On the Integrability of a New Class of Water Wave Equations

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Abstract

We examine two nonlinear partial differential equations (PDEs) from the point of view of their integrability. The first one is integrable and can be derived from the second equation through a local transformation recently introduced by Fokas [1]. Reductions of the first equation of the form $u = u(x)$ or $u = u(x - t)$ involve algebraic singularities, but it can be shown analytically (Painlevé analysis, Goriely transformations) and numerically (ATOMFT algorithm) that they are integrable. Moreover, the Painlevé analysis of the first PDE leads to a truncated expansion from which the Lax Pair can be obtained. We then make the reduction $\eta = \eta(x)$ for the second PDE and examine the resulting ordinary differential equation (ODE). The Painlevé analysis identifies some integrable cases for this ODE which need to be further checked in terms of their implication for integrability of the PDE. However, the violation of the Painlevé property in other cases together with a numerical study of the solutions in the complex x–plane give strong indications that, in general, this ODE and therefore the second PDE from which it is obtained are not integrable. Finally, we return to the second PDE and verify that, except from the cases for which it is reduced to the KdV or mKdV, this equation fails to pass the classical Painlevé test.

Introduction

The completely integrable nonlinear equation

$$u_t + u_x - \frac{3}{2}\rho_2\beta u_{xxt} + \left(1 - \frac{3}{2}\rho_2\right)\beta u_{xxx} + \alpha u u_x - \frac{1}{2}\rho_2\alpha\beta(u u_{xxx} + 2u_x u_{xx}) = 0 \quad (1)$$

can be obtained from the water wave equation

$$\eta_t + \eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} + \rho_1\alpha^2\eta^2\eta_x + \alpha\beta(\rho_2\eta\eta_{xxx} + \rho_3\eta_x\eta_{xx}) = 0 \quad (2)$$

through the local transformation

$$\begin{aligned} \eta &= u + \lambda_1\alpha u^2 + \lambda_2\beta u_{xx} \\ \lambda_1 &= -\rho_1 \\ \lambda_2 &= -3\rho_1 - \frac{7}{4}\rho_2 + \frac{1}{2}\rho_3 \end{aligned}$$

recently introduced by Fokas in [1], where the notion of the equivalence of two differential equations was discussed, up to cubic terms in the small parameters α and β.

Equation (1) was derived in [2] by using the method of bi–Hamiltonian systems and was rederived from physical considerations in [3], where also its Lax Pair was given.

1 Study of equation (1)

Here we first study some ODE reductions of equation (1) and show that they lead to one of the known integrable equations of second order identified by Painlevé and coworkers, nearly 70 years ago [4]. We then apply the WTC algorithm [5, 6] to equation (1) and arrive at its Lax Pair.

Considering $u = u(x)$ and integrating (1) with respect to x we obtain the equation

$$u + \left(1 - \frac{3}{2}\rho_2\right)\beta u_{xx} + \frac{1}{2}\alpha u^2 - \frac{1}{4}\alpha\beta\rho_2(u_x^2 + 2u u_{xx}) = c \quad (3)$$

where c is the integration constant. To study nontrivial cases, it will be assumed, from now on, that $\alpha, \beta, \rho_2 \neq 0$.

The standard Painlevé analysis (or P–test) is an algorithmic procedure for identifing equations which possess the Painlevé property, i.e. whose general solution is free from movable singular points other than poles, and are hence expected to be integrable [7]. The P–test consists of 3 steps: (a) Determination of all possible leading–order behaviours of the Laurent series in the neighbourhood of a movable singularity, (b) computation of the resonances, i.e. powers at which arbitrary constants enter into the Laurent series expansion of the solution and (c) examination of the compatibility conditions, i.e. verification that a sufficient number of arbitrary constants arise without the introduction of logarithmic terms in the Laurent series.

The application of the P–test to (3) yields the following results:
Leading order behaviour: $u = a_1 + a_2\chi^p$, where $\chi = x - x_*$ (x_* is the singularity), $p = 2/3$, $a_1 = (2 - 3\rho_2)/(\alpha\rho_2)$ and a_2 is an arbitrary parameter.
Free constants enter at resonances -1, 0.

The resonance -1 corresponds to the free location of the singularity, while 0 corresponds to the arbitrary parameter a_2. Therefore, in this case, there is no need to move on to the third step. Note however that, here, eventhough no logarithms appear, the singularities are algebraic and appear in the series in such a way that it is not possible to remove them by simply raising the series to some integer power. Thus we need to study further this equation.

