VARIATIONAL APPROACH TO SOLITON DYNAMICS IN THE DISCRETE NONLINEAR SCHRÖDINGER EQUATION

Boris Malomed[1]

Department of Applied Mathematics, School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel

Michael Weinstein[2]

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

ABSTRACT

Using a variational technique, based on an effective Lagrangian, we analyze static and dynamical properties of solitons in the 1D discrete nonlinear Schrödinger equation with an arbitrary power nonlinearity. We demonstrate that, at low values of the nonlinearity power, there is no bistability and no threshold for formation of the soliton. Above a critical value of the power, which lies between the cubic and quintic nonlinearities, but at the powers which are smaller than that corresponding to the quintic nonlinearity, there is still no threshold. However, in a certain interval of the energies, the soliton becomes multistable; there are two stable and one unstable solutions at the same energy. The quintic nonlinearity plays another critical role: in this case, and at stronger nonlinearities, there exists a static energy threshold for formation of the soliton. For all the nonlinearities stronger than quintic, one always has two solitons above the threshold, one stable and one unstable, the stable solution corresponding to a narrower soliton. We also find dynamical energy threshold for formation of a soliton from an arbitrarily narrow initial configuration.

[1]e-mail malomed@leo.math.tau.ac.il

[2]e-mail miw@math.lsa.umich.edu

1 INTRODUCTION

This is an abridged version of a paper to be published elsewhere, in which we will present basic results obtained by means of a variational approximation for the discrete nonlinear Schrödinger (DNLS) equation. Since postscript files of relevant figures were not ready in a final form by a deadline for submission of this electronic contribution, figures are not included here. Whenever possible, most essential results which would be normally presented in the form of the plots will be replaced by some verbal description.

The DNLS equation (also known as the self-trapping equation) is a well-known nonintegrable model which has a number of applications in molecular physics, nonlinear optics, and in other fields [1, 2]. In one dimension, the equation takes the form

$$i\frac{du_n}{dt} + \frac{1}{2}\left(u_{n+1} + u_{n-1} - 2u_n\right) + |u|^{2\sigma}u \,=\, 0 \tag{1}$$

where u is a complex function of time t and of the discrete coordinate n, and σ is the nonlinearity power. Eq. (1) can be naturally derived from the Lagrangian

$$L \,=\, \sum_{n=-\infty}^{+\infty}\left[\frac{i}{2}\left(u_n^*\frac{du_n}{dt} - u_n\frac{du_n^*}{dt}\right) - \frac{1}{2}|u_n - u_{n-1}|^2 + \frac{1}{\sigma+1}|u|^{2(\sigma+1)}\right]. \tag{2}$$

The existence of the Lagrangian representation suggests a possibility of application to Eq. (1) of the known technique based on the variational approximation, which has proved to be quite effective, e.g., in the analysis of the continuum NLS equation [3]. The objective of the present work is to apply this technique to soliton-like configurations. Only quiescent solitons will be considered; however, internal vibrations of the soliton will be admitted.

2 THE VARIATIONAL TECHNIQUE

To implement the variational approximation, we first specify an Ansatz which is both simple for explicit analytical computations yet sufficiently flexible so as to incorporate a large class of dynamical phenomena. We define:

$$u_n(t) \;=\; A\,\exp\left(i\phi + ib|n| - \alpha|n|\right)\,, \tag{3}$$

where A and ϕ are the amplitude and overall phase of the soliton, α^{-1} is its width, and the parameter b is the direct analog of the so-called "chirp", which plays a well-known role in the dynamics governed by the continuum NLS equation [3]. Notice that we assume the same ansatz (3) to be valid both at $n \neq 0$ and at the soliton's center, $n = 0$.

The next step is, letting all the free parameters in the ansatz (3) be functions of time, to insert it into the Lagrangian (2) and, explicitly performing the summation, to obtain the *effective* Lagrangian:

$$L \;=\; -A^2\frac{d\phi}{dt}\,\mathrm{coth}\alpha - \frac{1}{2}\frac{db}{dt}\,\frac{A^2}{\sinh^2\alpha} - A^2\mathrm{coth}\alpha + A^2\frac{\cos b}{\mathrm{sinh}\alpha} + (\sigma+1)^{-1}A^{2(\sigma+1)}\mathrm{coth}\left((\sigma+1)\alpha\right)\,. \tag{4}$$

Varying the effective Lagarangian in all the four variables yields a set of ODE's which furnish a natural basis for approximate description of the soliton's dynamics.

First of all, varying the phase ϕ yields the conservation law

$$\frac{d}{dt}W \;\equiv\; \frac{d}{dt}\left(A^2\mathrm{coth}\alpha\right) \;=\; 0\,. \tag{5}$$

We will call the conserved quantity W the soliton's energy (which is different from the Hamiltonian, see below).

