A New Integrable Reduction of the Matrix NLS Equation

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Abstract

In this paper we perform a multiple scales analysis on the coupled KdV systems associated with an energy-dependent Schrödinger operator. We derive the corresponding matrix NLS equation, together with the zero-curvature respresentation. This constitutes a new, integrable NLS system. We present a Hamiltonian structure and constants of motion.

We also present the stationary flows of these equations, giving a new integrable generalisation of the Garnier system, having a $2N \times 2N$ Lax matrix.

1 Introduction

It is well known that a multiple scales analysis of the KdV equation (and, indeed, many other equations) leads to the NLS equation for the modulated amplitude. In [7] Kuznetsov and Zakharov showed a much deeper connection between these equations by showing that a multiple scales analysis of the Schrödinger spectral problem leads to the Zakharov-Shabat problem for the NLS equation.

In this paper we consider a coupled KdV system, which is isospectral to an energy-dependent Schrödinger operator. We perform a multiple-scales analysis on this coupled KdV system to find the analogous integrable NLS system. These constitute a new integrable reduction of the matrix NLS equation. We use the technique of Zakharov and Kuznetsov to construct the corresponding spectral problem, giving a new generalisation of the Zakharov-Shabat spectral problem for the scalar NLS equation. We discuss the first integrals and the stationary reduction, leading to a new integrable generalisation of the Garnier system.

In section 2 we present some background material on the isospectral flows of an energy-dependent Schrödinger operator [1, 2]. In section 3 we present background material on multiple scales, with the derivation of our 2−component system. Sections 4 and 5 are respectively concerned with the $N-$component NLS hierarchy and its stationary flows.

2 An Energy Dependent Scalar Schrödinger Operator and its Matrix Form

In this paper we consider the coupled KdV system which is isospectral to the energy-dependent Schrödinger equation:

$$L\psi \equiv (\partial^2 + \sum_{i=0}^{N-1} u_i \lambda^i)\psi = \lambda^N \psi. \tag{1}$$

In [1, 2], the hierarchy of isospectral flows were shown to be Hamiltonian with respect to $(N+1)$ local Hamiltonian structures and to have a remarkable sequence of Miura maps and modified systems. It was shown that several known, but disparate, systems could be embedded into this general scheme (and its generalisations), including the dispersive water wave, shallow water wave and Ito equations. For $N = 1$, this reduces to the usual KdV hierarchy.

It was shown in [3] that (1) could be written in matrix Schrödinger form:

$$\Psi_{xx} + \mathcal{U}\Psi = \lambda^N \Psi, \tag{2}$$

where $\Psi = (\psi, \lambda\psi, \cdots, \lambda^{N-1}\psi)^T$ and $\mathcal{U} = \sum_{i=0}^{n-1} u_i \tilde{\Lambda}^i$, where

$$\tilde{\Lambda} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ \lambda^N & \cdots & \cdots & 0 \end{pmatrix}, \quad \text{with} \quad \tilde{\Lambda}^N = \lambda^N I_N,$$

where I_N is the $N \times N$ identity matrix.

The hierarchy of polynomial isospectral flows of (2) correspond to a time evolution of Ψ given by:

$$\psi_{t_m} = P\Psi \equiv 2A_{(m)}\Psi_x - A_{(m)x}\Psi, \tag{3}$$

where $A_{(m)} = \sum_{i=0}^{m-1} A_{m-i}\tilde{\Lambda}^i$, with A_i a function of u_i; their x-derivatives and λ. The integrability conditions of (1) and (3) are:

$$L_t = [P, L] + A_x L,$$

which can be written in Hamiltonian form:

$$\mathbf{u}_{t_m} = \mathbf{B}_N \delta\mathcal{H}_m$$

where

$$\mathbf{B}_N = \begin{pmatrix} 0 & \cdots & J_0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ J_0 & \cdots & J_{N-1} \end{pmatrix}, \quad \delta\mathcal{H}_m = (A_{m-N+1}, \cdots, A_m)^T,$$

with $J_0 = \partial^3 + 4u_0\partial + 2u_{0x}$, $J_k = 4u_k\partial + 2u_{kx}, k \neq 0$. When $m = N$, the flow takes the form of a coupled KdV system.

