RATIONAL APPROXIMATION METHODS FOR LARGE AMPLITUDE SOLITARY WAVES ON NONLINEAR ELECTRICAL LATTICES.

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ABSTRACT. In this paper we describe an approximate method to characterise solitary wave solutions of nonlinear lattice equations. It is based upon one and two point Padé approximations to a series of the real exponential traveling wave solutions of the underlying dispersive system.
The theory is applied to an example of a lattice system which models an experimental nonlinear transmission line and the results obtained are consistent with numerical simulations even for relatively large amplitude solitary waves. The speed–amplitude relation is investigated and compared with that derived using quasi-continuum methods and amplitude versus half–width graphs are also presented. An extended version of most of this work has been submitted to Physica D.

1. Introduction

The study of solitons on discrete lattices dates back to the early days of soliton theory. Toda [1] demonstrated that a lattice of particles with nearest neighbour interactions corresponding to a potential of exponential form, supported exact multi-soliton solutions. Furthermore the Toda lattice is a completely integrable nonlinear differential difference equation solvable by the inverse scattering transform [2] and also possesses periodic solutions. There are a few other examples of integrable lattices [3–6] but most lattice systems are not solvable explicitly even though they may support solitary waves. Traditionally, nonlinear transmission lines (NLTL's) have been used to study the properties of solitons on lattices, including pulse interactions and collisions [7–9]. Usually, only small amplitude pulse interactions were studied because then various continuum approximations to the nonlinear lattice equations yielded well known equations such as the Korteweg-de-Vries (KdV) and Boussinesq equations, which have been comprehensively studied, and direct comparisons were then possible. Modulational instability of envelope solitons and wave packets on NLTL's have also been investigated [10].

Many numerical studies of one dimensional nonlinear lattices have been made and it is has been shown [11, 12] that narrow soliton-like states can propagate stably with interaction potential other than the Toda potential. Indeed in the continuum limit low order polynomial potentials can be related to generalised-Boussinesq equations for which solitary wave solutions exist. In recent work [13,14] analytic approximations to solitary waves on lattices have been derived using generalisations of the continuum approximation based around various Padé expansions of the difference operator involved. These approximations may be used to verify numerical simulations and to study properties of these solitary waves on non-integrable lattices. For instance for many potentials, at lowest order, amplitude-speed relations may be derived, but the method has the disadvantage that to obtain the wave profile often an implicit equation has to be solved, which can only be done numerically. Also, it would appear to be difficult to solve the equations arising from higher order approximations.

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When nonlinear lattices are studied from the engineering viewpoint as pulse compression devices, relatively large amplitude pulses are used [15] and the continuum methods previously employed yield equations which may no longer accurately model the physical situation. Importantly, nonlinear transmission lines now have a range of applications, including their use in sampling oscilloscopes and network analysers and they can also be manufactured in monolithic form [15]. We are interested in such devices for pulse compression in radar systems and in that case we will need to solve the lattice equations for large amplitude travelling waves.

Our approach is based upon the direct algebraic method for obtaining exact solitary wave solutions of nonlinear evolution and wave equations [16]. It involves summing a series of the real exponential traveling wave solutions of the underlying dispersive system. This method can be used to generate multiple soliton solutions by expanding the solution of the nonlinear wave equation simultaneously in several of these traveling wave solutions. The procedure can be used for integrable lattices to obtain soliton and kink solutions [17–19]. For example, in [17] bright and dark solitary wave solutions of a discrete modified KdV (DMKdV) equation are found and in [19] a version of the Toda lattice is treated and a solitary wave solution generated. For the Toda lattice the series obtained when expanding in powers of a single exponent is a geometric progression which may be summed to give the exact one-soliton solution [20].

Non-integrable lattices may be treated in the same way and again one can expand in a power series of an exponential traveling wave. In general this series will not be summable to an exact analytic form. Our method is then to use Padé approximants to obtain an approximate analytical solution. For the Toda lattice, the lowest approximant gives the one-soliton solution exactly, while in the case of a general lattice, the traveling wave solution will be built up from such solutions. Best results are obtained by using *two point* Padé approximants [21] where power series expansions of a given function are matched about two points (usually taken to be origin and infinity).

The layout of this paper is as follows. In section 2 we derive the basic (lossless) circuit equation governing the nonlinear transmission line. We then turn our attention to the case of a practical NLTL fabricated in the laboratory and we present our method of obtaining approximate solitary wave solutions to the circuit equations based around Padé approximants and a direct algebraic procedure. In section 3 we use the quasi-continuum method of Wattis [13] and Duncan *etal.* [14] to derive speed–amplitude relations for solitary wave solutions of the non–integrable equations. In section 4 we compare the results obtained for our transmission line using the methods described in sections 2 and 3 with numerical simulations. In addition, we show how the width of the pulse varies with amplitude. In the final section we summarise our conclusions from this work and suggest how it may be extended.

2. A Nonlinear Electrical Lattice, Padé Approximation and Solitary Wave Solutions

2.1. The Basic Circuit. A transmission line is usually defined to be an electric cable or waveguide that conveys electrical signals from one point to another in a telecommunications system and which forms a continuous path between the two points. Hirota and Suzuki (1973) [7] examined a transmission line consisting of periodic sections of an inductor L and a diode D joined in series (see figure 1). For a linear transmission line the diode is replaced by a capacitor C. However, in an NLTL the diode is usually a reverse-biased varactor diode which varies its capacitance when the voltage applied across its terminals is changed.

