The Dirac Equation and the Davey-Stewartson I Equation: Deformation of the Dromion Solutions

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Abstract

Our contribution to this conference regards some of the recent results obtained jointly with F. Guil, which can be found in the papers *The Dirac Equation and Integrable Systems of KP Type* submitted to J. Phys. A: Math. & Gen. and *Deformation of the Dromion and Solitoff Solutions of the Davey-Stewartson I Equation* to appear in Phys. Lett. A. We summarize these results: The propagator for the 2D heat equation in an arbitrary linear space give solutions of the two-component Kadomtsev-Petviashvilii equations. This propagator is subject to the Klein-Gordon equation and its right-derivatives are required to be of rank-one, that imply that it can be expressed in terms of solutions of the Dirac equation. Large families of solutions of the two-component Kadomtsev-Petviashvilii equations are constructed in terms of solutions of the heat and Dirac equations. Particular attention is paid to the real reductions of the Davey-Stewartson type, recovering in this way the dark line solitons and the multidromion solutions.

A new solution of the Davey-Stewartson I equation is presented. This is a one parameter deformation of the dromion solution, that is contained in as particular example. The deformation corresponding to the 1-dromion solution is studied in detail. We analyze its asymptotic behaviour, appearing as a nonlinear superposition of the 1-dromion with the a 2-line dark soliton type solution. The 1-solitoff degeneration of the 1-dromion and its deformation is studied in a similar fashion. The 1-line dark soliton solution appears as a deformation of the 1-solitoff solution.

We present here only a part of these results, the way to do it is more close to that of a talk or poster than the one of a paper.

The two component Kadomtsev-Petviashvilii (2c-KP)

$$\frac{\partial^2 U}{\partial x_1 \partial x_2} + p_{12} p_{21} = 0,$$
$$\frac{\partial p_{12}}{\partial t} - \frac{\partial^2 p_{12}}{\partial x_1^2} + \frac{\partial^2 p_{12}}{\partial x_2^2} - 2 p_{12} \left(\frac{\partial^2 U}{\partial x_1^2} - \frac{\partial^2 U}{\partial x_2^2} \right) = 0,$$
$$\frac{\partial p_{21}}{\partial t} + \frac{\partial^2 p_{21}}{\partial x_1^2} - \frac{\partial^2 p_{21}}{\partial x_2^2} + 2 p_{21} \left(\frac{\partial^2 U}{\partial x_1^2} - \frac{\partial^2 U}{\partial x_2^2} \right) = 0.$$

x_1, x_2: complex coordinates

$p_{12}, p_{21}, U(x_1, x_2, t)$: complex functions on x_1, x_2, t

Davey-Stewartson I (DSI)

$x_1 = \xi, x_2 = \eta \in \mathbb{R}$

ξ, η: light cone coordinates

$p(\xi, \eta, t)$: amplitude of the surface wave packet

U: velocity potential

$\varepsilon = -1, 1$ distinguish the focusing and defocusing cases

$$\frac{\partial^2 U}{\partial \xi \partial \eta} + \varepsilon |p|^2 = 0,$$
$$\mathrm{i} \frac{\partial p}{\partial t} - \Delta p - 2 p \Delta U = 0.$$

Klein-Gordon & 2c-KP

- V: complex linear space
- $\psi(x_1, x_2, t)$: invertible linear operator over V
- Q, A_1, A_2: constant linear operators over V, $Q^2 = A_1 A_2 = A_2 A_1$
- e_1, e_2: constant vectors in V, $A_1 e_2 = \lambda_{12} e_1$, $A_2 e_1 = \lambda_{21} e_2$, $\lambda_{12}, \lambda_{21} \in \mathbb{C}$
- $\alpha_1(x_1, x_2, t), \alpha_2(x_1, x_2, t)$: linear functionals over V

2D Klein-Gordon equation

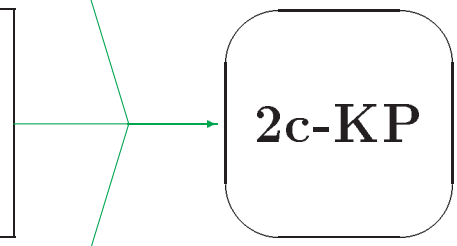
$$\frac{\partial^2 \psi}{\partial x_1 \partial x_2} - Q\psi = 0$$

