borne out by experimental evidence (see the references cited in [12]). For the propagation of compressive solitary ion-acoustic waves with $\phi \geq 0$, Schamel derived the following expression for the electron number density:

$$n_e(\phi, \beta) = \exp(\phi)\mathrm{erfc}(\phi^{1/2}) + \beta^{-1/2}\exp(\beta\phi)\mathrm{erf}(\{\beta\phi\}^{1/2}), \qquad \beta \geq 0. \tag{1.2}$$

The parameter β is defined by $\beta = T_{ef}/T_{et}$, where T_{ef} is the constant temperature of the free electrons and T_{et} is the constant temperature ot the trapped electrons. For $\beta = 1$ (1.2) reduces to (1.1). Hence $1 - \beta$ is a measure of the deviation from isothermality. For $\beta = 0$ the trapped electrons have a flat-topped distribution. Schamel [11] argues that the case $\beta > 1$ is unlikely to be physically realistic. He also notes [12] that under certain exceptional circumstances β may be negative, and derives an appropriate expression for n_e. In the present paper, however, we consider the range $0 \leq \beta \leq 1$ only.

Schamel [12] shows how the nonlinear term in the KdV equation of Washimi & Taniuti [6] is modified when the electrons deviate from isothermality. Later work investigated the effect of the addition of cold negative ions [7] and the effect of having warm positive and negative ions (see [13, 14] for example). Higher order corrections have been considered in this context [15, 16].

For finite amplitude solitary ion-acoustic waves, the addition of negative ions with non-isothermal electrons has been investigated only in the context of rarefactive waves [17].

The purpose of the present paper is to consider the propagation of compressive solitary ion-acoustic waves in a plasma comprising cold positive and negative ions and hot electrons.

We focus attention on the consequences of deviation from isothermality of the electrons. In Section 2 the basic equations are given and the Sagdeev potential is derived. In Section 3 we consider finite amplitude waves. Our results are a natural extension of the work of Tajiri & Tuda [4]. In Section 4 we consider small amplitude waves when the deviation from isothermality is not small. We obtain our results directly from the appropriate Sagdeev potential rather than by using the more usual reductive perturbation method with renormalization. We recover the sech^4 form for the first order solitary wave. Previous results for the second order correction are corrected. We also obtain the third order correction. Our conclusions are given in Section 5.

2 The basic equations

Consider the one-dimensional flow of an unmagnetized, homogeneous, collisionless plasma comprising cold positive ions of charge number Z_p, cold negative ions of charge number Z_n and hot non-isothermal electrons. We assume a three fluid model and neglect the effects of Landau damping and electron inertia. In non-dimensional form the governing equations are

$$n_{pt} + (n_p u_p)_x = 0 \tag{2.1}$$

$$u_{pt} + u_p u_{px} = -Z_p \phi_x \tag{2.2}$$

$$n_{nt} + (n_n u_n)_x = 0 \tag{2.3}$$

$$u_{nt} + u_n u_{nx} = (Z_n/Q)\phi_x \tag{2.4}$$

$$\phi_{xx} = n_e - Z_p n_p + Z_n n_n, \tag{2.5}$$

where n_e is given by (1.2). In (2.1) – (2.5), n_p and u_p are respectively the non-dimensional number density and fluid velocity of the positive ions; n_n and u_n are the corresponding quantities for the negative ions. The reference density, speed, time, length and electrostatic potential are respectively the unperturbed electron number density N_0, the positive ion sound speed $c_s = (k_B T_{ef}/m_p)^{1/2}$, $(\omega_p)^{-1}$, the Debye length $\lambda_D = c_s/\omega_p$ and $k_B T_{ef}/e$. Here k_B is Boltzmann's constant, m_p is the mass of a positive ion, $-e$ is the electron charge, $\omega_p = (N_0 e^2/\varepsilon_0 m_p)^{1/2}$ is the positive ion plasma frequency and ε_0 is the vacuum permittivity. Finally $Q = m_n/m_p$, where m_n is the mass of a negative ion.