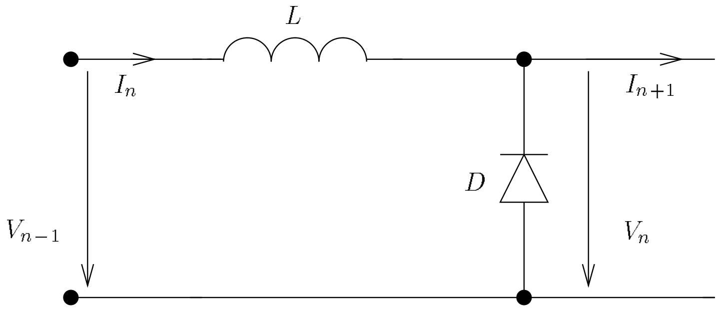

FIGURE 1. Representation of a typical section of a nonlinear transmission line.

In order to derive an equation governing the circuit we apply Kirchoff's Laws to the network shown in figure 1 and find

$$L\frac{\partial I_n}{\partial t} = V_{n-1} - V_n, \tag{2.1}$$

$$\frac{\partial Q_n}{\partial t} = I_n - I_{n+1}, \tag{2.2}$$

where V_n is the voltage, I_n the current and Q_n is the charge at the lattice point n. The charge Q_n is given by

$$Q_n = \int_0^{V_n} C(V)\,\mathrm{d}V \tag{2.3}$$

Eliminating I_n and I_{n+1} using (2.1) in (2.2) yields the circuit equation

$$L\frac{\partial^2 Q_n(V_n(t))}{\partial t^2} = V_{n+1}(t) + V_{n-1}(t) - 2V_n(t). \tag{2.4}$$

The usual method of obtaining soliton solutions is through the inverse scattering transform. However, this method cannot be used for the non-integrable lattices which we shall be studying. For our purposes a direct algebraic method, which has been applied to integrable lattices by Dash and Panigrahi [19], Xiao [17] and Xiao *et al.* [18], is more appropriate. Our ultimate aim here is to describe high voltage pulse solutions on nonlinear electrical lattices avoiding small amplitude approximations.

2.2. An Experimental Nonlinear Transmission Line. As we have indicated in the introduction there are various nonlinear lattices that have been shown to support the propagation of solitary waves. As can be seen from equation (2.4) the nonlinearity of the lattice equation depends critically on the nonlinearity of the diode.

In this section we model the circuit shown in figure 1 and use a commercially available diode, in this case a KV1235 varicap (varactor) diode. Figure 2 shows the measured capacitance (in picofarads (10^{-12}F)) of two of these reverse–biased diodes in parallel, against the applied voltage. The dots denote the measured values. Also shown are three least-squares fits of the data. The thin solid line is the function

$$C(V) = 36.5772 + 1715.54\exp\left(-\frac{10V}{21}\right), \tag{2.5}$$

the thin dotted line and dashed line the function

$$C(V) = \left[c_1 + \frac{c_0}{\left(1+\frac{V}{\gamma}\right)^2}\right], \tag{2.6}$$

with $c_1 = -76.3626pF$, $c_0 = 2517.1pF$, $\gamma = \frac{5}{2}V$ and $c_1 = 7.3684pF$, $c_0 = 22338.5pF$, $\gamma = \frac{1}{2}V$, respectively.

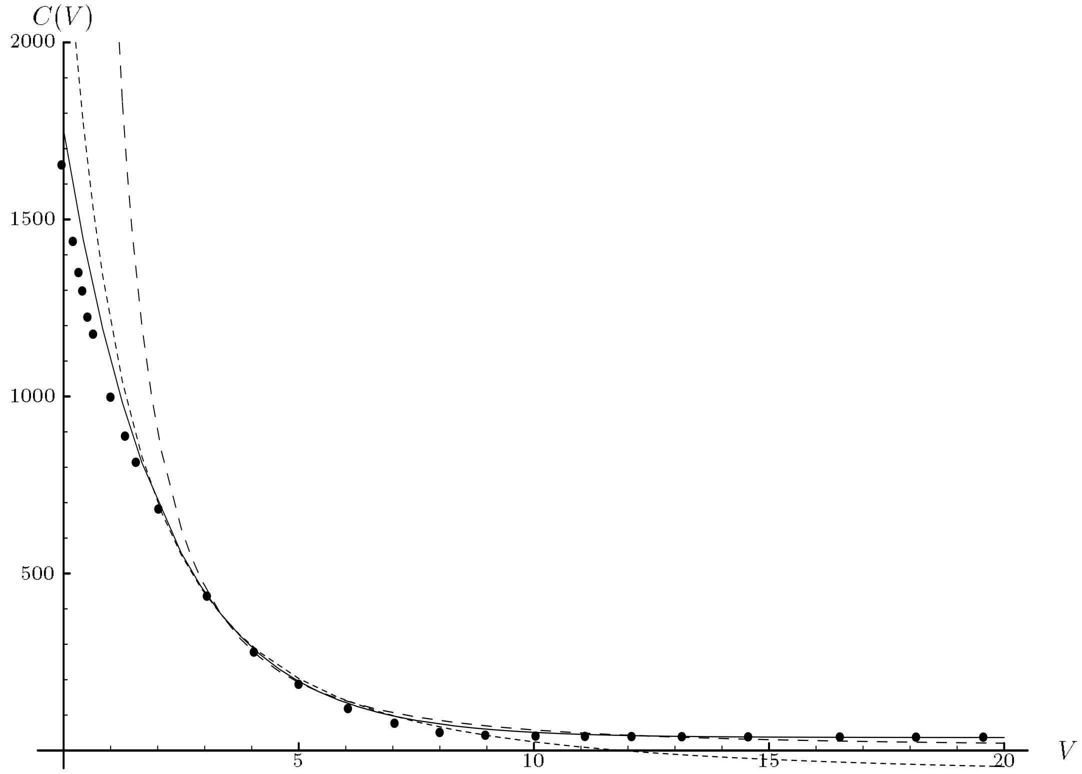

FIGURE 2. Various capacitance (in pF) versus voltage (in volts) models depicting the measured diode capacitance (dots), Exponential model: (solid line), $C(V) = 36.5772 + 1715.54 \exp(-10V/21)$ and Rational models $C(V) = -76.3626 + \frac{2517.1}{(1+V/2.5)^2}$ (dotted line) and $C(V) = 7.3684 + \frac{22338.5}{(1+2V)^2}$ (dashed line).