2D heat equation

$$\frac{\partial \psi}{\partial t} - \Delta\psi = 0$$

Rank-one constraint

$$\frac{\partial \psi}{\partial x_i} = (A_i + e_i \otimes \alpha_i)\psi$$



Functions p_{ij}, constructed in terms of ψ, which are solutions of the 2c-KP equations:

$$\begin{aligned} p_{ij} &:= \lambda_{ij} + \langle \alpha_i, e_j \rangle, \ i \neq j, \\ p_{11} &:= -\lambda_{12}\lambda_{21} x_2 + \langle \alpha_1, e_1 \rangle, \\ p_{22} &:= -\lambda_{12}\lambda_{21} x_1 + \langle \alpha_2, e_2 \rangle. \end{aligned}$$

One can easily conclude that

$$\begin{aligned} \frac{\partial p_{11}}{\partial x_2} + p_{12}p_{21} &= 0, \\ \frac{\partial p_{22}}{\partial x_1} + p_{12}p_{21} &= 0. \end{aligned}$$

That imply the local existence of a potential U: $p_{ii} = \dfrac{\partial U}{\partial x_i}$

$$p_{12}p_{21} + \frac{\partial^2 U}{\partial x_1 \partial x_2} = 0$$

In the 2c-KP appear three independent variables: x_1, x_2, t.

i. x_1, x_2: light cone coordinates in the complex Minkowski plane, $\mathrm{d}\, s^2 = (\mathrm{d}\, x_1 \otimes \mathrm{d}\, x_2 + \mathrm{d}\, x_2 \otimes \mathrm{d}\, x_1)/2$.

ii. The time variable t of 2c-KP must be understood as a deformation or symmetry of the Klein-Gordon equation.

If $\psi(x_1, x_2, t)$ is a solution of the 2D heat and Klein-Gordon equations as well as of the rank-one constraint modulo constants. Then, the functions p_{12}, p_{21} & U are a solution of 2c-KP.

Simultaneous solution of the 2D Klein-Gordon and heat equations and rank-one constraint $\Longrightarrow$ solutions of the 2c-KP.

One can find the general simultaneous solution!

Let $\beta_i(x_1, x_2, t) \in V^*$, $i = 1, 2$, be functions of x_1, x_2, t taking values in V^*, s. t.

$$\begin{pmatrix} \partial_2 & \lambda_{12} \\ \lambda_{21} & \partial_1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = 0,$$

$$\frac{\partial \beta_i}{\partial t} - \Delta \beta_i = 0.$$

A solution $\psi(x_1, x_2, t)$ of the 2D heat and Klein-Gordon equations, as well as of the rank-one constraint is

$$\psi = \psi_0 \cdot \varphi,$$

with ψ_0 given by

$$\psi_0(x_1, x_2, t) := \exp[A_1 x_1 + A_2 x_2 + (A_1^2 + A_2^2)t],$$

where $\varphi(x_1, x_2, t)$ is determined by the system of compatible equations:

$$\frac{\partial \varphi}{\partial x_i} = b_i \otimes \beta_i, \ \ i = 1, 2,$$

$$\frac{\partial \varphi}{\partial t} = \sum_{i=1,2} \left(b_i \otimes \frac{\partial \beta_i}{\partial x_i} - \frac{\partial b_i}{\partial x_i} \otimes \beta_i \right),$$

with $b_i = \psi_0^{-1} e_i$. The α's are given by $\alpha_i = \beta_i \psi$.

Dirac & Davey-Stewartson I

$x_1 = \xi, x_2 = \eta \in \mathbb{R}$
$p_{12} = \varepsilon \bar{p}_{21} =: p,\ \varepsilon = \pm 1,\ \nabla U(\xi, \eta, t) \in \mathbb{R}^2$

What A_i, β_i are associated with this reduction?