In order to investigate possible travelling solitary wave solutions to (2.1) – (2.5) we assume that the dependent variables depend only on $\eta = x - Ut$, where U is to be determined, and use the boundary conditions

$$\phi \to 0, \quad \frac{d\phi}{d\eta} \to 0, \quad n_p \to n_{p0}, \quad n_n \to n_{n0}, \quad u_p \to 0, \quad u_n \to 0,$$

where

$$1 - Z_p n_{p0} + Z_n n_{n0} = 0. \tag{2.6}$$

We obtain

$$\frac{1}{2}\Big(\frac{d\phi}{d\eta}\Big)^2 = -V(\phi, M, \beta). \tag{2.7}$$

$V(\phi, M, \beta)$ is the Sagdeev potential given by

$$\begin{aligned} V(\phi, M, \beta) &= 1 - p_e(\phi, \beta) + \frac{(1+\alpha\gamma)M^2}{(1-\alpha)^2}\Big[1 - \sqrt{1 - \frac{\phi}{YM^2}}\,\Big] \\ &\quad + \frac{\alpha(1+\alpha\gamma)M^2}{\gamma(1-\alpha)^2}\Big[1 - \sqrt{1 + \frac{\gamma\phi}{YM^2}}\,\Big], \end{aligned} \tag{2.8}$$

where

$$M = \frac{U}{U_0}, \quad U_0^2 = Z_p^2 n_{p0} + \frac{Z_n^2 n_{n0}}{Q}, \quad \alpha = \frac{Z_n n_{n0}}{Z_p n_{p0}}, \quad \gamma = \frac{Z_n m_p}{Z_p m_n}, \quad Y = \frac{1+\alpha\gamma}{2(1-\alpha)},$$

and

$$\begin{aligned} p_e(\phi, \beta) &= \int_0^\phi n_e(\phi', \beta)\, d\phi' \\ &= \exp(\phi)\mathrm{erfc}(\phi^{1/2}) + 2(1-\beta^{-1})(\phi/\pi)^{1/2} + \beta^{-3/2}\exp(\beta\phi)\mathrm{erf}(\{\beta\phi\}^{1/2}). \end{aligned}$$

(Note that α and γ correspond to the quantities α and β in [4].)

3 Finite amplitude waves

A compressive solitary wave solution to (2.7) of amplitude ϕ_m exists provided the following conditions hold [1]:

$$V(0, M, \beta) = 0, \qquad \frac{\partial V}{\partial \phi}(0, M, \beta) = 0, \tag{3.1}$$

$$\frac{\partial^2 V}{\partial \phi^2}(0, M, \beta) < 0, \tag{3.2}$$

$$V(\phi_m, M, \beta) = 0 \quad \text{and} \quad V(\phi, M, \beta) < 0 \quad \text{for} \quad 0 < \phi < \phi_m. \tag{3.3}$$

Condition (3.1) is identically satisfied. Condition (3.2) yields $M^2 > 1$. Condition (3.3) then determines ϕ_m. In addition, for V to be real, $\phi \le YM^2$ so that we also require

$$V(YM^2, M, \beta) \ge 0. \tag{3.4}$$

The equality in (3.4) gives an upper bound M_{max} for M which then gives an upper bound $\phi_{max} = YM_{max}^2$ for ϕ_m. (For $\alpha = 0$ and $\beta = 1$, $M_{max} \approx 1.58$ as in [1].) As $M^2 > 1$, α, γ and β must be such that

$$V(Y, 1, \beta) > 0. \tag{3.5}$$

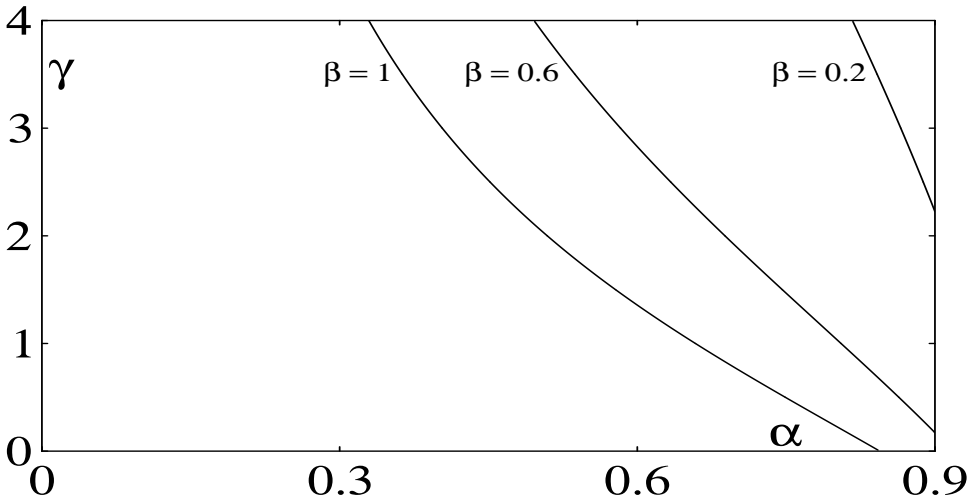

Figure 1: Each curve is given by (3.6). For each value of β a compressive solitary wave exists for the uniform state characterized by the region in α-γ space to the left of the corresponding curve.