It is observed that the exponential model (2.5) agrees well with the measured capacitance over a wide range of voltage. The rational model (2.6) is also a good approximation and leads to a simpler analysis, and in the remainder of this paper we concentrate our attention on the latter model in the case when $c_1 > 0$; the case of $c_1 < 0$ is discussed in [20]. It is seen that with $C(V)$ given by (2.6) then

$$Q(V_n) = c_0\gamma + c_1 V_n - \frac{c_0\gamma^2}{\gamma + V_n}, \tag{2.7}$$

and the circuit equation (2.4) becomes

$$Lc_0 \frac{\partial^2}{\partial t^2}\left[c_3 V_n - \frac{\gamma^2}{\gamma + V_n}\right] = V_{n+1} + V_{n-1} - 2V_n, \tag{2.8}$$

where $c_3 = \frac{c_1}{c_0}$ and $L = 220$ nano-henries ($10^{-9}H$). We now make the transformations

$$t \equiv \sqrt{Lc_0}\,\tau, \tag{2.9}$$

$$\frac{\partial \phi_n}{\partial \tau} \equiv \dot{\phi}_n \equiv V_n + \gamma, \tag{2.10}$$

to arrive after one integration at the equation,

$$\ddot{\phi}_n\left[c_3 + \frac{\gamma^2}{\dot{\phi}_n^2}\right] = \phi_{n+1} + \phi_{n-1} - 2\phi_n + \theta. \tag{2.11}$$

In a practical NLTL the diodes are *biased* by a constant voltage source in order to keep them in a non–conducting state, i.e. one in which no direct current is flowing. Therefore any voltage pulse on the line will tend to V_0 as time increases, where V_0 is the value of the bias voltage, so $\phi_n \to (V_0 + \gamma)\tau$ and hence in (2.11) we must take $\theta = 0$.

The procedure now is to look for traveling wave solutions of (2.11) (with $\theta = 0$) in the form of a series expansion. We try

$$\phi_n = a_0\tau + A_0 + \sum_{j=1}^{\infty} A_j g_n^j, \qquad 0 < g_n < 1, \tag{2.12}$$

$$\phi_n = a_0\tau - \sum_{j=1}^{\infty} A_{-j} g_n^{-j}, \qquad g_n > 1, \tag{2.13}$$

where

$$g_n = \exp\left(-\xi_n\right), \quad \xi_n = \kappa n - \omega\tau, \quad \kappa > 0, \tag{2.14}$$

and (2.12) corresponds to an expansion about $g_n = 0$ whilst (2.13) corresponds to expansion about $g_n = \infty$. Here, $a_0 = V_0 + \gamma$ and the A_0 term in (2.12) is necessary to match the values of the two expansions for ϕ_n at $g_n = 1$. Although A_0 will not appear in the resulting expression for V_n when it is computed from either (2.12) or (2.13), it plays a significant role in the construction of two point Padé approximants where the two series are used simultaneously and which will be described presently.

Substituting (2.12) and (2.13) into (2.11) and regrouping terms yields

$$\left[\sum_{j=1}^{\infty}\left[c_3\omega^2 j^2 - 4\sinh^2\left(\frac{\kappa j}{2}\right)\right] A_{\pm j} g_n^{\pm j}\right]\left[a_0^2 + 2\omega a_0 \sum_{j=1}^{\infty} j A_{\pm j} g_n^{\pm j} + \omega^2\left(\sum_{j=1}^{\infty} j A_{\pm j} g_n^{\pm j}\right)^2\right]$$
$$+ \gamma^2\omega^2 \sum_{j=1}^{\infty} j^2 A_{\pm j} g_n^{\pm j} = 0, \tag{2.15}$$

where care must be taken to preserve the order of the $\pm$ signs. We now read off the coefficients of $g_n^{\pm j}$. For $j = 1$ the dispersion relation is recovered,

$$\omega^2 = \frac{4a_0^2 \sinh^2\left(\frac{\kappa}{2}\right)}{\left(c_3 a_0^2 + \gamma^2\right)} = \frac{4c_0 \sinh^2\left(\frac{\kappa}{2}\right)}{C(V_0)}, \tag{2.16}$$

where $C(V_0)$ is found from (2.6). Thus the frequency ω depends crucially on the value of the capacitance at the bias voltage. In the remainder of this paper we take $V_0 = 3$ volts. The coefficients $A_{\pm j}$, $j = 2, 3, 4, \ldots$, can be determined in terms of the arbitrary coefficients $A_{\pm 1}$. It is found that

$$A_{\pm 2} = -\frac{\omega\gamma^2 A_{\pm 1}^2}{2a_0\left(c_3 a_0^2 + \gamma^2\right)\sinh^2\left(\frac{\kappa}{2}\right)}, \tag{2.17}$$

and in general $A_{\pm N}$ is complicated but has the property

$$A_{\pm N} = A_{\pm 1}^N B_{\pm N}, \qquad N = 2, 3, 4, \ldots, \tag{2.18}$$

where $B_{\pm N}$ is a function of ω, a_0, N, κ, c_3 and γ. In contrast to the Toda lattice case [20], it is not possible to obtain a simple expression for the ratio of consecutive terms $A_{\pm j}/A_{\pm(j-1)}$, $j = 2, 3, \ldots$, therefore we cannot sum equations (2.12) and (2.13) exactly to obtain an expression for a traveling wave solution. Our approach is to utilise the Padé method to derive approximate analytical expressions for the solution ϕ_n of (2.11) and hence the voltage $V_n(t)$, with the knowledge that the method is exact in the case of the Toda lattice.