V: linear space with a complex scalar product
$^\dagger : V \to V^*$ associated antilinear isomorphism
Choice:

$$\lambda_{12} = \varepsilon \bar{\lambda}_{21} =: \lambda \geq 0,$$
$$\beta_1 = b_1^\dagger H,$$
$$\beta_2 = \varepsilon b_2^\dagger H,$$
$$\varphi_0^\dagger H = H \varphi_0,$$

$H = H^\dagger$, H is Hermitian
φ_0: initial value of $\varphi(\xi, \eta, t)$ at $\xi = \xi_0, \eta = \eta_0, t = t_0$,

Deformations of the 1-dromion of DSI

$V = \mathbb{C}^2$, $\varepsilon = 1$

General expression for the operators A_i's

$$A_1 = \begin{pmatrix} q_1 & -\lambda k^2 \\ \lambda & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & \lambda \\ -\lambda k^{-2} & q_2 \end{pmatrix},$$

where $q_1, q_2 \in \mathbb{C}, k \in \mathbb{C}^\times$ satisfy $q_1 = k^2 q_2$.

For e_1 and e_2 we have

$$e_1 = \begin{pmatrix} a_1 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ a_2 \end{pmatrix},$$

with $a_1, a_2 \in \mathbb{C}^\times$ s. t. $k^2 = -a_1/a_2$.

Given $q_1, q_2 < 0$ define

$$Q := \sqrt{\frac{1}{4} - \frac{\lambda^2}{q_1 q_2}},$$

choose q_1, q_2 such that $0 \leq Q \leq 1/2$: $q_1 q_2 \geq 4|\lambda|^2$.

Introduce

- $Q_\pm := 1/2 \pm Q$
- $\xi(t) := \xi - \mathrm{i}\, q_1 t$, $\eta(t) := \eta + \mathrm{i}\, q_2 t$
- $\lambda_\pm := Q_\pm q_1 \xi(t) + Q_\mp q_2 \eta(t) + \mathrm{i}\, Q_+ Q_- (q_1^2 - q_2^2) t$

The vectors b_1 and b_2 are:

$$b_1 = \frac{q_1}{2Q}\begin{pmatrix} Q_+\exp(-\lambda_+) - Q_-\exp(-\lambda_-) \\ \frac{\lambda}{q_1}(\exp(-\lambda_+) - \exp(-\lambda_-)) \end{pmatrix},$$

$$b_2 = \frac{q_2}{2Q}\begin{pmatrix} \frac{\lambda}{q_2}(\exp(-\lambda_+) - \exp(-\lambda_-)) \\ Q_-\exp(-\lambda_+) - Q_+\exp(-\lambda_-) \end{pmatrix}.$$

The principal tau function is $\tau = \det\varphi$, that in our case reads, $H = (H_{ij})$, $H^\dagger = H$,

$$\det\varphi = 1 - \frac{1}{8Q^2}\exp(-(q_1\xi + q_2\eta))\Big[\Sigma_+\exp(-2Q(q_1\xi - q_2\eta)) + \Sigma_-\exp(2Q(q_1\xi - q_2\eta)) - \Sigma_0(t)\Big] + \frac{q_1q_2\det H}{4}\exp(-2(q_1\xi + q_2\eta)),$$