Using this conservation law, we can eliminate the amplitude A from the remaining equations; the phase does ϕ not appear in those equations either. Thus, we can bring the remaining evolutional equations for the parameters α and b to the form of a two-dimensional dynamical system which conserves the Hamiltonian

$$H = -W\left[\cos b\,\mathrm{sech}\alpha + (\sigma+1)^{-1}W^{\sigma}\tanh^{\sigma+1}\alpha\coth\left((\sigma+1)\alpha\right) - 1\right]. \qquad (6)$$

3 STATIONARY SOLITONS

Stationary states of the soliton correspond to fixed points (FP's) of the above-mentioned dynamical system, which, in turn, are critical points of the Hamiltonian function (6), defined by the conditions $\partial H/\partial\alpha = 0$ and $\partial H/\partial b = 0$. Straightforward analysis demonstrates, firstly, that the FP's are only possible at $b = 2\pi N$, $N = 0, 1, 2...$ (it is noteworthy that there is no FP at $b = \pi + 2\pi N$). The value of the width parameter α at the FP is related to the energy W as follows:

$$W^{-\sigma} = \frac{1}{2}\frac{\sinh^{\sigma-1}\alpha}{\cosh^{\alpha}}\cdot\frac{\sinh\left(2(\sigma+1)\alpha\right) - \sinh(2\alpha)}{\sinh^{2}((\sigma+1)\alpha)}. \qquad (7)$$

Equation (7) greatly simplifies in the limit $\alpha \to \infty$: $\alpha \approx \ln\left(2W^{\sigma}\right)$, the meaning of which is clear; a soliton with the large energy must be narrow.

In the opposite limit, $\alpha \to 0$ (very broad solitons), the asymptotic form of Eq. (7) is

$$W^{-\sigma} = \sigma(\sigma+1)^{-2}\alpha^{\sigma-2}. \qquad (8)$$

The situation at small α, and implications for the whole dependence $\alpha(W)$, strongly depend upon the value of the nonlinearity power σ. At $\sigma = 1$, which corresponds to the usual cubic nonlinearity, Eq. (8) takes the form $W = 4\alpha$. Further analysis demonstrates that, with increase of W from zero to infinity, α is also monotonically growing, having exactly one value for each value of W. Thus, in the case of the cubic DNLS, there is no energy threshold for formation of the soliton. Using Eq. (7), one can check that qualitatively the same picture remains valid in the interval $1 \leq \sigma < \sigma_{\rm cr} \approx 1.4$.

At the critical value, $\sigma_{\rm cr} \approx 1.4$, the dependence $W(\alpha)$ (it is more convenient to consider now this dependence rather than $\alpha(W)$) acquires a *critical point*, at which the derivative $dW/d\alpha$ vanishes. This point is found to be at $\alpha \approx 0.55$. A noteworthy fact is that this $\sigma_{\rm cr}$, produced by the variational approximation, is quite close to the minimum value of σ at which the numerical results presented in Ref. [2] signalize onset of an instability of some soliton states in the same model. As it will be shown below, the variational approach predicts an instability of some soliton states at $\sigma > \sigma_{\rm cr}$.

In the range

$$\sigma_{\rm cr} < \sigma < 2\,, \tag{9}$$

Eq. (8) still predicts that $W = 0$ at $\alpha = 0$. On the other hand, the dependence $W(\alpha)$ has, at these values of σ, a local minimum at some value $W_{\rm min}$ of W, and a local maximum at some other value, $W_{\rm max}$. It is important that $W_{\rm max}$ corresponds to a *smaller* value of α than $W_{\rm min}$. Thus, we arrive at the following conclusion: in the range (9), there is a single soliton state at $W < W_{\rm min}$ and at $W > W_{\rm max}$, and *three* soliton states at $W_{\rm min} < W < W_{\rm max}$. Moreover, using the explicit dynamical equations produced by the variational approximation, it is easy to show that, in accord with the natural expectations, the soliton is stable in the region in which it is single; in the region where three solutions exist, two of them corresponding to the positive slope $dW/d\alpha$ (at a fixed value of the energy, these are the solitons with the largest and smallest widths) are stable too, while the intermediate soliton, corresponding to the negative slope, is unstable.

A remarkable fact following from Eq. (8) is that, exactly at $\sigma = 2$ (which corresponds to the so-callled quintic NLS equation), the value of $W(\alpha = 0)$ jumps from zero to the finite value $W_0 = 3/\sqrt{2}$, and the local maximum in the dependence $W(\alpha)$ disappears, while the minimum survives with no dramatic change. This implies that, at $\sigma = 2$, the properties of the solitons are qualitatively different from what was described above for the range (9). Namely, there is no solution at $W < W_{\min}$, so that $W_{\min}$ is the *energy threshold* for formation of the soliton. We find that $W_{\min} \approx 1.427$ at $\sigma = 2$. At $W_{\min} < W < W_0$, there are two solutions, the one with larger α (i.e., corresponding to a narrower soliton) being stable, and the one with smaller α unstable. Lastly, at $W > W_0$ we have the single stable soliton.

We remark that this special value $\sigma = 2$ is the same as the critical value for the continuum NLS equation, at which the weak collapse (blow up) takes place. A sharp threshold phenomenon in this continuum problem was established in [4].