2.3. Simple Padé Approximation. The easiest method to derive an explicit expression for a traveling wave solution of (2.11) is to approximate the infinite series in (2.12) and (2.13). The (M, N) Padé approximant $f_{M/N}(X)$ to a function $f(X) = \sum_{j=0}^{\infty} f_j X^j$ is a rational function consisting of a polynomial in X of degree M divided by a monic polynomial of degree N, whose coefficients are determined by the requirement that

$$f_{M/N}(X) - f(X) = O\left(X^{N+M+1}\right) \tag{2.19}$$

To obtain an approximate form for ϕ_n, we take a factor g_n out of the series in (2.12) and similarly a factor g_n^{-1} out of the series (2.13) and construct Padé approximants to the series which remain. The $(0, 1)$ approximant to ϕ_n is given by

$$\phi_n \approx a_0\tau \pm \frac{A_{\pm 1} g_n^{\pm 1}}{1 - \alpha_{\pm 1} g_n^{\pm 1}}, \tag{2.20}$$

where $\alpha_{\pm 1} = \frac{A_{\pm 2}}{A_{\pm 1}}$. Thus the approximate voltage at the nth section of the lattice is given by

$$V_n(\tau) \approx a_0 - \gamma - \frac{\omega A_{\pm 1}^2}{4A_{\pm 2}} \operatorname{sech}^2\left[\frac{1}{2}(\kappa n - \omega\tau) - \delta_{\pm}\right], \tag{2.21}$$

where $\delta_{\pm} = \pm\frac{1}{2}\log\left(\frac{A_{\pm 2}}{A_{\pm 1}}\right)$. It is clear that g_n and g_n^{-1} can be scaled by appropriate factors so that $A_1 = A_{-1}$ and it turns out that $A_j = A_{-j}$ for all $j \geq 1$. The amplitudes of the two solutions represented in (2.21) are then equal and the phases match only if $\delta_+ = \delta_- = 0$ implying that $A_2 = A_1$. Using equations (2.16) and (2.17) A_1 is then found to be

$$A_1 = -\frac{\left(c_3 a_0^2 + \gamma^2\right)^{3/2} \sinh\left(\frac{\kappa}{2}\right)}{\gamma^2}. \tag{2.22}$$

Hence the voltage $V_n(t)$ is approximated by

$$V_n(t) \approx V_0 + \frac{C(V_0)\sinh^2\widetilde{\kappa}}{2(\gamma + V_0)c_0\gamma^2}\operatorname{sech}^2\left[\widetilde{\kappa}n - \frac{(\sinh\widetilde{\kappa})\, t}{\sqrt{LC(V_0)}}\right] \tag{2.23}$$

where $\widetilde{\kappa} = \frac{\kappa}{2}$.

Expressions for the width, amplitude and velocity of the pulses can be found from equation (2.23) (see [20]). Higher order approximants to $V_n(t)$ are of the form

$$V_n(t) \approx V_0 \pm \frac{\partial}{\partial\tau}\left[f_{M/N}(g_n^{\pm 1})\right]\Big|_{\tau = \frac{t}{\sqrt{Lc_0}}}, \tag{2.24}$$

where $f_{M/N}$ is an (M, N) Padé approximant to the series for $\phi_n(\tau)$.

In figure 3 we show a plot of the approximants to $V_n(t)$ given by (2.24) with the positive sign taken, which means that we have expanded in a series about $t = -\infty$. We have set $\kappa = 2.451$, $n = 1$ and for each approximant A_1 is found from (2.22). Recall that changing A_1 only affects the phase (position) of each solution. The approximants follow one another closely as t increases from $-\infty$, until near the point where the first maximum is reached, at say $t = t_{\max}$. After this the (1,2), (2,3) and (3,4) approximants diverge from one another with the latter two indicating the formation of extra humps. A similar behaviour was observed in [22] for large amplitude solitary waves where high order Padé approximants to perturbation series were plotted in the case of a lattice with interaction potential of Lennard–Jones type. Of course figure 3 only shows the solution when the series (2.12) is approximated. The plots of the solution when the series (2.13) is used to calculate $\phi_n(t)$ (and hence $V_n(t)$) is essentially a mirror image of figure 4, with the reflection being about the maximum value of the (0,1) Padé solution (equation (2.23)) which is the same for both expansions.

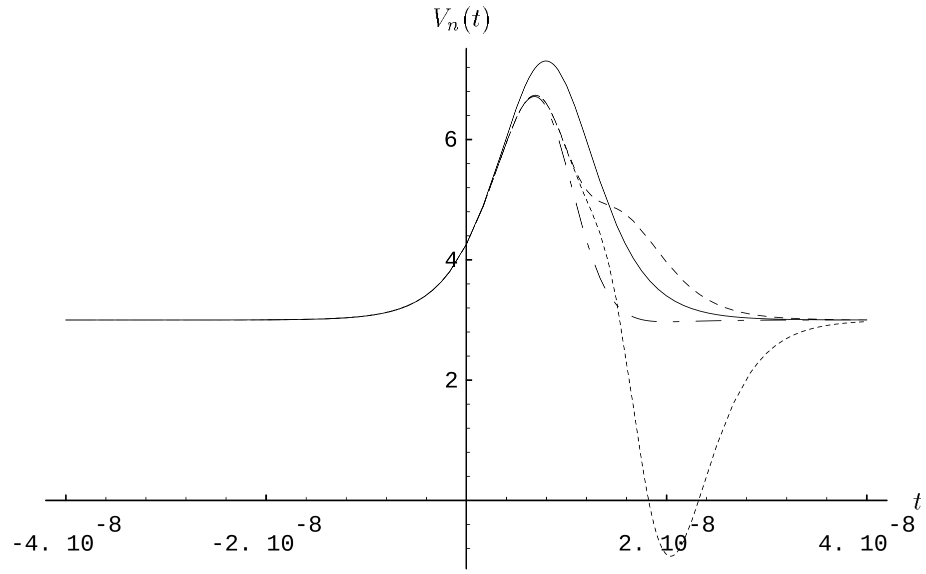

FIGURE 3. Approximants to $V_n(t)$ given by (2.24) corresponding to a series expansion about $t = -\infty$ with $c_1 = 7.36848pF$, $c_0 = 22338.5pF$ and $\gamma = \frac{1}{2}V$ for sets of values (M, N). The solid line corresponds to the (0,1) approximant given in equation (2.23), whilst the (1,2) approximant is represented by the dot-dashed line, (2,3) by a dashed line and (3,4) by a dotted line.