where

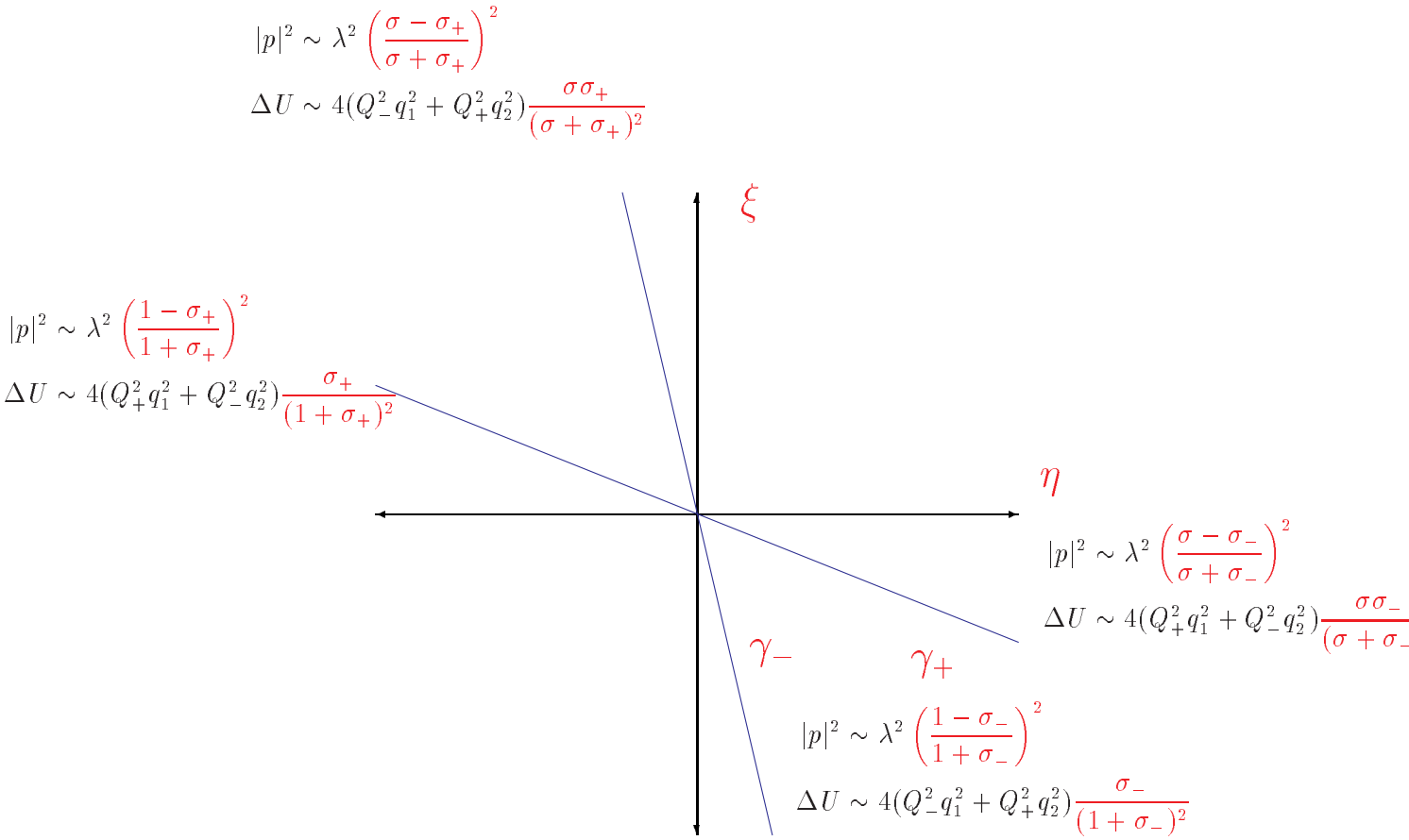
$$\begin{aligned}
\Sigma_+ :=& H_{11}q_1Q_+ + H_{22}q_2Q_- + 2\lambda\,\mathrm{Re}\,H_{21},\\
\Sigma_- :=& H_{11}q_1Q_- + H_{22}q_2Q_+ + 2\lambda\,\mathrm{Re}\,H_{21},\\
\Sigma_0(t) :=& 4\lambda\Big[(H_{11}q_1 + H_{22}q_2)\frac{\lambda}{q_1q_2}\cos(2Q(q_1^2+q_2^2)t)\\
&+ \mathrm{Re}\left\{H_{21}(\cos(2Q(q_1^2+q_2^2)t) + 2\,\mathrm{i}\,Q\sin(2Q(q_1^2+q_2^2)t))\right\}\Big].
\end{aligned}$$

The modulus of the amplitude $|p|$ is determined by

$$|p|^2 = \lambda^2 - \partial_\xi\partial_\eta \ln\det\varphi.$$

When $\lambda = 0$ one recovers the general formulae for the 1-dromion solution an exponential localized solution in all directions. As λ^2 goes from 0 to $q_1q_2/4$ one finds a one-complex-parameter deformation of the 1-dromion solution of DSI.