In Figure 1 the curve

$$V(Y, 1, \beta) = 0 \tag{3.6}$$

in α-γ space is plotted for different values of β. Note that (2.6) implies that $0 < \alpha < 1$. For each value of β condition (3.5) indicates that compressive solitary waves exist to the left of the appropriate curve in Figure 1. (For $\beta = 1$, (3.6) is equivalent to (17) in [4] and the curve for $\beta = 1$ in Figure 1 corresponds to the curve labelled C in Figure 1 of [4].) From Figure 1 we conclude that for deviations from isothermality, compressive solitary waves exist for uniform states that are characterized by larger regions of α-γ space.

In Figure 2 ϕ_{max} as calculated from

$$V(\phi_{max}, M_{max}, \beta) = 0 \tag{3.7}$$

is plotted as a function of α for a given value of γ and several values of β. The extreme right-hand end of each curve corresponds to the point (α, γ) on the corresponding curve in Figure 1. It can be seen that, for a given value of α and as β decreases, the value of ϕ_{max}

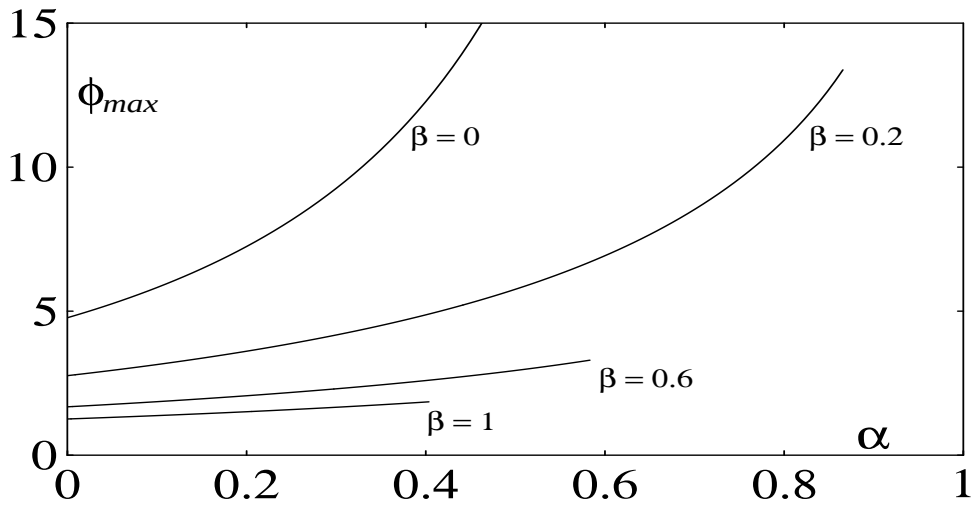

Figure 2: ϕ_{max} as a function of α as given by (3.7) for $\gamma = 3$.

increases. In other words larger amplitude compressive solitary waves are possible when the electrons are non-isothermal.

For a given value of α and γ it is of interest to see how ϕ_m varies as M increases from 1 to M_{max}. This is illustrated in Figure 3 for several values of β. The extreme right-hand end of each curve is the point (M_{max}, ϕ_{max}). Note that the curves for $\beta = 0$ and $\beta = 0.2$ have $\phi_m \to 0$ as $M \to 1$ so that small amplitude waves are possible. This is not so for the curves corresponding to $\beta = 0.6$ and $\beta = 1$. This phenomenon is discussed further in Section 4.

