Bose-gas with nontrivial interparticle interaction and exotic magnetic solitons.

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1 Introduction

In weakly nonlinear systems the solitonic excitations (dynamic two-parametric solitons) are studied usually in the framework of nonlinear Schrodinger equation (NSE) [1]:

$$i\partial\psi/\partial t + \partial^2\psi/\partial x^2 - \omega_0\psi + g \mid \psi \mid^2 \psi = 0. \quad (1)$$

Its motionless soliton solution has the simple form

$$\psi = \sqrt{2\epsilon/g}\cosh^{-1}(\epsilon x)\exp(-i\omega t), \epsilon = \sqrt{\omega_0 - \omega}. \quad (2)$$

The 1D Bose-gas with a pair δ-function interaction is a quantum analog of a classical system, which is described by NSE [2]. Schrodinger equation for N-particle wave-function $\Phi(x_1, x_2, .., x_N)$ has the following form ($\hbar = 1, m = 1/2$):

$$-\sum_i d^2\Phi/dx_i^2 + N\omega_0\Phi + \sum_{i<j} U_{ij}\Phi = E\Phi, U_{ij} = -g\delta(x_i - x_j). \quad (3)$$

This quantum system is completely integrable by the Bethe-anzats method [3].Classical soliton solution (2) corresponds to many-bosons bounded states with the energy spectrum $E = \omega_0 N - g^2 N(N^2-1)/48$, where $N = 4\epsilon/g$. This spectrum coincides with the result of quasi-classical quantization of (2). The distribution of boson density coincides with an envelope of the function $\mid \psi \mid^2$for the soliton (2) in limit $N \gg 1$.

But the description of solitons in framework of NSE valids only for small amplitudes (small N) and the soliton collapse takes place while $N \to \infty$. In reality the limitation of amplitude growth takes place and is connected with a saturation of nonlinearity. The sine- Gordon equation is a typical example of such system. The generalized NSE with a saturated nonlinearity has a form [1,4]:

$$i\partial\psi/\partial t+\partial^2\psi/\partial x^2-\omega_0\psi+g\mid\psi\mid^2\psi-\delta\mid\psi\mid^4\psi=0. \tag{4}$$

In this equation nonlinear terms contain only the field variable ψ but not its derivatives. The soliton solution of this equation has a form of a function with a plateous and corresponds to the Bose-drop of condensed matter. The 1D Bose-gas of particles with a two-bodies δ- function attraction and three-particles point repulsion is a quantum analog of classical nonlinear system (4) [2].

But Eq.(4) is not the unique generalization of Eq(1). The well-known Landau-Lifshitz equation for a ferromagnet with the easy-axis on-site anisotropy is another example of such generalization [5]:

$$i\partial\psi/\partial t-l^2\omega_0\left(m_z\partial^2\psi/\partial x^2-\psi\partial^2 m_z/\partial x^2\right)+\omega_0\psi m_z=0, \tag{5}$$

where $\psi=m_x+im_y, m_z=\sqrt{1-\mid\psi\mid^2}, l=\sqrt{\alpha/\beta}$ is a magnetic length, $\omega_0=2\mu_0 M_0\beta/\hbar$ is a frequency of homogeneous ferromagnetic resonance, α is an exchange constante and β is the constante of anysotropy. In small amplitude limit Eq.(5) can be rewritten as

$$\begin{gathered}i\partial\psi/\partial t-l^2\omega_0\partial^2\psi/\partial x^2+\omega_0\psi-\omega_0\mid\psi\mid^2\psi/2-\omega_0\mid\psi\mid^4\psi/8\\+l^2\omega_0/2\left[\mid\psi\mid^2\partial^2\psi/\partial x^2-\psi\left(\partial^2\mid\psi\mid^2/\partial x^2\right)\right]=0.\end{gathered} \tag{6}$$

This equation differs from Eq.(4) by the last two terms which are cubic in variaable ψ, but contain two spatial derivatives. In small- amplitude soliton the amplitude ψ and the spatial derivative $\partial/\partial x$ are of order of small parameter ϵ and additional terms have the same smallness as saturated nonlinearity $\mid\psi\mid^4\psi$. (An axistence of these terms is connected with the fact that magnons are not the exact Bose-particles and their interaction is more complicated that point interaction).