We now address the problem of how to obtain symmetric solitary wave solutions by utilising the excellent behaviour of the Padé approximants to (2.12) and (2.13), before the first maximum is reached as t increases/decreases from $\mp\infty$, respectively.

2.4. Two Point Padé Approximants. In the previous section it was shown that the Padé approximants (1,2), (2,3) and (3,4) to the series (2.12) lead to undesirable behaviour for the approximant to $V_n(t)$ as t increased from where the (first) maximum voltage occurred at $t_{\max}$. However, before $t_{\max}$ each approximation was very reasonable. We now describe how two point Padé approximants can be used to derive approximations to $V_n(t)$ which have good behaviour for the whole range of t values from minus infinity to plus infinity. Their definition and properties may be found in the work of McCabe and Murphy [21] and we will review those which will be needed here.

Consider an analytic function $f(z)$ with power series expansions at $z = g_n = 0$ and $z = g_n = \infty$, which for $|z|$ small is given by

$$f(z) = A_0 + A_1 z + A_2 z^2 + \dots \tag{2.25}$$

while for $|z|$ large

$$f(z) = -\left\{\frac{A_{-1}}{z} + \frac{A_{-2}}{z^2} + \frac{A_{-3}}{z^3} + \dots\right\}. \tag{2.26}$$

The function $f(z)$ is approximated by $f_{ij}(z)$ which consists of a polynomial in z of degree $T-1$ divided by a monic polynomial in z of degree T and $f(z)$ is such that

$$f(z) - f_{ij}(z) = O\left(z^i, z^{-(j+1)}\right), \tag{2.27}$$

where $i + j = 2T$. Then $f_{ij}(z)$ can be expressed in standard notation as a truncated continued fraction in the form

$$f_{ij}(z) = \frac{n_1^{(0)}}{1 + d_1^{(0)} z} + \frac{n_2^{(0)} z}{1 + d_2^{(0)} z} + \dots + \frac{n_T^{(0)} z}{1 + d_T^{(0)} z}, \tag{2.28}$$

where $n_T^{(0)}$ and $d_T^{(0)}$ are constants independent of z and are calculated using the QD algorithm. We define $\phi_n(\tau) = a_0\tau + f_{ij}(z)$ and hence $\phi_n(\tau)$ can be expressed in terms of the coefficients in the power series expansions about $\tau = \pm\infty$. The coefficients A_0 and $A_{\pm 1}$ are determined by matching the series expansions (2.25) and (2.26) and their first and second derivatives at $z = 1$; A_0 is then given by an infinite series and can be expressed as a Padé approximant (see [20]).

When the first term is taken in the continued fraction representation (2.28) ($T = 1$) then $V_n(t)$ is the same as that given by (2.23). Note that analytical expression for higher approximants can be given, but the formulas contain a large number of terms. Therefore we content ourselves by showing the numerical results of using various numbers of terms T in (2.28) with different approximations to A_0. The results are shown in figure 4 and we have taken $\kappa = 2.451$ and $n = 1$, as before. It is seen that as T increases the two point Padé approximants converge for the full range of values of t which was not the case when standard Padé approximants were used. It should be noted that in the work described in this entire section we have not made any assumptions about the amplitude of any pulse propagating on the nonlinear transmission line.