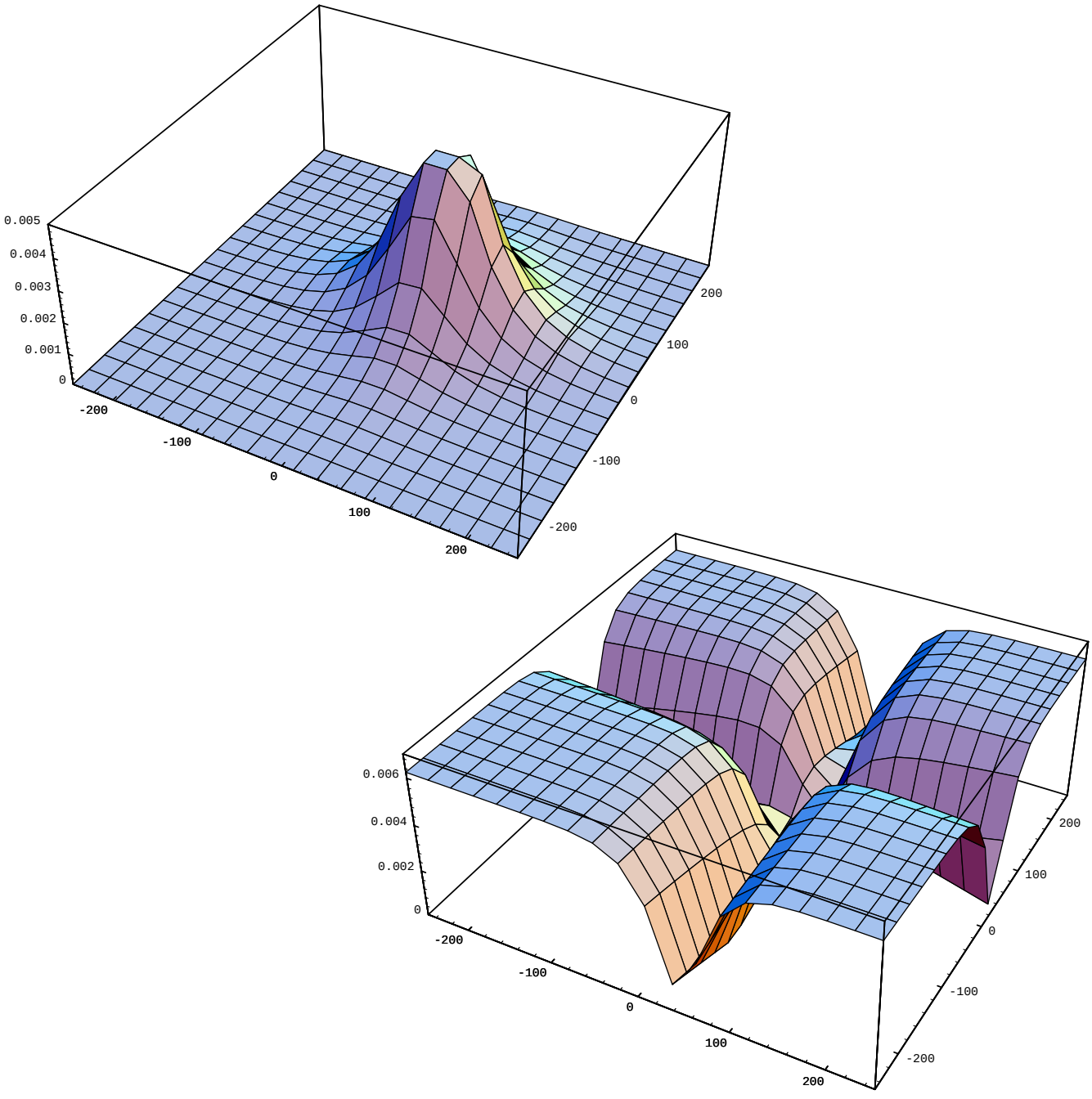
The asymptotic behaviour of this solution can be analyzed by using straight lines. In almost any straight ray from the origin we have the following asymptotics $|p|^2 \sim \lambda^2$, $\Delta U \sim 0$. The are two privileged lines $\gamma_\pm$ in where this asymptotic behaviour is different. These lines are given by $\gamma_\pm := \{(\xi,\eta)^t \in \mathbb{R}^2 : Q_\pm q_1\xi + Q_\mp q_2\eta = 0\}$.