Let us first perform a numerical integration of (3) using the ATOMFT package to study the analytic properties of the solution in the complex x–plane.

Setting $c = 0$, $\alpha = \rho_2 = 2$ and $\beta = 1$ and taking initial conditions $u(0) = 1$ and $u_x(0) = 0.1$ we calculate $u_P(0)$ (the initial value of u at point P), using analytic continuation (Figure 1). We then integrate equation (3) numerically in the complex x–plane along a rectangular contour, starting from P and going clockwise and after each turn we calculate the difference $\Delta u_P(N) = |u_P(N) - u_P(0)|$, where N is the number of turns (more details can be found in [8] – [12]).

Keeping $-1 \le Re(x) \le 1$ and increasing the limits of $Im(x)$, we observe that new singularities enter the contour, but always yield exact returns after a number of turns N, which increases as the limits of $Im(x)$ are increased (see for example Figures $2-5$). Similar results are found for different contours. This behaviour is refered to as the FSS property (finitely sheeted solutions) and constitutes a numerical indication of integrability, if it persists for arbitrary large contours.

Motivated by this evidence let us now show analytically that (3) is indeed integrable. Following Goriely [13] we set

$$
\begin{aligned}
& w(x) = 2\delta u(x) + e,\ \delta = \frac{1}{4}\alpha\beta\rho_2,\ e = \left(\frac{3}{2}\rho_2 - 1\right)\beta \\
& dx = w(t)dt \\
& z = t\sqrt{\frac{\alpha}{6\delta}}
\end{aligned}
$$

and obtain from (3) the equation

$$
\frac{d^2w}{dz^2} = \frac{1}{2w}\left(\frac{dw}{dz}\right)^2 + \frac{3w^3}{2} + \left(\frac{6\delta}{\alpha} - 3e\right)w^2 + \left(\frac{3e^2}{2} - \frac{6\delta e}{\alpha} - \frac{12\delta^2 c}{\alpha}\right)w
$$

which is Painlevé XXX [4]. This shows indeed that equation (3) posseses the P–property, as all the algebraic singularities have been transformed to simple poles.

We also mention that if we consider the wave reduction $u = u(x - t)$ and follow a similar procedure we again arrive at Painlevé XXX. Finally, it is worth noting that the integrability of the above ODE reductions is also suggested by the fact that they satisfy the so–called weak–Painlevé property (associated with algebraic singularities) as described in [7].