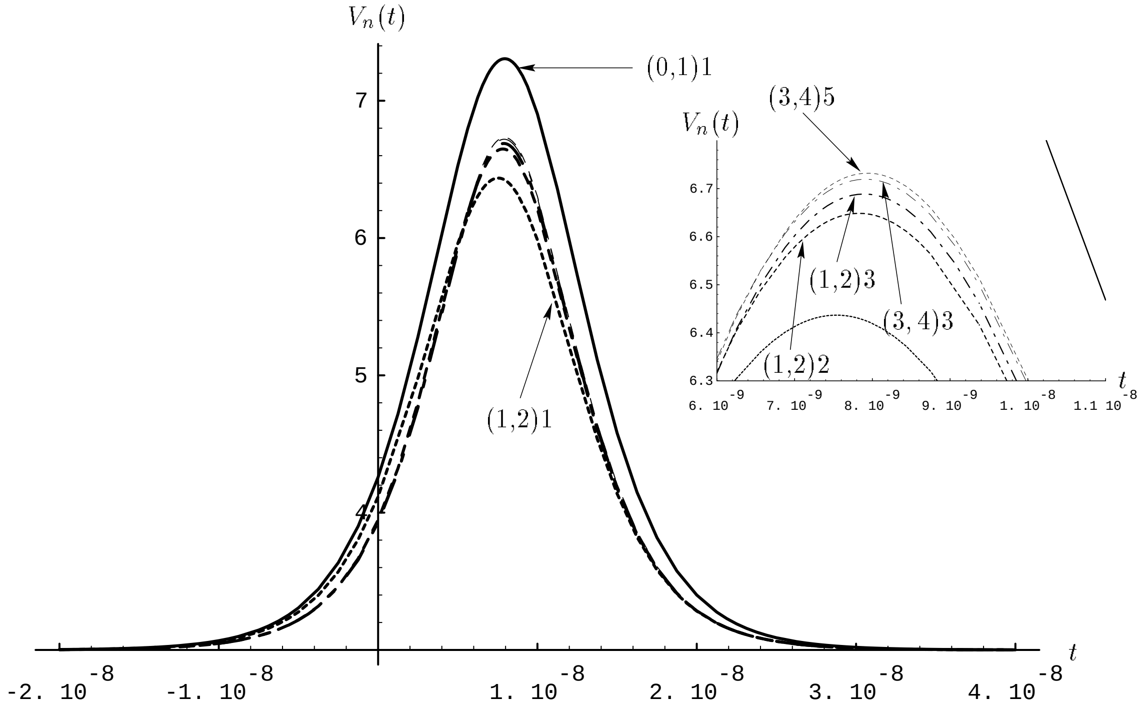

FIGURE 4. Plots of two point Padé approximants to $V_n(t)$, labelled as $(M, N)T$. (M, N) denotes the Padé approximation to A_0 and T the number of terms in the continued fraction expansion (2.28). Here, $\kappa = 2.451$, $n = 1$, $c_1 = 7.36848pF$, $c_0 = 22338.5pF$ and $\gamma = \frac{1}{2}V$. The insert clearly shows the convergent nature of the two point Padé approximants.

3. Quasi-continuum methods

When a solitary wave solution is written down there is often a relationship, sometimes implicit, between the speed of the wave and its amplitude. In this section we derive several speed–amplitude relationships for the experimental NLTL considered in section 2. The approach we follow here is that of Wattis [13] and Duncan *et al.* [14], who used quasi-continuum methods to derive approximate solitary wave solutions of a number of nonlinear lattices.

Consider a lattice equation

$$L\frac{\partial^2\psi_n(t)}{\partial t^2} = P^{(1)}\left(\psi_{n+1}(t)\right) + P^{(1)}\left(\psi_{n-1}(t)\right) - 2P^{(1)}\left(\psi_n(t)\right), \tag{3.1}$$

where $^{(r)}$ represents differentiation r times with respect to the argument and $P\left(\psi_n\right)$ is a potential energy function to be introduced below.

We search for a solution $\psi_n(t) = \psi(z)$ where $z = \kappa n - \omega t$, and so (3.1) becomes

$$L\omega^2 \frac{\mathrm{d}^2\psi(z)}{\mathrm{d}z^2} = P^{(1)}\left(\psi(z+\kappa)\right) + P^{(1)}\left(\psi(z-\kappa)\right) - 2P^{(1)}\left(\psi(z)\right), \tag{3.2}$$

so that (3.2) can be written as

$$L\omega^2\psi = \Lambda(\mathrm{D})P^{(1)}(\psi) + \alpha z + \beta, \tag{3.3}$$

where $\Lambda(\mathrm{D}) = 4\sinh^2\left(\frac{\kappa \mathrm{D}}{2}\right)/\mathrm{D}^2$, $\mathrm{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$, $\psi = \psi(z)$ is identified as the charge Q_n, $P^{(1)}(\psi)$ is the voltage V_n and α and β are constants of integration. In previous work [13, 14] these integration constants were found to be zero because there $P^{(1)}(\psi)$ and $\psi \to 0$ as $z \to \pm\infty$. However this is not the case here since these variables correspond to voltage and charge which have non–zero limits at infinity.

We assume $V_n(t) \to V_0$ so that $P^{(1)}(\psi) \to V_0$ and

$$\psi \to \psi_\infty = c_0\gamma + c_1 V_0 - \frac{c_0\gamma^2}{\gamma + V_0}, \qquad \text{as} \quad |z| \to \infty. \tag{3.4}$$

Therefore we find

$$\beta = \omega^2 L\psi_\infty - \kappa^2 V_0, \tag{3.5}$$

since $\psi^{(2)}(z) \to 0$ as $|z| \to \infty$ and $\alpha = 0$ assuming there exists a bounded solution. The method described in [13] and [14] is to take Padé approximations to $\Lambda(\mathrm{D})$ and to try to obtain solutions of the resulting ordinary differential equation. In the ensuing working we need to express V_n in terms of Q_n or equivalently, $P^{(1)}(\psi)$ in terms of ψ. The expression for Q_n in terms of V_n is given in section 2, equation (2.7), and solving the relation for V_n we find

$$P^{(1)}(\psi) \equiv V_n = \frac{\psi - \gamma(c_1 + c_0) + \sqrt{\mathrm{R}}}{2c_1}, \tag{3.6}$$

where $\mathrm{R} = \psi^2 + 2\gamma(c_1 - c_0)\psi + \gamma^2(c_1 + c_0)^2$ and we have taken the positive square root in (3.6) to ensure that $P^{(1)}(\psi)$ is positive. The potential $P(\psi)$ is then given by

$$P(\psi) = \frac{\psi^2}{4c_1} - \frac{\gamma\psi(c_1 + c_0)}{2c_1} + \frac{\psi + \gamma(c_1 - c_0)}{4c_1}\sqrt{\mathrm{R}} + \gamma^2 c_o \log\left[2\sqrt{\mathrm{R}} + 2\left(\psi + \gamma(c_1 - c_0)\right)\right] + p_0, \tag{3.7}$$

where p_0 is a constant chosen to make $P(0) = 0$. Thus $P(\psi)$ is a highly nonlinear function of $\psi(z)$ and as a result we are unable here to determine exact solutions to equation (3.3), though approximate speed-amplitude relations can be derived. As in Wattis [13] we will take (0,2), (2,0) and (2,2) Padé approximations to the operator $\Lambda(\mathrm{D})$ in (3.3). Given $V_{\max}$ the maximum amplitude, $\psi_{\max}$ is determined from (3.4) by replacing V_0 by $V_{\max}$ and $P(\psi_{\max})$ and $P(\psi_\infty)$ are found from (3.7). Detailed calculations are given in [20] and here we only present the results.