Example 2.1 $[N = 2]$
Here the coupled KdV system is:

$$\begin{aligned} u_{0t_2} &= (\partial^3 + 4u_0\partial + 2u_{0x})(u_0 + \frac{3}{4}u_1^2), \\ u_{1t_2} &= (\partial^3 + 4u_0\partial + 2u_{0x})u_1 + (4u_1\partial + 2u_{1x})(u_0 + \frac{3}{4}u_1^2), \end{aligned} \tag{4}$$

and is a member of the dispersive water wave (DWW) hierarchy.

3 The Multiple Scales Analysis

In this section we consider the multiple scales analysis of nonlinear evolution equations. We are particularly interested in equations which are isospectral to some linear eigenvalue problem. Applying the technique of Zakharov and Kuznetsov [7], we can then derive NLS type equations and their corresponding linear spectral problems. We first briefly remind the reader of the original Zakharov-Kuznetsov calculation with the KdV and then present the simplest nontrivial generalisation, corresponding to the system (4).

Consider the general evolution equation:

$$u_t = K(u, u_x, \cdots),$$

and let $L[\partial_x]u$ be the linear part of K. The *linear dispersion relation* is obtained by substituting the plane wave solution $u_k = Ae^{i(kx-\omega(k)t)} \equiv Ae^{i\theta}$ into the linear equation:

$$u_t = L[\partial_x]u \quad \Rightarrow \quad \omega(k) = iL[ik].$$

A nonlinear equation modulates the amplitude of this plane wave solution in such a way that we may consider it to depend upon the *slow variables*:

$$\xi = \varepsilon(x - \omega'(k)t) \quad \text{and} \quad \tau = -\frac{1}{2}\varepsilon^2\omega''(k)t. \tag{5}$$

Our multiple scales analysis starts with the assumption:

$$u = u(x, t, \xi, \tau) \quad \text{and} \quad u = \varepsilon u_1 + \varepsilon^2 u_2 + \cdots.$$

Then

$$D_t u = \left(\partial_t - \varepsilon\omega'(k)\partial_\xi - \frac{1}{2}\varepsilon^2\omega''(k)\partial_\tau\right)u, \quad D_x u = (\partial_x + \varepsilon\partial_\xi)\,u.$$

Example 3.1 (The KdV Equation) The particular case:

$$u_t = u_{xxx} + 6uu_x$$

gives:

$$\omega(k) = k^3\,, \quad \xi = \varepsilon(x - 3k^2t)\,, \quad \tau = -3\varepsilon^2 kt, \tag{6}$$

so:

$$D_x = \partial_x + \varepsilon\partial_\xi \text{ and } D_t = \partial_t - 3k^2\varepsilon\partial_\xi - 3k\varepsilon^2\partial_\tau.$$

We find:

$$\begin{aligned}
u_{1t} - u_{1xxx} &= 0, \\
u_{2t} - u_{2xxx} &= 3(u_1^2)_x + 3(u_{1xx} + k^2 u_1)_y, \\
u_{3t} - u_{3xxx} &= 3k^2 u_{2\xi} + 3k u_{1\tau} + 3u_{2xx\xi} + 3u_{1x\xi\xi} + 6(u_1 u_2)_x + 3(u_1^2)_\xi,
\end{aligned}$$

and so on. These equations can be solved iteratively to give:

$$\begin{aligned}
u_1 &= v_1(\xi,\tau)e^{i\theta} + v_{-1}(\xi,\tau)e^{-i\theta} \quad \text{where} \quad \theta = kx - k^3 t, \\
u_2 &= k^{-2}\left(-2v_1 v_{-1} + v_1^2 e^{2i\theta} + v_{-1}^2 e^{-2i\theta}\right), \\
u_3 &= k^{-3}\left(2iv_1 v_{1\xi} e^{2i\theta} - 2iv_{-1}v_{-1\xi}e^{-2i\theta}\right) + \frac{3}{4}k^{-4}\left(v_1^3 e^{3i\theta} + v_{-1}^3 e^{-3i\theta}\right).
\end{aligned}$$