Of course, the addition terms $\mid\psi\mid^4\psi,\mid\psi\mid^2\partial^2\psi/\partial x^2,\psi\partial^2\mid\psi\mid^2/\partial x^2$ are much smaller that the main nonlinear term $\mid\psi\mid^2\psi$. But situation is quite different in ferromagnets with a strong exchange anisotropy. If the symmetry of on-site and exchange anisotropies is the same (for example, easy-axis anisotropy) the energy density has a form [5]:

$$W=(\alpha/2)\left(\partial\mathbf{M}/\partial x\right)^2+(\alpha_1/2)\left(\partial M/\partial x\right)^2-(\beta/2)\,M_z, \tag{7}$$

where $\mathbf{M}=M_0\mathbf{m}$ and $\mathbf{m}$ is the unit vector. The corresponding dynamical equation has a form [5]:

$$i\partial\psi/\partial t-l^2\omega_0\left[m_z\partial^2\psi/\partial x^2-\left(1+\frac{\alpha_1}{\alpha}\right)\psi\partial^2 m_z/\partial x^2\right]+\omega_0\psi m_z=0, \tag{8}$$

and in some limiting cases (for some values of $\frac{\alpha_1}{\alpha}$) the above-mentioned terms are of great importance. In the limit of Ising magnet ($\alpha=0$) and classical

XY-magnet ($\alpha_1 = -\alpha$) the differential part of Eq.(8) reduces to the expressions $L^2\omega_0\psi\partial^2 m_z/\partial x^2$ and $-l^2\omega_0 m_z\partial^2\psi/\partial x^2$ correspondingly ($L^2 = \alpha_1/\beta$). In the Ising limit the exchange part of equation does not contain the linear terms and in linear approximation the dispersive terms are absent. This fact leads to essential modification of soliton solution and arising of new type of solitons (*exotic solitons*) in which the deviation of magnetization from the equilibrium state is exactly nonzero only in finite domain of space. Such solitons were investigated previously by P.Rosenau [6,7] for the equations of KdV-type and by Yu.Kosevich [8,9] for nonlinear elastic chains and received the name *"compactons"*. Later Holm and Camassa investigated another type of exotic solitons-*"peakons"*.

The aim of this article is to investigate magnetic compactons and peakons in anisotropic ferromagnets and to study the linkage between exotic solitons and bounded states of bosons in Bose-gas of interacting particles.

2 Generalized model of Bose-gas of interacting particls

We shall investigate 1D Bose-gas with two-particle but more complicated than in (3) potential of interparticle interaction to understand the quantum-mechanical nature of nonlinear dispersion terms in Eqs.(5,6,8). Let us choose the potential function U_{ij} in the following form:

$$U_{ij} = 2V\delta\left(x_i - x_j\right) - 2U\left[\delta\left(x_i - x_j + a\right) + \delta\left(x_i - x_j - a\right)\right]. \tag{9}$$

We consider that the integral interaction is attractive, i.e. $2U - V = W > 0$. Then there are two different cases with $U, V > 0$ (A) and $U, V < 0$ (B). The equation (3) with the potential (9) is completely integrable, but in the limit $a^2UW \ll 1$ we can restrict ourselves to the simple potential

$$U_{ij} = -2W\delta\left(x_i - x_j\right) - 2Ua^2\frac{\partial^2}{\partial x^2}\delta\left(x_i - x_j\right), \tag{10}$$

which differes from zero only on the system of planes $x_i < x_j$ in N - dimensional coordinate space $(x_1, x_2, ...x_N)$. Using the evident boundary conditions on these planes we can find the Bethe-type wave function $\Phi = \exp\left(-\sum_{i=1}^{N} k_i x_i\right)$ (for sector $-\infty < x_N \leq x_{N-1} \leq ... \leq x_2 \leq x_1 < \infty$) with the set of parameters $\{k_i\}$,which form the "strings" $k_i - k_{i-1} = W$ and $k_i - k_{i-1} = 1/a^2U$ in the case A and string $k_i - k_{i-1} = W$ in the case B. The same result is valid for potential (9) as well.