A measure of the width of a solitary wave is given by $W(M, \beta)$, where from (2.7)

$$W(M, \beta) = 2 \int_{\phi_m/2}^{\phi_m} \frac{d\phi}{\sqrt{-2V(\phi, M, \beta)}} \,. \tag{3.8}$$

In Figure 4, W^{-1} is plotted as a function of M for the same parameter values as in Figure 3. From Figures 3 and 4 it can be seen that, for a given M, the more the deviation from isothermality the larger the amplitude and width of the solitary wave. (Note that the curves in Figure 4 for $\beta = 0$ and $\beta = 0.2$ are difficult to compute accurately as $M \to 0$. However since the solitary waves are then very shallow the concept of 'width' is not then

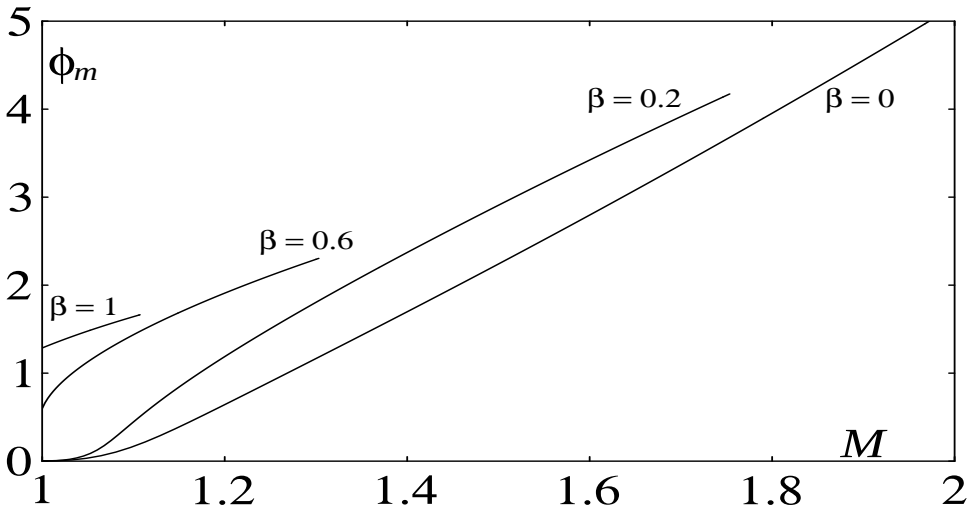

Figure 3: ϕ_m as a function of M as given by (3.3) for $\alpha = 0.3$ and $\gamma = 3$

particularly meaningful.)

4 Small amplitude waves

First note that, for small ϕ, n_e as given by (1.2) may be expanded as

$$n_e(\phi, \beta) = 1 + \phi - \tfrac{4}{3} b_1 \phi^{3/2} + \tfrac{1}{2}\phi^2 - \tfrac{8}{15} b_2 \phi^{5/2} + \cdots , \tag{4.1}$$

where $b_1 = (1-\beta)/\sqrt{\pi}$ and $b_2 = (1-\beta^2)/\sqrt{\pi}$. In this Section we assume that b_1 and b_2 are $O(1)$ quantities; in physical terms this implies that the electrons are strongly non-isothermal.

The usual method of investigating small amplitude waves is to use the reductive perturbation method with renormalization [10]. For the system described by (2.1) – (2.5) and (4.1), the transformations

$$\xi_0 = \varepsilon^{1/4}(x - U_0 t), \qquad \tau = \varepsilon^{3/4} U_0 t,$$

where ε is a smallness parameter, and expansions for ϕ, $n_p - n_{p0}$, $n_n - n_{n0}$, u_p and u_n of the form

$$\phi = \varepsilon\phi_1 + \varepsilon^{3/2}\phi_2 + \varepsilon^2\phi_3 + \cdots \tag{4.2}$$

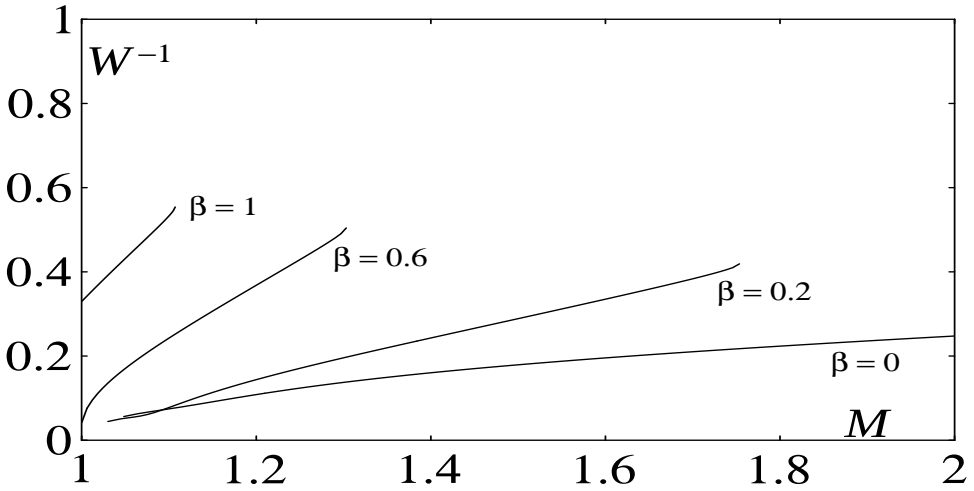

Figure 4: W^{-1} as a function of M as given by (3.8) for $\alpha = 0.3$ and $\gamma = 3$.