The non-oscillating term in u_2 was chosen to avoid a secularity at the level ε^3. Further secular terms are avoided by choosing v_1 and $v_{-1} = v_1^*$ to satisfy an amplitude equation, which, in terms of $q = v_1/k$ is the NLS equation:

$$iq_\tau = q_{\xi\xi} - 2 \mid q \mid^2 q.$$

The Spectral Problem The KdV equation has Lax representation $L_t = [P, L]$, which is the integrability condition of:

$$L\psi \equiv (\partial^2 + u)\psi = -\lambda^2\psi, \quad \text{and} \quad \psi_t = P\psi \equiv (4\partial^3 + 6u\partial + 3u_x)\psi.$$

Our expansion for u starts with εu_1, so our spectral problem and time evolution can be considered as perturbations of:

$$\psi_{xx} + \lambda^2\psi = 0 \quad \text{and} \quad \psi_t = 4\psi_{xxx},$$

with plane wave solutions:

$$\psi_0(x,t) = \varphi_1 e^{i(\lambda x - 4\lambda^3 t)} + \varphi_{-1}e^{-i(\lambda x - 4\lambda^3 t)}.$$

With the perturbation:

$$u = \varepsilon u_1 + \cdots \quad \text{and} \quad \psi = \psi_0 + \varepsilon\psi_1 + \cdots,$$

we get a resonance at $\lambda = \pm\frac{1}{2}k$, so we perturb about

$$\psi_0 = \varphi_1 e^{i\theta/2} + \varphi_{-1}e^{-i\theta/2}.$$

In a multiple scales analysis, with ξ and τ as before, and $\lambda = \frac{1}{2}k + \varepsilon\mu$, $(D_x^2 + \lambda^2 + u)\psi = 0$ gives at order ε:

$$\begin{aligned}
\psi_{1xx} + \frac{1}{4}k^2\psi_1 &= -2\psi_{0x\xi} - u_1\psi_0 - \mu k\psi_0 \\
&= -(ik\varphi_{1\xi} + k\mu\varphi_1 + v_1\varphi_{-1})e^{i\theta/2} + (ik\varphi_{-1\xi} - k\mu\varphi_{-1} - v_{-1}\varphi_1)e^{-i\theta/2} \\
&\quad -(v_1\varphi_1 e^{3i\theta/2} + v_{-1}\varphi_{-1}e^{-3i\theta/2}),
\end{aligned}$$

To avoid secular terms at this level, we need.

$$\begin{pmatrix} \varphi_1 \\ \varphi_{-1} \end{pmatrix}_\xi = i\begin{pmatrix} \mu & q \\ -q^* & -\mu \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_{-1} \end{pmatrix}, \tag{7}$$

where $v_1 = kq$, $v_{-1} = v_1^*$. This is the spectral problem for the NLS equation.

When we similarly perturb the time-evolution for ψ, using:

$$D_t\psi = 4D_x^3\psi + 6uD_x\psi + 3(D_xu)\psi,$$

we find:

$$\begin{pmatrix} \varphi_1 \\ \varphi_{-1} \end{pmatrix}_\tau = \begin{pmatrix} 2i\mu^2 + iqq^* & q_\xi + 2i\mu q \\ q_\xi^* - 2i\mu q^* & -2i\mu^2 - iqq^* \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_{-1} \end{pmatrix}. \tag{8}$$

The calculation for the coupled KdV system, isospectral to the energy-dependent Schrödinger problem, is very similar. Further details can be found in [6]. We illustrate this with the simplest nontrivial case.

Example 3.2 (The DWW Hierarchy) Here we consider the $N = 2$ case of the energy-dependent Schrödinger operator (2). In this case the coupled KdV system is the t_2-flow (4).

The linear dispersion relation is the same as for the KdV equation, so, dropping the suffix 2, we use (6). With the substitution : $u_n = \varepsilon u_{n1} + \varepsilon^2 u_{n2} + \cdots$, for $n = 1, 2$, we find:

$$\begin{aligned}
u_{01t} - u_{01xxx} &= 0, \\
u_{11t} - u_{11xxx} &= 0, \\
u_{02t} - u_{02xxx} &= 3(k^2u_{01} + u_{01xx})_y + 6u_{01}u_{01x} + \frac{9}{2}u_{11x}u_{11xx} + \frac{3}{2}u_{11}u_{11xxx}, \\
u_{12t} - u_{12xxx} &= 3(k^2u_{11} + u_{11xx})_y + 6(u_{01}u_{11})_x,
\end{aligned}$$

and so on.