In Hartree approximation ($\Phi = \prod_i \varphi\left(x_i\right)$) in the case of potential (10) for one-particle normalized ($\int \mid \varphi \mid dx = 1$) wave-function $\varphi(x_i)$ for stationary states we have the following nonlinear equation:

$$\frac{d^2\varphi}{dx^2} - \epsilon\varphi + 2\,(N-1)\,W\varphi^3 + 2\,(N-1)\,Ua^2\varphi\frac{d^2}{dx^2}\varphi^2 = 0, \tag{11}$$

where the Hartree energy $\epsilon = \epsilon\,(N)$ is equal to

$$\epsilon = \int dx \left[-\left(\frac{d\varphi}{dx}\right)^2 + 2\,(N-1)\,W\varphi^4 - 2\,(N-1)\,Ua^2\varphi^2\left(\frac{d\varphi}{dx}\right)^2 \right]. \tag{12}$$

We see that Hartree equation (11) includes the nonlinear term of the form $\varphi\left(\frac{d^2}{dx^2}\varphi^2\right)$ as well as Eqs.(6,8) for the dynamics of magnetization.

3 Compactons and peakons in Bose-gas.

For small amplitude solitons in long-wave approximation ($d/dx \ll 1$) the last term in Eq.(11) is small and it is naturally to leave it only for large value of parameter Ua^2/W. As we supposed above that $WUa^2 \ll 1$, we have the system of inequalities: $W^2 \ll WUa^2 \ll 1$.

The soliton solution of Eq.(11) exists only for positive values of Hartree energy. After the first integration this equation reduces to

$$\left(\frac{d\varphi}{dx}\right)^2 = \varphi^2\frac{\epsilon - NW\varphi^2}{1 + 4NUa^2\varphi^2}. \tag{13}$$

In the case (A) the soliton solution of Eq.(13) in unevident form is

$$\frac{1}{\sqrt{A}}\ln\frac{\sqrt{A}-f}{\sqrt{A}+f} - \arcsin\frac{f^2-1}{f^2+1} = 2x\sqrt{\epsilon_\star}, \tag{14}$$

where A is proportional to the Hartree energy $A = \epsilon/\epsilon_\star$, where $\epsilon_\star = W/4Ua^2$ is the characteristic value of this energy, and $f = \left(1/\sqrt{\epsilon_\star}\right)\frac{d}{dx}\ln\varphi$. From Eq.(13) and normalization condition we can find the dependence $N = N\,(\epsilon)$:

$$\frac{N}{N_\star} = \sqrt{\frac{\epsilon}{\epsilon_\star}} + \left(1 + \frac{\epsilon}{\epsilon_\star}\right)\arcsin\sqrt{\frac{\epsilon}{\epsilon+\epsilon_\star}}, \tag{15}$$

where $N_\star = \sqrt{\frac{1}{4WUa^2}}$ is the characteristic number of bounded in soliton bosons. It is evident from (15) that $(\epsilon/\epsilon_\star) \simeq (N/2N_\star)^2$ for $N \ll N_\star$ and $(\epsilon/\epsilon_\star) \simeq (2N/\pi N_\star)$for $N \gg N_\star$.

The amplitude of soliton is $\varphi_{\max} = \sqrt{\epsilon/NW}$ and asymptotes on infinity have the form $\varphi \sim \exp\left(-x\sqrt{\epsilon}\right)$. For small N soliton has a standard form of small-amplitude soliton. But in the limit $N \gg N_\star$ the amlitude $\varphi^2_{\max}$ tends to limited value $\sqrt{W/\pi Ua^2}$ and the soliton localizes in the region $\Delta = 2x_0 = 2\sqrt{\pi^2 Ua^2/W}$. For $N \to \infty$ we have

$$\varphi = \left(\tfrac{W}{\pi U a^2}\right)^{1/4} \cos\left(\sqrt{\tfrac{W}{4Ua^2}}x\right), for \mid x \mid \leq x_0,$$
$$\varphi = 0, for \mid x \mid > x_0. \tag{16}$$

This is the typical form of compacton solution. Of course, the density of bosons $\rho = N\varphi^2$ increases to infinity in this compacton when $N \to \infty$, but the collapse differes from those in Bose- gas with usual δ- function interaction and the size of soliton remaines finite.