The (0,2) and (2,0) Padé approximations to $\Lambda(\mathrm{D})$ generate the speed-amplitude relations

$$\frac{\omega^2}{\kappa^2} = \frac{2}{L}\left[\frac{P(\psi_{\max}) - P(\psi_\infty) + V_0\left(\psi_\infty - \psi_{\max}\right)}{\left(\psi_\infty - \psi_{\max}\right)^2}\right], \tag{3.8}$$

$$\frac{\omega^2}{\kappa^2} = \frac{(V_{\max} - V_0)^2}{2L\left[P(\psi_{\max}) - P(\psi_\infty) + V_{\max}(\psi_{\max} - \psi_\infty)\right]}. \tag{3.9}$$

Using the (2,2) approximant to $\Lambda(\mathrm{D})$ we find

$$\begin{aligned} &2c^4L^2(\psi_\infty - \psi_{\max})^2 \\ &+ 2c^2L\left\{5P(\psi_\infty) - 5P(\psi_{\max}) + (\psi_{\max} - \psi_\infty)(3V_{\max} + 2V_0)\right\} - 3(V_{\max} - V_0)^2 = 0, \end{aligned} \tag{3.10}$$

which is a quadratic equation for the square of the wavespeed, c^2.

We now have three approximate relations to find the wavespeed $c = \omega/\kappa$ and each will be plotted in the next section where we also show how the speed varies with amplitude for the two point Padé approximants considered in the previous section.

4. Theoretical and Numerical Results

In this section we will compare the speed–amplitude relationships derived in the previous section with those obtainable by the two point Padé (TPP) approximant procedure described in section 2.3. We also compare with measured values found by modeling the NLTL using HSPICE, which is a commercially available circuit simulator. The amplitude half-width relations are also investigated. Further information can be found in [20].

In order to plot the speed–amplitude relations from the TPP procedure the method adopted was to let the wavenumber κ vary over a range of values and the maximum value of the pulse for each κ was found using some simple MATHEMATICA code. Note that for given κ, ω is known from the dispersion relation (2.16), hence the wavespeed can be found.

In figures 5 and 6 overleaf, the dots represent the values found using the circuit simulator. In figure 5 the thick lines are the quasi-continuum (QC) approximants and the thinner lines the TPP approximants. We have chosen to plot the speed–amplitude and half–width–amplitude relations using $(M, N)T = (0, 1)1$, (1,2)2, (1,2)3 and (3,4)5 and again, $n = 1$.

In figure 5 we see that both the TPP and QC approximants give speed–amplitude relationships close to the results from numerical simulations for waves of amplitude less than 5 volts. For larger amplitudes, even though the QC approximants are improving with order, the TPP approximants are converging much faster. Even the relatively simple (1,2)2 approximant agrees well with simulated values for solitary waves as large as 25 volts. It may also be noted that the lowest order QC and TPP approximants give nearly the same (but inaccurate) amplitude–speed relationships over the whole range of voltages under consideration.

From the results presented in the figure 5 it appears that the wave amplitude $V_{\max}$ increases rapidly with its speed c. In fact from a log log plot of the data we estimate that $V_{\max} \propto c^3$. This may be compared with the Toda lattice [1] where the growth is approximately quadratic for sufficiently large κ. Figure 6 shows how the half–width of the pulse varies with amplitude. Again, the TPP approximants converge, though for larger amplitudes they do not compare quite as well with numerical simulations as the speed–amplitude results have done.

5. Conclusions

In this work we have described a way of approximating solitary waves on non–integrable lattices based on a direct algebraic method [17] for obtaining solitons on integrable lattices. In the non–integrable case the resulting series cannot be summed exactly and we have used Padé approximants to obtain estimates of these sums. Best results are obtained by using *two point* Padé approximants introduced by McCabe and Murphy [21].

The method has been applied to the lattice equations (2.11) corresponding to an electrical circuit under investigation for *pulse compression* properties. They are closely related to those describing behaviour of the Toda lattice which has an exact soliton solution. Results presented in section 5 show that our approximate method yields good results for appreciably larger waves than those for which the quasi–continuum method works and we are also able to obtain approximate solitary wave solutions. In [22] two methods are used to obtain solitary wave profiles of lattices with Lennard–Jones interaction potential. The first method which uses Padé approximants fails to describe high amplitude solitary waves correctly whilst in the second method it is assumed that because the pulses are narrow, higher difference terms can be neglected. In contrast, the two