At zeroth order we get:

$$u_{01} = v_{01}e^{i\theta} + v_{0-1}e^{-i\theta}, \quad u_{11} = v_{11}e^{i\theta} + v_{1-1}e^{-i\theta}, \tag{9}$$

where $v_{0-1} = v_{01}^*$ and $v_{1-1} = v_{11}^*$. Analogous notation will be used at higher orders below. No secular terms arise at ε^2, giving u_{02} and u_{12}:

$$u_{02} = v_{02} + (v_{01}^2k^{-2} - v_{11}^2)e^{2i\theta} + (v_{0-1}^2k^{-2} - v_{1-1}^2)e^{-2i\theta}, \tag{10}$$

$$u_{12} = v_{12} + \frac{2}{k^2}(v_{01}v_{11}e^{2i\theta} + v_{0-1}v_{1-1}e^{-2i\theta}). \tag{11}$$

The 'constant' terms are determined by non-secularity conditions at ε^3:

$$v_{02} = -\frac{2}{k^2}v_{01}v_{0-1}, \quad v_{12} = -\frac{2}{k^2}(v_{11}v_{0-1} + v_{1-1}v_{01}). \tag{12}$$

Further secularities are avoided at this order by requiring that $q_1 = v_{01}k^{-1}$, $q_2 = \frac{1}{2}v_{11}$ and their complex conjugates satisfy:

$$\begin{aligned}
iq_{1\tau} &= q_{1\xi\xi} - 2(|q_1|^2 - 2|q_2|^2)q_1 + 2q_1^*q_2^2, \\
iq_{2\tau} &= q_{2\xi\xi} + 2(|q_2|^2 - 2|q_1|^2)q_2 - 2q_1^2q_2^*,
\end{aligned} \tag{13}$$

together with their complex conjugate equations. These are the coupled NLS equations associated with the coupled KdV equations (4).

The Spectral Problem The spectral problem we use is (2) with $N=2$. The time evolution (3), with $A_{(2)}=(2\lambda^2+u_0+\frac{3}{4}u_1^2)I+u_1\tilde{\Lambda}$, can be written:

$$\Psi_t=4\Psi_{xxx}+6\left((u_0+\frac{1}{4}u_1^2)I+u_1\tilde{\Lambda}\right)\Psi_x+3\left((u_0-\frac{1}{4}u_1^2)I+u_1\tilde{\Lambda}\right)_x\Psi. \tag{14}$$

Once again we get a resonance at $\lambda=\pm\frac{1}{2}ik$, so perturb λ by $\lambda=i(\frac{1}{2}k+\varepsilon\mu)$. With ξ defined as before, u_i perturbed as above and $\Psi=\Psi_0+\varepsilon\Psi_1+\cdots$, we get:

$$\begin{aligned}\Psi_{0xx}+\frac{1}{4}k^2\Psi_0 &= 0,\\ \Psi_{1xx}+\frac{1}{4}k^2\Psi_1 &= -2\Psi_{0x\xi}-\mu k\Psi_0-(u_{01}+u_{11}\tilde{\Lambda}_0)\Psi_0,\end{aligned}$$

where $\tilde{\Lambda}_0$ is the zeroth order part of $\tilde{\Lambda}$:

$$\tilde{\Lambda}_0=\begin{pmatrix}0 & 1\\ -\frac{1}{4}k^2 & 0\end{pmatrix}.$$

Thus, at zeroth order we find:

$$\Psi_0=\varphi_1e^{i\theta/2}+\varphi_{-1}e^{-i\theta/2}.$$

To avoid secular terms at order ε, we find:

$$ik\varphi_{1\xi}=-\mu k\varphi_1-(v_{01}+v_{11}\tilde{\Lambda}_0)\varphi_{-1},$$

together with its complex conjugate. Defining:

$$\Phi_{\pm1}=\begin{pmatrix}\frac{1}{2}k & 0\\ 0 & 1\end{pmatrix}\varphi_{\pm1},$$

and q_1,q_2 as before, we find:

$$\begin{pmatrix}\Phi_1\\ \Phi_1^*\end{pmatrix}_\xi=i\begin{pmatrix}\mu I & Q\\ -Q^* & -\mu I\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_1^*\end{pmatrix}, \tag{15}$$

where $Q=q_1I+q_2\Lambda$ with $\Lambda=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. A perturbation expansion of (14) to order ε^2 leads to:

$$\begin{pmatrix}\Phi_1\\ \Phi_1^*\end{pmatrix}_\tau=\begin{pmatrix}i(2\mu^2+QQ^*) & Q_\xi+2i\mu Q\\ Q_\xi^*-2i\mu Q^* & -i(2\mu^2+QQ^*)\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_1^*\end{pmatrix}. \tag{16}$$

The integrability conditions of these give rise to the matrix NLS equations:

$$iQ_\tau=Q_{\xi\xi}-2Q^2Q^*, \tag{17}$$

which is just the coupled system (13). It should be noted that Q and Q^* commute, so the order of the matrices in the nonlinear term is unimportant.

4 The N-component Case and Hamiltonian Formulation

The multiscales calculation for the N-component case is very similar and leads to the matrix NLS equation (17), with zero curvature representation given in terms of (15) and (16) but where now:

$$Q = \sum_{i=1}^{N} q_i \Lambda^{i-1}\,, \quad \text{where} \quad \Lambda = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ -1 & \cdots & \cdots & 0 \end{pmatrix},$$

with I the $N \times N$ identity matrix and Φ_1 an N-vector. Matrices (15) and (16) are $2N \times 2N$, but have the appearance of the usual 2×2 matrices of Zakharov and Shabat. This is true for the entire isospectral hierarchy of (15). By diagonalising Λ we can simultaneously diagonalise Q and Q^*. However, since the transformation is *complex*, the equation (17) does not decouple in diagonalised form. The analogous *real* MKdV and KdV hierarchies are trivial, since they decouple into N copies of the single component equations.

4.1 Hamiltonian Formulation

Let $\mathcal{H}(q, q^*)$ be a conserved density of the usual scalar NLS hierarchy. Then, since the matrices Q, $Q_\xi, \cdots$ all commute, $\mathcal{H}(Q, Q^*)$ is a *matrix* of conserved densities. Indeed, if $\mathcal{F}(q, q^*)$ is the flux corresponding to the NLS flow, then we have the matrix conservation law:

$$\partial_\tau \mathcal{H}(Q, Q^*) = \partial_\xi \mathcal{F}(Q, Q^*),$$

since all manipulations are identical to the scalar case, so no traces are required. The first few are:

$$\begin{aligned} &\mathcal{H}_0 = QQ^*, \quad \mathcal{H}_1 = \frac{1}{2} i \left(Q Q_\xi^* - Q^* Q_\xi \right), \\ &\mathcal{H}_2 = Q_\xi Q_\xi^* + (Q\, Q^*)^2, \quad \mathcal{H}_3 = i \left(Q_\xi^* Q_{\xi\xi} - Q_\xi Q_{\xi\xi}^* + 3 Q Q^* (Q^* Q_\xi - Q Q_\xi^*) \right). \end{aligned}$$

Thus, each conserved density $\mathcal{H}_n$ of the scalar NLS hierarchy gives rise to N scalar conserved densities $\{\mathcal{H}_{nk}\}_{k=0}^{N-1}$ for our hierarchy, where $\mathcal{H}_n = \sum_{k=0}^{N-1} \mathcal{H}_{nk} \Lambda^k$. These can be written in terms of the matrices:

$$\sigma_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & -1 \\ \vdots & \vdots & ⋰ & \vdots \\ 0 & -1 & \cdots & 0 \end{pmatrix}, \tag{18}$$

$$\sigma_k = \sigma_1 \Lambda^{k-1} = -\Lambda^{N-k+1} \sigma_1 = \begin{pmatrix} 0 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & ⋰ & \vdots & \vdots & & \vdots \\ 1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & -1 \\ \vdots & & \vdots & \vdots & ⋰ & \vdots \\ 0 & \cdots & 0 & -1 & \cdots & 0 \end{pmatrix}, \tag{19}$$

where the top block of σ_k is $k\times k$. In terms of these matrices:

$$\begin{aligned}
\mathcal{H}_0 &= \sum_{k=1}^{N}\rho_k\Lambda^{k-1} \quad \text{where} \quad \rho_k=\mathbf{q}^T\sigma_k\mathbf{q}^*,\\
\mathcal{H}_1 &= \frac{1}{2}i\sum_{k=1}^{N}\left(\mathbf{q}^T\sigma_k\mathbf{q}_\xi^*-\mathbf{q}^{*T}\sigma_k\mathbf{q}_\xi\right)\Lambda^{k-1},\\
\mathcal{H}_2 &= \sum_{k=1}^{N}\left(\mathbf{q}_\xi^T\sigma_k\mathbf{q}_\xi^*+\rho^T\sigma_k\rho\right)\Lambda^{k-1} \quad \text{where} \quad \rho=(\rho_1,\cdots,\rho_N).
\end{aligned}$$

The Hamiltonian structure:

$$i\dot{Q}_{ij}=-\frac{\delta\mathcal{H}}{\delta Q_{ji}^*}$$

reduces to:

$$i\dot{q}_k=-(\sigma_1)_{kj}\frac{\delta\mathcal{H}}{\delta q_j^*},$$

giving a Poisson bracket

$$\{H,G\}=i\int_{-\infty}^{\infty}\left[\left(\frac{\delta\mathcal{G}}{\delta\mathbf{q}}\right)^T\sigma_1\left(\frac{\delta\mathcal{H}}{\delta\mathbf{q}^*}\right)-\left(\frac{\delta\mathcal{G}}{\delta\mathbf{q}^*}\right)^T\sigma_1\frac{\delta\mathcal{H}}{\delta\mathbf{q}}\right]d\xi.$$

The NLS equation (17) is generated by $\mathcal{H}_2^0$. A complex MKdV flow is generated by the Hamiltonian $\mathcal{H}_3^0$.

Example 4.1 ($N=2$) Here

$$\Lambda=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad \sigma_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

giving the first few densities as:

$$\begin{aligned}
\mathcal{H}_0^0 &= |\, q_1\,|^2-|\, q_2\,|^2, \quad \mathcal{H}_0^1=q_1q_2^*+q_2q_1^*,\\
\mathcal{H}_1^0 &= q_1q_{1\xi}^*+q_2^*q_{2\xi}-q_1^*q_{1\xi}-q_2q_{2\xi}^*, \quad \mathcal{H}_1^1=q_1q_{2\xi}^*+q_2q_{1\xi}^*-q_1^*q_{2\xi}-q_2^*q_{1\xi},\\
\mathcal{H}_2^0 &= |\, q_{1\xi}\,|^2-|\, q_{2\xi}\,|^2+(|\, q_1\,|^2-|\, q_2\,|^2)^2-(q_1q_2^*+q_1^*q_2)^2,\\
\mathcal{H}_2^1 &= q_{1\xi}q_{2\xi}^*+q_{2\xi}q_{1\xi}^*+2\left(|\, q_1\,|^2-|\, q_2\,|^2\right)(q_1q_2^*+q_1^*q_2).
\end{aligned}$$

Thus we get two hierarchies $\{\mathcal{H}_n^k\}_{n=0}^{\infty}$ for $k=1,2$, labelling the corresponding 'time' evolution parameters by τ_{nk}. These hierarchies are not independent, since, for given n, $\mathbf{q}$ satisfies:

$$\partial_{\tau_{n1}}\mathbf{q}=\Lambda\partial_{\tau_{n0}}\mathbf{q},$$

which in the present case of $N=2$, takes the form of Cauchy-Riemann equations for q_1 and q_2:

$$(\partial_{\tau_{n1}}+i\partial_{\tau_{n0}})(q_1+iq_2)=0.$$

Thus, we only need consider the $k=0$ hierarchy. The coupled NLS flow (13) is generated by $\mathcal{H}_2^0$. The next member of this hierarchy is:

$$\begin{aligned}
\mathcal{H}_3^0 &= \frac{1}{2}i\left(q_{1\xi\xi}q_{1\xi}^*-q_{1\xi}q_{1\xi\xi}^*+q_{2\xi}q_{2\xi\xi}^*-q_{2\xi\xi}q_{2\xi}^*+6\,|\, q_1\,|^2(q_2q_{2\xi}^*-q_{2\xi}q_2^*)+6\,|\, q_2\,|^2(q_1q_{1\xi}^*-q_{1\xi}q_1^*)\right.\\
&\left.+3(q_1^2-q_2^2)(q_2^*q_{2\xi}^*-q_1^*q_{1\xi}^*)+3(q_2^{*2}-q_1^{*2})(q_2q_{2\xi}-q_1q_{1\xi})\right),
\end{aligned}$$

which generates a coupled, complex MKdV system:

$$\begin{aligned} q_{1\tau_3} &= q_{1\xi\xi\xi} + 6(|\, q_2\,|^2 - |\, q_1\,|^2)q_{1\xi} + 6(q_1 q_2^* + q_2 q_1^*)q_{2\xi}, \\ q_{2\tau_3} &= q_{2\xi\xi\xi} + 6(|\, q_2\,|^2 - |\, q_1\,|^2)q_{2\xi} - 6(q_1 q_2^* + q_2 q_1^*)q_{1\xi}, \end{aligned}$$

together with the complex conjugate equations.

5 A Generalised Garnier System

The stationary flow of (17):

$$Q_{\xi\xi} - 2Q^2Q^* = 0\,, \quad Q^*_{\xi\xi} - 2Q^{*2}Q = 0,$$

admits the *nontrivial* real reduction:

$$Q^* = Q^T \quad \text{or, equivalently} \quad \mathbf{q}^* = \sigma_1 \mathbf{q}.$$

With this substituion, the PDE Hamiltonian density $\mathcal{H}_2^0$ plays the role of a Lagrangian for the stationary flow:

$$\mathcal{L} = \frac{1}{2}\mathbf{q}_\xi^T \mathbf{q}_\xi + \frac{1}{2}\rho^T\sigma_1\rho\,, \quad \text{where} \quad \rho_i = \mathbf{q}^T\sigma_i\sigma_1\mathbf{q} = -\mathbf{q}^T\Lambda^{N-i+1}\mathbf{q},$$

as follows from formula (19) and $\sigma_1^2 = I$. More explictly, $\mathcal{L}$ takes the form

$$\mathcal{L} = \frac{1}{2}\mathbf{q}_\xi^T \mathbf{q}_\xi + \frac{1}{2}\left(\rho_1^2 - \sum_{k=2}^{N}\rho_k\rho_{N-k+2}\right)$$

and

$$\rho_k\rho_{N-k+2} = \rho_{N-k+2}\rho_k = -(\mathbf{q}^T\Lambda^{k-1}\mathbf{q})^2,$$

since $(\Lambda^{N-k+1})^T = -\Lambda^{k-1}$. Thus

$$\mathcal{L} = \frac{1}{2}\mathbf{q}_\xi^T \mathbf{q}_\xi + \frac{1}{2}(\mathbf{q}^T\mathbf{q})^2 + \sum_{i=1}^{n}(\mathbf{q}^T\Lambda^i\mathbf{q})^2$$

for $N = 2n+1$ or $N = 2(n+1)$.

Remark 5.1 *When $N = 2(n+1)$, there would appear to be an extra term $\frac{1}{2}\mathbf{q}^T\Lambda^{n+1}\mathbf{q}$, but this vanishes since Λ^{n+1} is skew symmetric.*

The Legendre transformation gives a Hamiltonian for a generalised Garnier system:

$$h = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - \frac{1}{2}\left(\sum_{i=1}^{N} q_i^2\right)^2 - \sum_{i=1}^{n}\left(\mathbf{q}^T\Lambda^i\mathbf{q}\right)^2. \tag{20}$$

For $N = 1$ and $N = 2$ this is just the 1 and 2 component Garnier sytems. The first true generalisation is $N = 3$:

$$h = \frac{1}{2}\sum_{i=1}^{3} p_i^2 - \frac{1}{2}\left(\sum_{i=1}^{3} q_i^2\right)^2 - (q_1q_2 + q_2q_3 - q_3q_1)^2,$$

with a similar expression for $N = 4$. For $N = 5$ the Garnier potential is supplemented by:

$$V_\Lambda = -\left(\sum_{i=1}^{4} q_iq_{i+1} - q_5q_1\right)^2 - \left(\sum_{i=1}^{3} q_iq_{i+2} - q_4q_1 - q_3q_2\right)^2,$$

and so on.