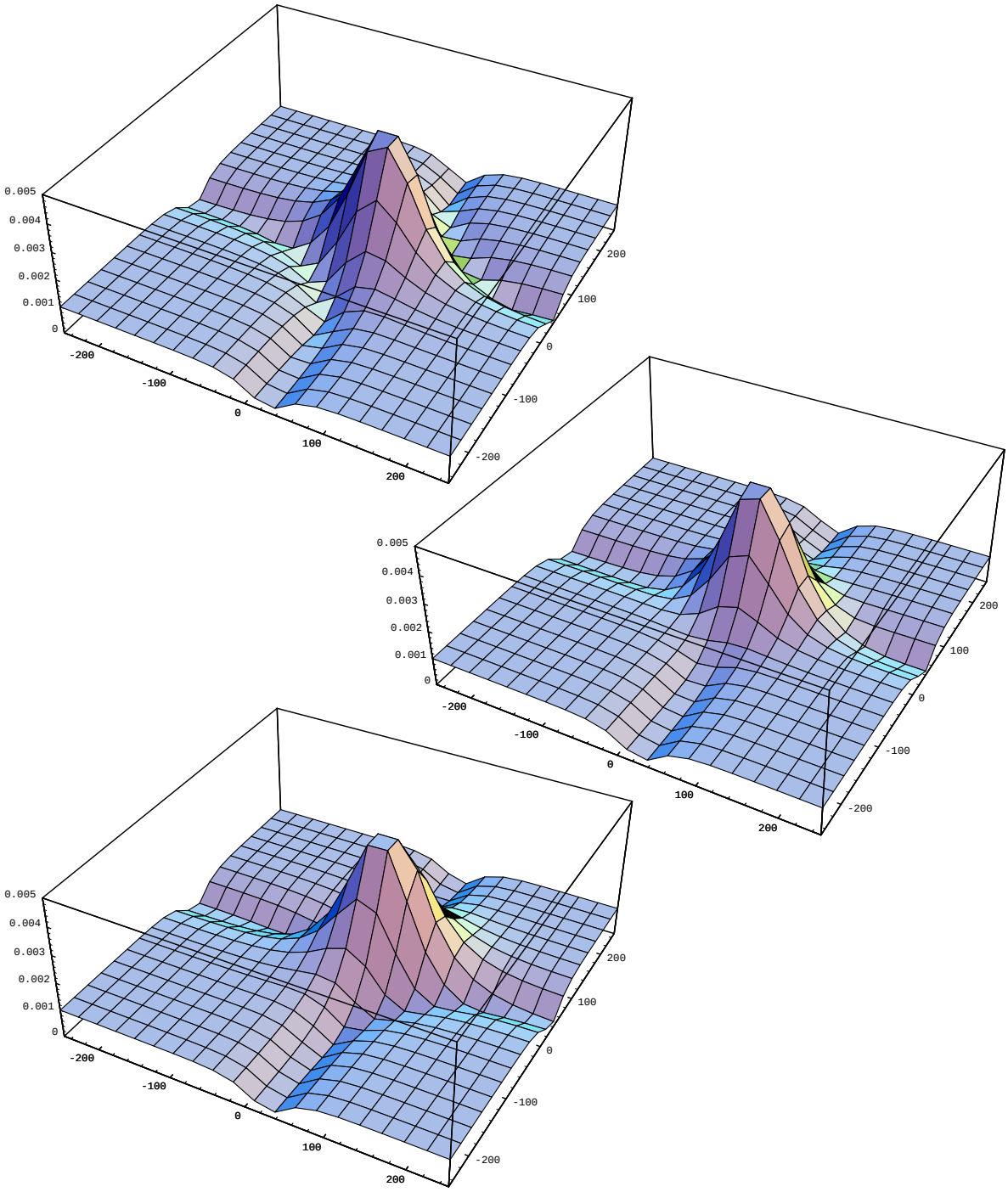
Observe that the lines $\gamma_\pm$ always lay on the regions where $\xi\eta \leq 0$, picking the point $(\xi_0, \eta_0) = (-\infty, \infty)$, the lines will lay on the regions with $\xi\eta \geq 0$. The asymptotic value of the amplitude $|p|^2$ on $\gamma_\pm$ is always less (with the exception of the dromion $\lambda = 0$) than in the other lines –where it takes the value λ^2–, and in general different in each on the directions defined by $\gamma_\pm$. The boundaries associated with ΔU are asymptotically localized over the lines $\gamma_\pm$ assuming in general different values in each of the directions defined by them. Note also that asymptotically the time dependent factor $\Sigma_0(t)$ plays no rôle. This time dependence, gives to $|p|$ a localized pulsation with period given by $T = \pi/(Q(q_1^2+q_2^2))$. This breather character for $|p|$ disappears as soon as $\Sigma_0 = 0$, that happens when $\lambda = 0$ or $H_{12} \in \mathbb{R}^+$ and $\lambda = \lambda_* := -\frac{H_{12}q_1q_2}{H_{11}q_1+H_{22}q_2}$ that is an allowed value of λ if $(H_{11}q_1 + H_{22}q_2)^2 \geq q_1q_2H_{12}^2$. These two values of λ gives the cases when the τ function is of the type $1 + \exp\eta_1 + \exp\eta_2 + D_{12}\exp(\eta_1+\eta_2)$. In general 4 independent exponentials appear generically in $\det\varphi$, from where one readily concludes that this solution can not be a two line-soliton solution.