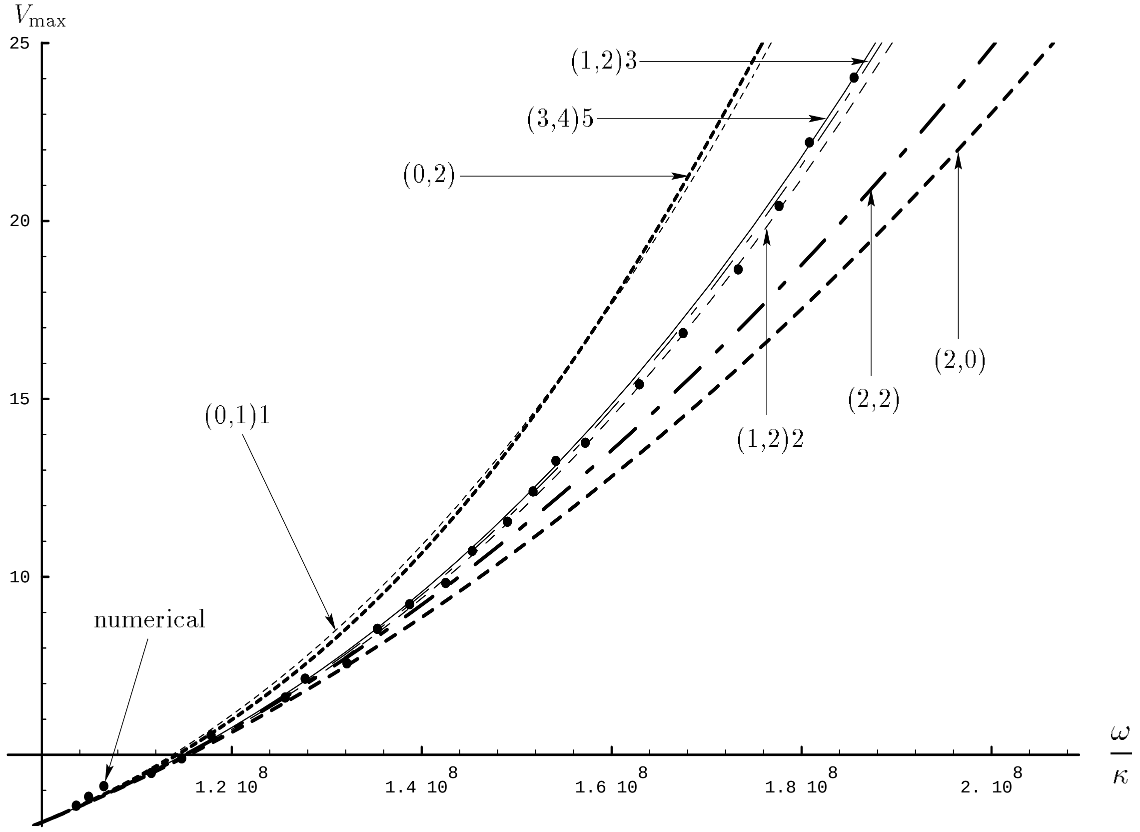

FIGURE 5. Plots of speed–amplitude relationships for the rational $C(V)$ model (equation (2.6)). The QC approximations are the thick lines and labelled $(P,Q) = (0,2)$, $(2,0)$ and $(2,2)$. The thinner lines are the TPP approximations and are labelled $(M,N)T=$ (0,1)1, (1,2)2, (1,2)3 and (3,4)5. The dots (•) are the numerical estimates.

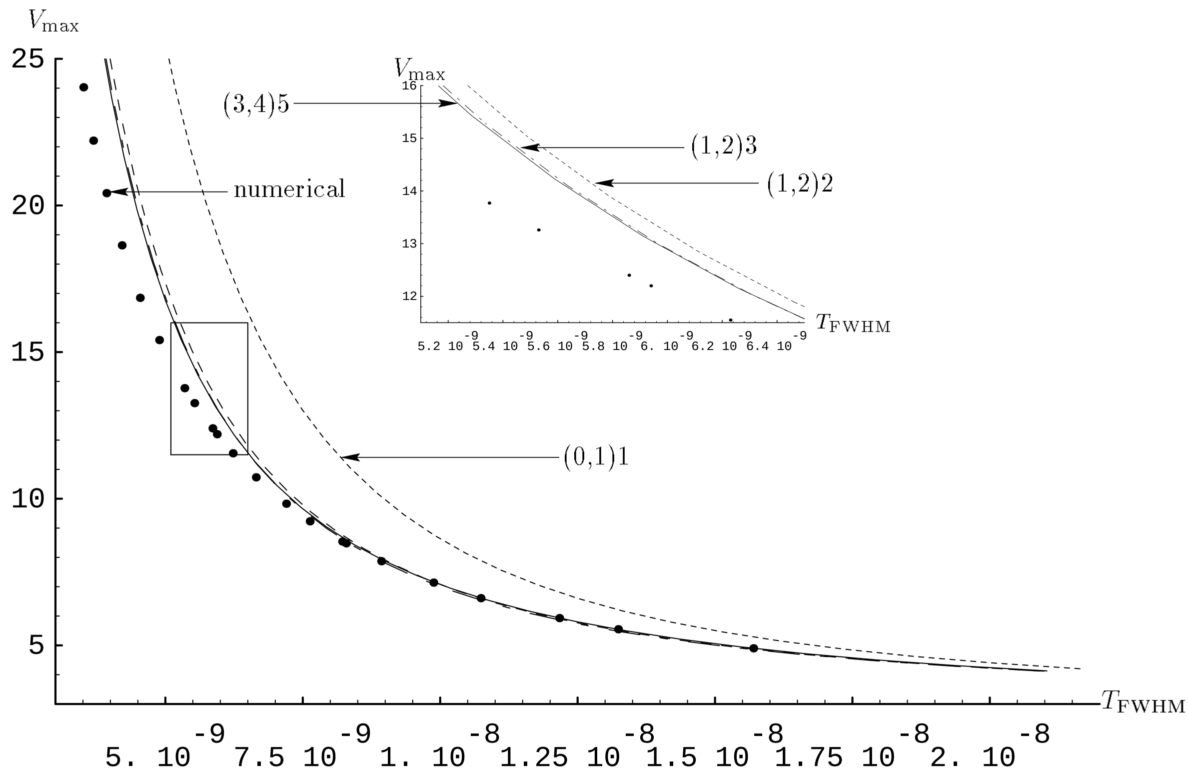

FIGURE 6. Plots of half–width ($T_{\rm FWHM}$) versus amplitude for the rational $C(V)$ model (equation (2.6)). The TPP approximations are labelled $(M,N)T=$ (0,1)1, (1,2)2, (1,2)3 and (3,4)5. The dots (•) are the numerical estimates. The insert shows the convergent nature of the TPP approximants

point Padé approximation procedure we have employed uses all the terms in the lattice equation (2.11) and yields good results for both large and small amplitude solitary waves. It is also very straightforward to apply.

Our technique has the advantage of being easily extendible to the situation of *two soliton* solutions. Numerical experiments indicate that non–integrable lattices can support such states with the solutions scattering *almost* elastically [11]. To approximate such solutions we intend to use *two–variable* Padé approximants which have been studied in some detail [23]. Finally it should not be difficult to extend the methods to other nonlinear lattices of interest.

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