5.1 The Lax Representation

The Lax representation for the stationary flow is just the 'time part' of the PDE zero curvature representation, but written in terms of the canonical co-ordinates (q_i, p_i):

$$L = \begin{pmatrix} i(2\mu^2 I + QQ^T) & P + 2i\mu Q \\ P^T - 2i\mu Q^T & -i(2\mu^2 I + QQ^T) \end{pmatrix},$$

where $P = Q_\xi$ is the matrix of momenta: $P = \sum_{k=1}^N p_k \Lambda^{k-1}$, with $p_k = q_{k\xi}$.

The first integrals of (20) can be obtained in the usual way by considering TrL^k. This can be reduced to a calculation with only $N \times N$ matrices since:

$$L^2 = \begin{pmatrix} l_2 & 0 \\ 0 & l_2 \end{pmatrix}, \quad \text{where} \quad l_2 = 4\mu^4 I_N - 2i\mu(QP^T - PQ^T) + PP^T - (QQ^T)^2.$$

Noting that odd powers of L have zero trace and that $Tr\, L^{2k} = 2\, N\, Tr\, l_2^k$, we need ony consider:

$$Tr\, l_2^k, \quad k = 1, \cdots, N.$$

Furthermore, since l_2 depends upon the spectral parameter μ, a full set of first integrals can be extracted from the lower order traces.

An alternative construction of these integrals is to write the PDE fluxes (with respect to the NLS flow) of h_{0k} and h_{1k} in terms of the stationary manifold cordinates (q_i, p_i). These are not all independent and some degenerate to zero.

Example 5.1 (N=2) This is just the 2 component Garnier system, with energy and angular momentum. The traces $\mathrm{Tr}\, l_2$ and $\mathrm{Tr}\, l_2^2$ give respectively h and Ω^2, where:

$$h = \frac{1}{2} Tr(PP^T - (QQ^T)^2) = \frac{1}{2}\sum_{i=1}^2 p_i^2 - \frac{1}{2}\left(\sum_{i=1}^2 q_i^2\right)^2, \quad \Omega = (q_2 p_1 - q_1 p_2).$$

In terms of fluxes, Ω and h are respectively derived from h_{01} and h_{10}.

Example 5.2 (N=3) For any N, Trl_2 just gives the Hamiltonian (20). For $N = 3$ $\mathrm{Tr} l_2^2$ gives two further first integrals (which are in involution):

$$k_1 = \mathbf{p}^T \Lambda \mathbf{q} - \mathbf{q}^T \Lambda \mathbf{p}, \quad k_2 = \mathbf{p}^T \Lambda \mathbf{p} - 2(\mathbf{q}^T \mathbf{q})(\mathbf{q}^T \Lambda \mathbf{q}) + (\mathbf{q}^T \Lambda \mathbf{q})^2.$$

The integral k_1, h and k_2 correspond respectively to the fluxes of h_{01}, h_{10} and h_{11}.

Example 5.3 (N=4) In addition to h, for $N = 4$ we have the additional involutive integrals:

$$k_1 = \mathbf{p}^T \Lambda \mathbf{q} - \mathbf{q}^T \Lambda \mathbf{p}, \quad k_2 = \mathbf{p}^T \Lambda^2 \mathbf{q} \quad \text{and} \quad k_3 = \mathbf{p}^T \Lambda \mathbf{p} - 2(\mathbf{q}^T \mathbf{q})(\mathbf{q}^T \Lambda \mathbf{q}).$$

The integral k_1, k_2, h and k_3 correspond respectively to the fluxes of h_{01}, h_{02}, h_{10} and h_{11}.

6 Conclusions

We have used the multiple scales technique to derive a new integrable system of NLS equations, together with the corresponding spectral problem. The starting point was the energy-dependent Schrödinger operator (1), together with an isospectral coupled KdV system. Starting with any other integrable KdV system, a multiple scales analysis will lead to a corresponding integrable NLS system, but not, of course, always new. For instance, starting with the KdV systems associated with a given Hermitian symmetric space [4], we derive the NLS system [5], associated with the same Hermitian symmetric space. The details are given in [6].

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