Note that in compacton limit solutions satisfy the following equation:

$$\varphi^2 \left[\left(\frac{d\varphi}{dx}\right)^2 + \epsilon_\star \varphi^2 - 2\epsilon_\star^{3/2}/\pi\right] = 0. \tag{17}$$

In the case B with $U < 0$ the soliton solution of Eq.(13) can be found exactly as well:

$$\frac{1}{\sqrt{A}} \ln \frac{f - \sqrt{A}}{f + \sqrt{A}} - \ln\left[\left(1 - 4N \mid U \mid a^2\varphi^2\right)(1-f)^2\right] = 2x\sqrt{\epsilon_\star}, \tag{18}$$

where we must replaces $U \to \mid U \mid$ in $\epsilon_\star, A,\ f$ and $N_\star$.The relation between the Hartree energy and the number of bosons is now somewhat different from (15):

$$\frac{N}{N_\star} = \sqrt{\frac{\epsilon}{\epsilon_\star}} + \left(1 - \frac{\epsilon}{\epsilon_\star}\right) \ln \frac{\sqrt{\epsilon_\star} + \sqrt{\epsilon}}{\sqrt{\epsilon_\star - \epsilon}}, \tag{19}$$

where $0 < \epsilon < \epsilon_\star$ and $0 < N < N_\star$. For small values of N the amplitude of soliton $\varphi_{\max}$ grows ($\varphi_{\max} = WN/4$) and peaks at $N = N_\star$, where $\varphi^2_{\max} = \sqrt{\frac{W}{4|U|a^2}} = \sqrt{\epsilon_\star}$ and amplitude of soliton has the same order as in previous case of compacton with $N \to \infty$.In this limit Eq.(13) reduces to the form:

$$\left(\sqrt{\epsilon_\star} - \varphi^2\right) \left[\left(\frac{d\varphi}{dx}\right)^2 - \epsilon_\star \varphi^2\right] = 0. \tag{20}$$

The soliton solution of this equation

$$\varphi = \epsilon_\star^{1/4} \exp\left(-\sqrt{\epsilon_\star} x\right), x > 0,$$
$$\varphi = \epsilon_\star^{1/4} \exp\left(\sqrt{\epsilon_\star} x\right), x < 0 \tag{21}$$

has a form of peakon. This solution is expressed in terms of exponential functions, as in the case of linear equations, but the amplitude of peakon solution is nonzero and fixed.

It is important that the functions (16) and (21) determine a field distribution in solitons (compactons and peakons) only in some definite intervals. In the sewpoints there are the jumps of derivatives, but we must not intriduce δ-

functions in these points because of the first factors in Eqs.(17,20). It is connected with the fact that the solutions (16,21) for compactons and peakons are the limiting expressions for analytical functions (14,18).

4 Solitons in ferromagnets with anisotropic exchange interaction.

Let us turn to the investigation of uni-axis ferromagnets with the exchainge anisotropy and with the same symmetry of exchange and uni-ion anisotropies, i.e. to the dynamical equation (8). After the first integration in the simplist case of immovable soliton of the type $\psi = \sin\theta\,(x)\exp\,(-i\omega t)$ we have

$$\left(l^2 + L^2 \sin^2\theta\right)\left(\frac{d\theta}{dx}\right)^2 = 4\sin^2\left(\frac{\theta}{2}\right)\left[\cos^2\left(\frac{\theta}{2}\right) - \frac{\omega}{\omega_0}\right]. \tag{22}$$

For weak anisotropy $\alpha_1/\alpha \ll 1$ small amplitude solitons have the same structure as in the case of isotropic exchange. But in the case of strong exchange anisotropy the situation is quite different. It is convenient to use the new dynamical variable $\varphi = 1 - \cos\theta$, which describes the density of spin deviations (magnon density). In the terms of this variable Eq.(22) is rewritten as follows:

$$L^2\left(\frac{d\varphi}{dx}\right)^2 = \varphi^2\frac{(2-\varphi)\left[2\,(1-\omega/\omega_0)-\varphi\right]}{(l/L)^2 + 2\varphi - \varphi^2}. \tag{23}$$

For small deviation of frequency from the frequency of homogeneous ferromagnet resonance ($1 - \omega/\omega_0 \ll 1$) and for small difference from Ising limit ($l/L \ll 1$) this equation is very close to Eq.(13) for many-boson bounded states:

$$L^2\left(\frac{d\varphi}{dx}\right)^2 = \varphi^2\frac{2\,(1-\omega/\omega_0)-\varphi}{l^2/2L^2+\varphi}. \tag{24}$$

In the Ising limit ($l \to 0$) Eq.(22) has a compacton solution [10]:

$$\begin{gathered}\cos\theta = \tfrac{\omega}{\omega_0} - \left(1 - \tfrac{\omega}{\omega_0}\right)\cos\left(\tfrac{x}{L}\right), for\ |\, x\,| \le \pi L,\\ \cos\theta = 1, for\ |\, x\,| > \pi L.\end{gathered} \tag{25}$$

It is the magnetic analog of compact bounded state of bosons (16).