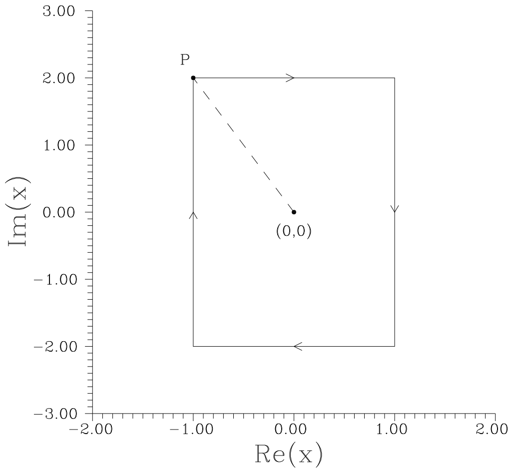

Figure 1

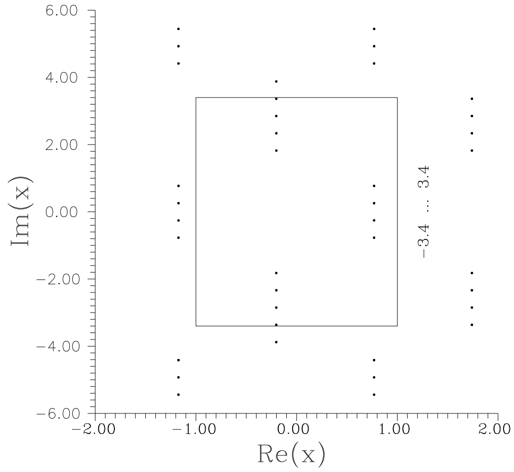

Figure 2

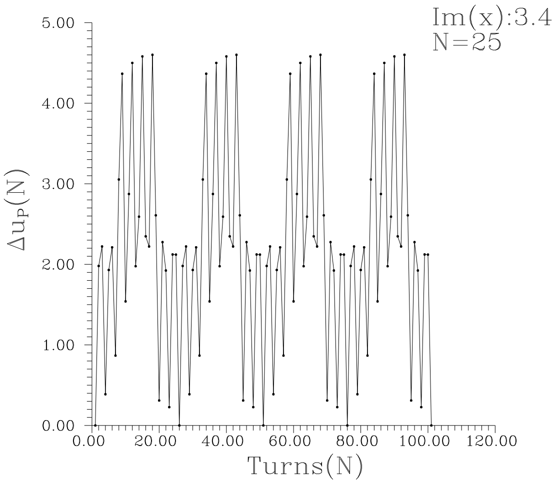

Figure 3

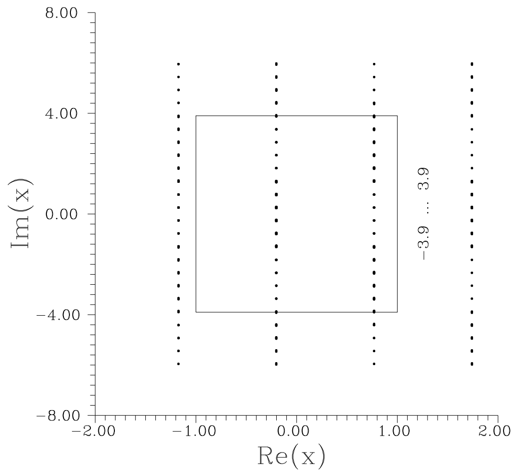

Figure 4

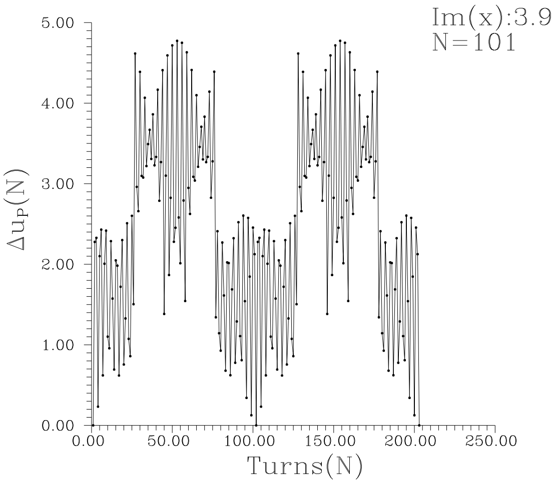

Figure 5

We now return to the full PDE (1) and apply again the P–test. The Painlevé analysis for PDEs is similar to the one for ODEs and has been developed by Weiss, Tabor and Carnevale in [5, 6]. In what follows u_i, φ, V and ψ are functions of x and t.

Let us exhibit the results of the WTC analysis as applied to equation (1):

Step 1: $u = u_c + u_0\varphi^p$ where $\varphi = 0$ is the singularity manifold, $p = 2/3$,
$u_c = (2-3\rho_2)/(\alpha\rho_2) - (3\varphi_t)/(\alpha\varphi_x)$ and u_0 is an arbitrary function.

Step 2: Resonances occur at -1, 0, 2/3.

Step 3:

$$u = u_c + \varphi^{\frac{2}{3}}(u_0 + u_1\varphi^{\frac{1}{3}} + u_2\varphi^{\frac{2}{3}} + \ldots) \tag{4}$$

where u_2 is again an arbitrary function and u_1 satisfies

$$9u_{0,t}\varphi_x - 9u_{0,x}\varphi_t + 2\alpha u_0 u_1 \varphi_x^2 = 0$$

If we now set $u_0 = u_2 = 0$, u_1 becomes arbitrary and it follows that $u_i = 0$ for $i > 2$, whence relation (4) truncates to

$$u = \frac{2-3\rho_2}{\alpha\rho_2} - \frac{3\varphi_t}{\alpha\varphi_x} + u_1\varphi \tag{5}$$