are introduced. This leads to a hierarchy of partial differential equations which are solved sequentially. At first order Schamel's equation

$$\frac{\partial \phi_1}{\partial \tau} + b_1 \phi^{1/2} \frac{\partial \phi_1}{\partial \xi_0} + \frac{1}{2} \frac{\partial^3 \phi_1}{\partial \xi_0^3} = 0$$

is obtained [7]. It has the solitary wave solution

$$\phi_1 = \left(\frac{15 c_1}{8 b_1} \right)^2 \mathrm{sech}^4 \left[\sqrt{\frac{c_1}{8}} (\xi_0 - c_1 \tau) \right], \tag{4.3}$$

where c_1 is an arbitrary positive constant. The elimination of secularity producing terms in the higher order equations leads to a renormalization of the velocity c_1. These equations are solved to obtain ϕ_2 etc..

Watanabe & Jiang [18] described a more efficient alternative to the reductive perturbation method and used ion-acoustic waves in a plasma comprising cold positive ions and hot isothermal electrons as an illustrative example. They recovered ϕ_1 and ϕ_2 as found in [10] and went on to find ϕ_3 and ϕ_4. Here we use the method to find ϕ_1, ϕ_2 and ϕ_3 for the system governed by (2.1) – (2.5) and (4.1). The starting point is the equation

$$\frac{d^2 \phi}{d \eta^2} = -\frac{\partial V}{\partial \phi}, \tag{4.4}$$

where V is the Sagdeev potential (2.8). Into (4.4) we substitute ϕ as given by (4.2) and

$$\xi = \varepsilon^{1/4}\eta, \qquad M = 1 + \varepsilon^{1/2}c_1 + \varepsilon c_2 + \varepsilon^{3/2}c_3 + \cdots .$$

At first order we obtain

$$\frac{d^2\phi_1}{d\xi^2} = 2c_1\phi_1 - \tfrac{4}{3}b_1\phi_1^{3/2}$$

which has the solitary wave solution

$$\phi_1 = A^2S^4, \quad \text{where} \quad A = \frac{15c_1}{8b_1}, \quad S = \operatorname{sech}\Big(\sqrt{\frac{c_1}{8}}\xi\Big). \tag{4.5}$$

With U replaced by $U_0(1+\varepsilon^{1/2}c_1)$, (4.5) is equivalent to (4.3). At second order we obtain

$$L\phi_2 = (2c_2 - 3c_1^2)\phi_1 - q_0\phi_1^2, \tag{4.6}$$

where

$$L = \frac{d^2}{d\xi^2} - 2c_1 + 2b_1\phi_1^{1/2}$$

and

$$q_0 = \frac{1}{2}\Big[\frac{3n_{p0}Z_p^3}{U_0^4} - \frac{3n_{n0}Z_n^3}{U_0^4Q^2} - 1\Big]. \tag{4.7}$$

The first term on the right hand side of (4.6) would give a secular contribution to ϕ_2. Hence we take

$$c_2 = \tfrac{3}{2}c_1^2 \quad \text{and then} \quad \phi_2 = -\frac{5q_0A^3S^4}{2b_1}(1 - \tfrac{1}{2}S^2).$$

(This expression for ϕ_2 corrects the ones given by (2.41) in [15] and by the cold limit of (33) in [16].) Similarly, at third order we obtain

$$L\phi_3 = (2c_1c_3 - 6c_1c_2 + 4c_1^3)\phi_1 + q_1c_1\phi_1^2 - 2q_0\phi_1\phi_2 - \tfrac{1}{2}b_1\phi_1^{-1/2}\phi_2^2 - \tfrac{8}{15}b_2\phi_1^{5/2},$$

where $q_1 = 2(1+2q_0)$, from which we derive

$$c_3 = \tfrac{5}{2}c_1^3 \quad \text{and} \quad \phi_3 = [e_0 + e_1S^2 + e_2S^4](15A^4S^4/b_1),$$

where

$$e_0 = \frac{4b_1q_1}{45} + \frac{5q_0^2}{12b_1} - \frac{32b_2}{315}, \quad e_1 = -\frac{2b_1q_1}{45} - \frac{35q_0^2}{96b_1} + \frac{16b_2}{315}, \quad e_2 = \frac{5q_0^2}{64b_1} + \frac{4b_2}{315}.$$