In the following examples we take $q_1 = q_2 = -1/40$ and

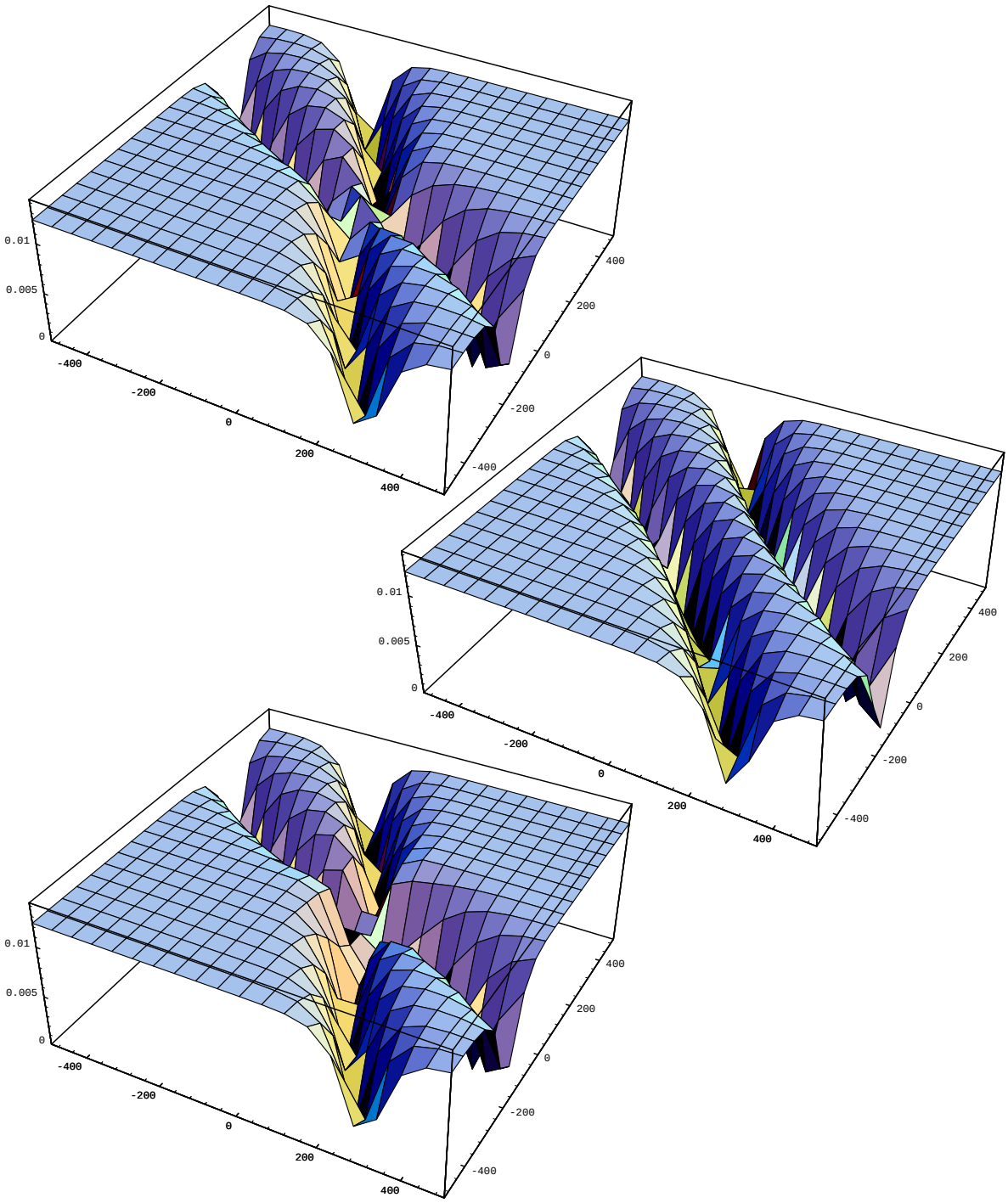
$$H = \begin{pmatrix} 20 & 10 \\ 10 & 20 \end{pmatrix}.$$



Amplitude modulus $|p|$ of a typical 1-dromion, $\lambda = 0.$ and of a typical 2-line soliton, $\lambda = 0.00625$. Now $\Sigma_0 = 0$, both are static.