Considering

$$u_1 = \frac{-2}{\alpha\rho_2(6\beta\rho_2\lambda + 1)\varphi}$$

in order to eliminate φ from relation (5) we can now apply the procedure of Weiss, Tabor and Carnevale [5, 6] and find the following Lax Pair for (1):

$$\begin{aligned} \psi_{xx} &= a_1(x,t)\psi \\ \psi_t &= b_1(x,t)\psi_x + c_1(x,t)\psi \end{aligned}$$

where

$$\begin{aligned} a_1(x,t) &= \lambda(V-1) + \frac{V}{6\rho_2\beta} \ , \ V = -\rho_2\alpha\left(u - \frac{3}{2}\rho_2\beta u_{xx}\right) \\ b_1(x,t) &= \frac{2-3\rho_2}{3\rho_2} - \frac{2}{3\rho_2(6\beta\rho_2\lambda+1)} - \frac{\alpha}{3}u \\ c_1(x,t) &= \frac{\alpha}{6}u_x + c \ , \ c \text{ arbitrary parameter} \end{aligned}$$

This is exactly the Lax Pair found by Camassa and Holm in [3] (see also [14]). It should also be mentioned that, as far as we know, the classical WTC algorithm has given the Lax Pair only for PDEs that pass the standard P–test, where only poles are found. Equation (1) is a rather exceptional case, in which we can obtain the Lax Pair from the classical Painlevé analysis, even if the equation has algebraic singularities.

2 Study of equation (2)

Let us now use the method of ODE reductions to examine whether there exist parameter values at which the original water wave equation (2) might be integrable. In order to study nontrivial cases we will again assume that α, $\beta \neq 0$.

First we apply the transformation

$$\eta(x,t) = \frac{1}{\alpha}\, w\left(\frac{1}{\sqrt{\beta}}x, \frac{1}{\sqrt{\beta}}t\right)$$

and write equation (2) in the simpler form

$$w_t + w_x + ww_x + w_{xxx} + \rho_1 w^2 w_x + \rho_2 w w_{xxx} + \rho_3 w_x w_{xx} = 0 \qquad (6)$$

where we have used for simplicity the same notation for the independent variables x, t.

If $\rho_2 = \rho_3 = 0$ then equation (6) is the mKdV. If also $\rho_1 = 0$ then (6) reduces to the KdV. Therefore we will exclude these cases. We will also exclude the case $\rho_2 = 0$, because if we make the reduction $w = w(x)$ and integrate once with respect to x, the resulting equation has logarithmic singularities

$$w = a_0 ln\chi + \sum_{i=2}^{\infty} ln^i\chi \sum_{j=0}^{\infty} {a_j}^{(i)}\chi^j \quad , \quad \chi = x - x_*$$

where x_* and a_0 are the two arbitrary constants, and is therefore most likely not integrable, as it possesses infinitely sheeted solutions around every singularity x_* [9] – [12].

We now eliminate the w_{xxx} term setting

$$w(x,t) = u(x,t) - 1/\rho_2$$

and obtain

$$u_t + \frac{1}{{\rho_2}^2}\left(\rho_1 - \rho_2 + {\rho_2}^2\right) u_x + \frac{1}{\rho_2}(\rho_2 - 2\rho_1) u u_x + \rho_1 u^2 u_x + \rho_2 u u_{xxx} + \rho_3 u_x u_{xx} = 0 \qquad (7)$$

Finally, considering $u = u(x)$ and integrating once with respect to x, we obtain the equation

$$c + \frac{1}{{\rho_2}^2}\left(\rho_1 - \rho_2 + {\rho_2}^2\right) u + \frac{1}{2\rho_2}(\rho_2 - 2\rho_1) u^2 + \frac{1}{3}\rho_1 u^3 + \rho_2 u u_{xx} + \frac{1}{2}(\rho_3 - \rho_2) u_x^2 = 0 \qquad (8)$$

where c is the integration constant.

The application of the P–test to (8) yields the following results: In the most general case we have three leading order behaviours, namely $u \sim a_0\chi^p$ where
(a) $a_0 = -6(\rho_3 + 2\rho_2)/\rho_1, \;\; p = -2$
(b) a_0 : arbitrary, $\;-2 < p = 2\rho_2/(\rho_2 + \rho_3) < 1$
(c) ${a_0}^2 = 2c/(\rho_2 - \rho_3), \;\; p = 1$
and the resonances are -1 and $4 + 2\rho_3/\rho_2$, -1 and 0, -1 and $1 - \rho_3/\rho_2$ respectively.

Obviously the Painlevé analysis is now much more complicated, as the ρ_i parameters appear in both leading order behaviours and resonances. Therefore we will search only for the cases that pass the classical P–test, i.e. all leading order behaviours and resonances must be integers (and of course compatibility conditions must be satisfied).