With $B = \varepsilon^{1/2}A$ the above results may be combined to give

$$M \approx 1 + (8b_1B/15) + \tfrac{3}{2}(8b_1B/15)^2 + \tfrac{5}{2}(8b_1B/15)^3 \tag{4.8}$$

and

$$\phi_m \approx B^2 - \frac{5q_0B^3}{4b_1} + \Big[\frac{2q_1}{3} + \frac{125q_0^2}{64b_1^2} - \frac{4b_2}{7b_1}\Big]B^4. \tag{4.9}$$

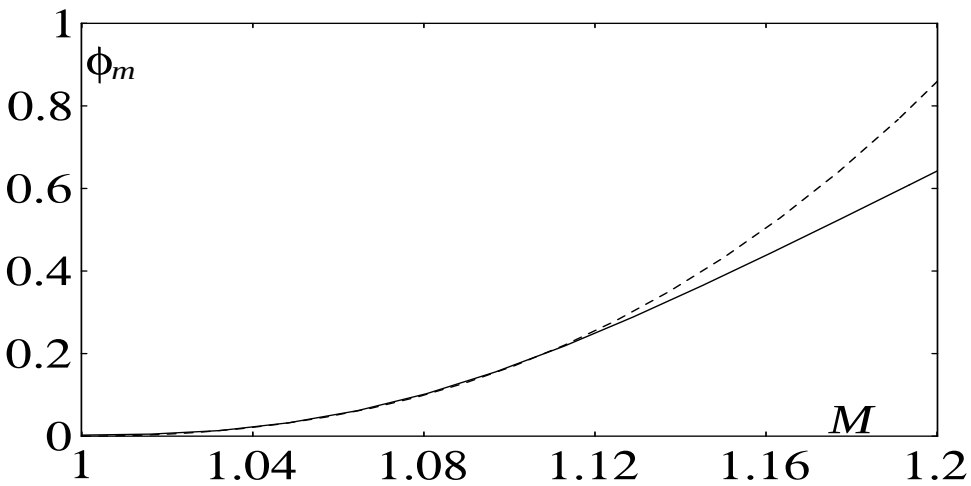

Figure 5: ϕ_m as a function of M for $\alpha = 0.3$, $\gamma = 3$ and $\beta = 0$. Solid line: exact result as given by (3.3). Dashed line: approximate result as given by (4.8) and (4.9).

In Figure 5 ϕ_m for $\beta = 0$, $\alpha = 0.3$ and $\gamma = 3$ is plotted as a function of M using the parametric approximations (4.8) and (4.9) and compared to the exact result. (The curve for the latter result is also show in Figure 3.) The agreement is, of course, good only for small ϕ_m.

Finally, note that for isothermal electrons, i. e. with $\beta = 1$ so that $b_1 = 0$, ϕ_1 is given by [7]

$$\phi_1 = \frac{3c}{q_0}\,\mathrm{sech}^2\left[\sqrt{\frac{c}{2}}\,\varepsilon^{1/2}\eta\right],$$

where c is an arbitrary positive constant, so that small amplitude waves are compressive only if $q_0 > 0$. By expressing q_0 as given by (4.7) in terms of α and γ we find that $q_0 > 0$ is equivalent to $\alpha < \alpha_c$, where

$$\alpha_c = [3\gamma^2 + 2\gamma + 3 - (1+\gamma)\sqrt{9\gamma^2 - 6\gamma + 9}\,]/(4\gamma^2).$$

Figure 3 is plotted for $\gamma = 3$ for which $\alpha_c = 0.057$; hence $\alpha > \alpha_c$ in Figure 3. This explains why ϕ_m does not tend to zero as $M \to 1$ for the curve corresponding to $\beta = 1$ in Figure 3.

5 Conclusions

We have shown that deviation from isothermality of the electrons has a significant effect on the propagation characteristics of compressive solitary ion-acoustic waves in a plasma comprising cold positive and negative ions. It would be straightforward to extend our analysis to include other influences such as the effect of warm ions and weak relativistic effects.

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