Amplitude modulus $|p|$ for small mass, $\lambda = 0.001$. Over the original dromion structure a 2-line soliton appears, giving a nonzero asymptotic value. We can appreciate the breather character of the function.



Finally, $\lambda = 1/81 \lessapprox .0125$, the critical value. With our data we have $\lambda \in (0, 1/80)$.

Let s_1 y s_2 be functions of ξ, η, t taking values in W that with respect to the linear splitting $W = \mathsf{W} \oplus \mathsf{W}$ can be written as

$$s_1 = \mathsf{s}_1 + \varepsilon \bar{\lambda} S,$$
$$s_2 = \lambda S + \mathsf{s}_2,$$

with $S(0) = 0$ & s. t.

$$\begin{pmatrix} \partial_2 & \varepsilon\lambda \\ \lambda & \partial_1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = 0,$$
$$\mathrm{i}\frac{\partial s_i}{\partial t} + \frac{\partial^2 s_i}{\partial \xi^2} - \frac{\partial^2 s_i}{\partial \eta^2} = 0.$$

Let $\Phi(\xi, \eta, t) \in \mathrm{GL}(\mathsf{W})$ be a solution of

$$\frac{\partial \Phi}{\partial \xi} = s_1 \otimes s_1^\dagger H,$$
$$\frac{\partial \Phi}{\partial \eta} = \varepsilon s_2 \otimes s_2^\dagger H,$$
$$\mathrm{i}\frac{\partial \Phi}{\partial t} = \Big[s_1 \otimes \frac{\partial s_1}{\partial \xi}^\dagger - \frac{\partial s}{\partial \xi} \otimes s_1^\dagger - \varepsilon (s_2 \otimes \frac{\partial s}{\partial \eta}^\dagger - \frac{\partial s}{\partial \eta} \otimes s_2^\dagger) \Big] H,$$

where H is a Hermitian operator, with the initial value Φ_0 s. t.

$$\Phi_0^\dagger H - H \Phi_0 = 0,$$

then

$$p = \lambda + \langle s_1^\dagger, H \Phi^{-1} s_2 \rangle,$$
$$U = \ln \det \Phi - \varepsilon \lambda^2 \xi \eta$$

solves

$$\frac{\partial^2 U}{\partial \xi \partial \eta} + \varepsilon |p|^2 = 0,$$
$$\mathrm{i}\frac{\partial p}{\partial t} - \Delta p - 2p \Delta U = 0.$$

The case $\lambda = 0$ contains the double Grammian determinant formulae of Degasperis-Gilson-Nimmo, and allows an extension by including double Wronskian determinants. (Wronskians are particular examples of Grammians.) Dromions and Gaussons can be obtained by using these formulae.

Thus, we have exposed a scheme for massive deformations of the dromions.

Mass = asymptotic values.

- Do not confuse algebraic mass with the energy or mass of the dromions which diverges when $\lambda \neq 0$.
- The massive deformations of the dromions correspond as dromions to a solution of an boundary-initial value problem:

$$\mathrm{i}\,\partial_t p + \Delta p + p(U_1 + U_2) = 0,$$

$$U_1(\xi,\eta,t) = -\frac{\varepsilon}{2}\int_{-\infty}^{\xi} \mathrm{d}\,\xi \frac{\partial |p|^2}{\partial \eta} + u_1(\eta,t),$$

$$U_2(\xi,\eta,t) = -\frac{\varepsilon}{2}\int_{-\infty}^{\eta} \mathrm{d}\,\eta \frac{\partial |p|^2}{\partial \xi} + u_2(\xi,t),$$

but now the initial value $p(\xi,\eta,0)$ do not vanish asymptotically, we only have:

$$\frac{\partial |p|^2}{\partial \eta}, \frac{\partial |p|^2}{\partial \xi} \xrightarrow[\xi,\eta\to-\infty]{} 0$$

- The functions s_1, s_2 are chosen s. t. they vanish at $\xi_0 = -\infty$, $\eta_0 = -\infty$.