The second resonance of branch (a), $R = 4 + 2\rho_3/\rho_2$, will play a crucial role in the analysis. First we observe that

$$R < -2 \;\Rightarrow\; -1 < \frac{2\rho_2}{\rho_2 + \rho_3} < 0$$

and

$$R > 6 \;\Rightarrow\; 0 < \frac{2\rho_2}{\rho_2 + \rho_3} < 1$$

which means that branch (b) is algebraic. Therefore we have to check one by one all the cases for $-2 \leq R \leq 6$ and integer R:

$R = -2 \Rightarrow \rho_3 = -3\rho_2$.
In this case branch (a) has negative resonances. However branches (b) and (c) pass the test (for branch (b) $p = -1$, while the resonances of branch (c) are -1, 4 and the compatibility conditions are satisfied). Moreover, using the Ince transformations [4], equation (8) can be reduced to equation Painlevé VIII (PVIII).
$R = -1 \Rightarrow \rho_3 = -5\rho_2/2$.
In this case branch (b) is algebraic ($p = -4/3$) and the resonances of branch (c) are -1 and 7/2. Therefore this case doesn't pass the P–test.
$R = 0 \Rightarrow \rho_3 = -2\rho_2$.
Branches (a) and (b) are combined into one and a_0 is arbitrary if $\rho_1 = 0$. The resonances of branch (c) are -1, 3 and the compatibility conditions are satisfied. The equation can now be transformed to PXXX.
$R = 1 \Rightarrow \rho_3 = -3\rho_2/2$.
Now we have only branches (a) and (c). The resonances of (c) are -1, 5/2.
$R = 2 \Rightarrow \rho_3 = -\rho_2$.
Branch (b) doesn't exist. For both (a) and (c) the resonances are -1, 2 and the compatibility conditions are satisfied if $\rho_1 = \rho_2/2$. (8) $\Rightarrow$ PXII.
$R = 3 \Rightarrow \rho_3 = -\rho_2/2$.
Branch (b) doesn't exist. The resonances of (c) are -1, 3/2.
$R = 4 \Rightarrow \rho_3 = 0$.
Again branch (b) doesn't exist. The resonances of (c) are -1, 1 and for both (a) and (c) the compatibility conditions are satisfied if $\rho_1 = \rho_2(1 - \rho_2)$. (8) $\Rightarrow$ PXXXIII.
$R = 5 \Rightarrow \rho_3 = \rho_2/2$.
Again branch (b) doesn't exist and the resonances of (c) are -1, 1/2.
$R = 6 \Rightarrow \rho_3 = \rho_2$.
In this case the compatibility condition for branch (a) is satisfied if $c = 0$. But this cannot be done, because c is the integration constant and has to by an arbitrary number.

Thus we conclude with the following Painlevé integrable cases:
1) $\rho_3 = -3\rho_2$. (8) $\Rightarrow$ PVIII.
2) $\rho_3 = -2\rho_2,\;\; \rho_1 = 0$. (8) $\Rightarrow$ PXXX.
3) $\rho_3 = -\rho_2,\;\; \rho_1 = \rho_2/2$. (8) $\Rightarrow$ PXII.
4) $\rho_3 = 0,\;\; \rho_1 = \rho_2(1 - \rho_2)$. (8) $\Rightarrow$ PXXXIII.