The dynamical equation of easy-axis ferromagnet in Ising limit has very simple form in terms of ψ and m_z:

$$i\frac{1}{\omega_0}\partial\psi/\partial t + L^2\psi\partial^2 m_z/\partial x^2 + \psi m_z = 0. \tag{26}$$

Note that this equation is dispersionless in linear approximation, i.e. the dispersion of linear excitations $\left(\partial^2\omega/\partial k^2\right)_{\psi\to 0}$ is zero for the dispersion relation of magnons $\omega = \omega(k)$, where ω and k are the frequancy and wave number of magnon. This is the jeneral property of all systems with compacton solutions.

For dynamical compacton Eq.(26) can be rewritten in the form

$$\left(1-m_z^2\right)\left[L^2\left(\frac{dm_z}{dx}\right)^2+\left(m_z-\frac{\omega}{\omega_0}\right)^2-\left(1-\frac{\omega}{\omega_0}\right)^2\right]=0, \qquad (27)$$

similar to Eq.(17), and for domain wall of compacton type Eq.(26) may be reduced to the same form for the variable ψ:

$$\psi^2\left[L^2\left(d\psi/dx\right)^2+\psi^2-1\right]=0. \qquad (28)$$

Compact domain wall has an evident form[10]:

$$m_z = 1, x > \pi/2,$$

$$m_z = \sin\left(x/L\right), \mid x \mid < \pi/2, \qquad (29)$$

$$m_z = -1, x < -\pi/2.$$

In easy-axis Ising-type ferromagnets both compact dynamical solitons and compact domain walls can not move.

In the opposite case of Ising ferromagnets with on-site anisotropy of easy-plane type the dynamical equation (26) changes in the following way:

$$i\frac{1}{\omega_0}\partial\psi/\partial t + L^2\partial^2 m_z/\partial x^2 - \psi m_z = 0. \qquad (30)$$

where $L=\sqrt{\frac{\alpha_1}{|\beta|}}$. In this system we have degenerated ground state with $m_z = 0$.

In easy-plane case there are no dynamical exotic solitons but the analog of stationary rotary wave solution [5] has an exotic form in this limit. For stationary rotary wave one can rewrite Eq.(30) in the form quite similar to equation (20) for peakons in Bose-gas:

$$\left(1-m_z^2\right)\left[L^2\left(dm_z/dx\right)^2-m_z^2\right]=0. \qquad (31)$$

Magnetic peakon-like soliton has the trivial profile:

$$m_z = \exp\left(- \mid x \mid /L\right), \qquad (32)$$

the same as peakon (21) in Bose-gas.

In the limit of classical XY-ferromagnet ($\alpha_1 = -\alpha, L = 0$)we have the following dynamical equations for a magnetization instead of Eqs.(26,30):

$$i\frac{1}{\omega_0}\partial\psi/\partial t - l^2 m_z\partial^2\psi/\partial x^2 + m_z\psi = 0, (easy-axis), \qquad (33)$$

$$i\frac{1}{\omega_0}\partial\psi/\partial t - l^2 m_z \partial^2\psi/\partial x^2 - m_z\psi = 0, (easy - plane)\,. \tag{34}$$

In easy-axis XY-ferromagnet there are the peakon-like domain walls and dynamic solitons. In the case of domain wall the profile of the field variable ψ has a peakon form: $\psi = \exp\left(- \mid x \mid /l\right)$, and z-component of magnetization has a usual form of kink. But in the center of this domain wall there is the singularity of the spatial derivative: $dm_z/dx \mid_{x=0} = \infty$. The dynamical soliton of the simple type $\psi = \psi\left(x\right)\exp\left(i\omega t\right)$ has a form of usual soliton in the frequency interval $1/2 < \omega/\omega_0 < 1$, but the soliton with small frequency $0 < \omega/\omega_0 < 1/2$ has the exotic structure: it represents the bounded state of a pair of peakons of ψ-field.

In easy-plane XY-ferromagnet there are compacton-type stationary rotary waves of magnetization.The equation for ψ is very similar to Eq.(28) :

$$\left(1-\psi^2\right)\left[l^2\left(d\psi/dx\right)^2 + \psi^2 - 1\right] = 0, \tag{35}$$

and soliton solution has a form (29) but for the variable ψ instead of m_z.

So there is the symmetry in existance of compactons and peakons in Ising -type and XY-type ferromagnets.

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