At $\sigma > 2$, the picture is generally similar to that at $\sigma = 2$, but with the essential difference that, as it follows from Eq. (8), $W(\alpha = 0) = \infty$. Therefore, at $\sigma > 2$ we always have one stable and one unstable solitons at $W > W_{\min}$. For example, at $\sigma = 3$, $W_{\min} \approx 1.375$. For large values of σ, one can obtain from Eq. (7) the approximate formula $(W_{\min})^\sigma \approx 12\sqrt{3(2\sigma - 1)}$.

The above results were represented in terms of the parameters W and α, which determine, respectively, the energy and inverse width of the soliton. In Ref. [2], the numerical results for the soliton's stability were presented in terms of the nonlinearity power σ and the soliton's frequency. From the viewpoint of the ansatz (3), the frequency is simply the constant value of $\frac{d\phi}{dt}$, which can be easily found in the framework of the variational approximation. It will be quite interesting, summarizing the results outlined above, to draw a border between the stable and unstable solitons on the plane "σ - frequency", and compare this with the results of Ref. [2]. By the deadline for submission of this contribution, this plot was not yet ready, but preliminary sketches show that, at least qualitatively, the agreement is quite decent.

4 DYNAMICAL STATES OF THE SOLITONS

The variational approximation can be effectively used not only for prediction of the static soliton configurations and their stability, but also for description of their dynamical properties. The simplest dynamical problem that can be analyzed is the following one: does a given initial configuration produce a soliton? Here, we will consider this problem for the configuration (3) with $b = 0$ and arbitrary A and α, i.e., with the arbitary initial energy $W \equiv A^2 \mathrm{coth}\alpha$. If the soliton does not show up, the initial configuration will eventually decay into the "lattice radiation" (small-amplitude vibrations spreading along the lattice). To analyze the possibility of the decay, one can use the Hamiltonian (6), or, which is more convenient technically, the conserved quantity $h \equiv 1 - H/W$. In terms of our ansatz, the decay into radiation obviously implies $\alpha \to 0$. As it follows from Eq. (6), in this limit the function h takes the value $h_0 = \cos b$. If one can find a value of the chirp b such that the inital value $h(0)$ of h is equal to $\cos b$, the decay into radiation is possible. Thus, the condition prohibiting the decay, and therefore guaranteeing the existence of the soliton, is $h(0) > 1$. In particular, in the limit of a very narrow initial configuration, $\alpha \to \infty$, $h = (\sigma + 1)^{-1} W^{\sigma}$. Thus, the no-decay condition takes for this particular initial configuration the form

$$W^{\sigma} > \sigma + 1 \,. \tag{10}$$

In particular, at the critical value $\sigma = 2$, this implies $W > \sqrt{3}$, which, in turn, is surely larger (although not much larger) than the value $W_{\min} \approx 1.427$ found in the previous section, which is the *absolute* energy threshold for the existence of the soliton in the system. We have checked that, at all the values $\sigma \geq 2$, at which the absolute threshold exists, it is smaller than the special dynamical threshold given by Eq. (10).

5 CONCLUSION

In this work, we have developed a variational technique for analysis of static and dynamical properties of solitons in the 1D discrete nonlinear Schrödinger equation with an arbitrary power nonlinearity. A major part of the consideration was performed analytically. We have demonstrated that, at low values of the nonlinearity power, there is no multistability and no threshold for formation of the soliton. Above a critical value of the power, which lies between the cubic and quintic nonlinearities, but at the powers which are smaller than that corresponding to the quintic nonlinearity, there is still no threshold, but, in a certain interval of the energies, the soliton becomes multistable: there are two stable and one unstable solutions. The quintic nonlinearity plays another critical role: for nonlinearities stronger than it, there is an energy threshold for formation of the soliton, and above the threshold one always have two solutions, one stable and one unstable, the stable solution corresponding to a narrower soliton. The energy threshold for formation of the soliton out of an arbitrarily narrow initial configuration was found too.

Recently, similar ideas, but based on essentially different Ansätze, were put forward in the preprints [5].

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References

[1] J.C. Eilbeck, P.S. Lomdahl, and A.C. Scott, Physica D **16**, 318 (1985).

[2] E.W. Laedke, K.H. Spatschek, and S.K. Turitsyn, Phys. Rev. Lett. **73**, 1055 (1994).

[3] D. Anderson, Phys. Rev. A **27** (1983); D. Anderson, M. Lisak, and T. Reichel, J. Opt. Soc. Am. B **5** 207 (1988).

[4] M.I. Weinstein, Contemporary Math. **99**, 213 (1989)

[5] A.B. Aceves, C. De Angelis, T. Peschel, R. Muschall, F. Lederer, S. Trillo, and S. Wabnitz, *"Discrete Self-Trapping, Soliton Interactions, and Beam-Steering in Nonlinear Waveguide Arrays"*, preprint;

T. Peschel, R. Muschall, and F. Lederer, *Variational Approach to the Localization od Spatial Solitary Waves in Arrays of Nonlinear Waveguides*, preprint.