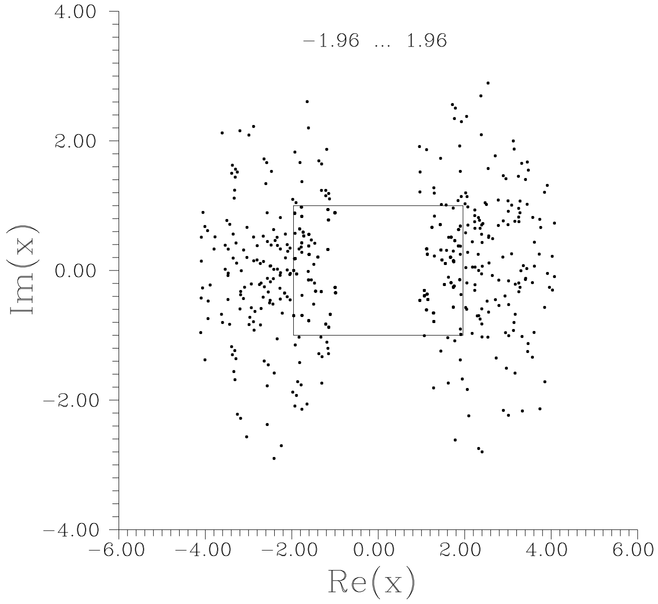

Figure 6

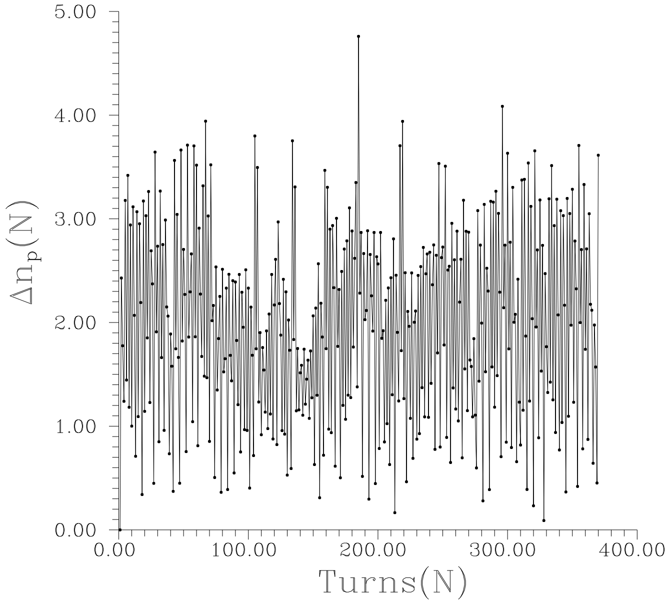

Figure 7

Equation (8) is not integrable in general. There are many cases, for example $\rho_3 = -\rho_2$, $\rho_1 \neq \rho_2/2$, for which the compatibility conditions are not satisfied, therefore logarithmic terms enter the Laurent expansion.

Moreover, if we make the reduction $\eta = \eta(x)$ directly in (2), integrate once with respect to x and apply again the ATOMFT algorithm for $\alpha = \beta = 1$, $\rho_1 = -1/6$, $\rho_2 = 5/3$, $\rho_3 = 23/6$ (i.e. the ρ_i parameter values that appear in [1]), $c = 0$ (integration constant) and initial conditions $\eta(0) = 1$ and $\eta_x(0) = 0.1$ the following results are obtained: When

$-1 \leq Im(x) \leq 1$ and $-1.95 \leq Re(x) \leq 1.95$ no singularities enter the contour and we have exact return after $N = 1$. However, when we take $-1.96 \leq Re(x) \leq 1.96$ a pattern of singularities enter the contour (Figure 6) and η doesn't return to its original value, even after hundreds of turns (Figure 7). This behaviour is refered to as the ISS property (infinitely sheeted solutions) and constitutes a numerical indication for non–integrability.

Now it is time to return again to the full PDE (7) to check whether any of the cases identified above are indeed integrable. Unfortunately, none of the above candidates for integrability passes the WTC Painlevé test for this PDE, because some compatibility condition is not satisfied. Therefore, we can conclude that there is no combination of the ρ_i parameters for which equation (2) has the Painlevé property (except, of course, for the cases which lead to KdV or mKdV). However, we must stress the fact that the above search is not exhaustive, because, as we mentioned before, there are many cases for which the equation has algebraic singularities, which cannot be checked by the classical P–test.

3 Conclusions and Discussion

The Painlevé analysis shows clearly (together with some numerical results) that equation (1) is integrable in general, while equation (2) doesn't pass the classical P–test for any combination of the ρ_i parameters, thus stressing the significance of the local transformation mentioned in the Introduction of this paper. Moreover, the Painlevé analysis gives rise to relation (5) from which the Lax Pair can be derived, even if equation (1) has algebraic singularities.

As far as this stage it seems reasonable for someone to apply a local transformation and study the integrable equation (1) rather than the nonintegrable equation (2). Knowing that equation (1) admits soliton solutions some interesting questions are raised whether equation (2) also admits some wave solutions, and if yes, whether these solutions preserve their form after a collision with each other and to what extent they resemble the solitons of (1). These questions are currently under investigation and results will appear in a future publication [15].

4 Acknowledgements

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