Global Control of a Nonlinear Wave Equation

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Abstract

We consider the existence and numerical estimation of the distributed control of a nonlinear wave equation with an integral performance criterion and fixed final state. The problem is modified into one consisting of the minimization of a linear form over a set of pairs of positive measures satisfying linear constrains. The minimizer in the new problem is global; it can be approximated by means of the solution of a finite-dimensional linear programming problem. The solution of this problem is used to construct a nearly optimal control. An example is given to illustrate the procedure.

1 Introduction

Let D be a bounded region of the n-dimensional space $I\!R^n$ with smooth boundary ∂D, and T a positive real number, and define:
$Q_T := D \times (0,T)$, a bounded cylinder of height $T > 0$;
$\Gamma_T := \partial D \times (0,T)$, the lateral surface of the cylinder Q_T;
$D_0 := D \times \{0\}$, the base of the cylinder Q_T;
$D_T := D \times \{T\}$, the top of the cylinder Q_T.
We shall consider the following wave equation which appears in Relativistic Quantum Mechanics. (Lions [4]).

$$u_{tt} - \triangle u + |u|^\rho u = v, \tag{1}$$

for the real function $u = u(x,t)$, $(x,t) \in Q_T$, where

$$\triangle u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2},$$

and $\rho > 0$, with the boundary condition

$$u = 0 \text{ on } \Gamma_T, \tag{2}$$

and the initial conditions

$$u(x,0) = u_0(x), x \in D, \tag{3}$$

$$u_t(x,0) = u_1(x), x \in D, \tag{4}$$

the continuous function u_0 and u_1 are given.
The problem is *nonlinear* due to the presence of the term $|u|^\rho u$.
In (1), the function $(x,t) \in Q_T \longrightarrow v(x,t) \in V \subset I\!R$ is the *Distributed Control Function*, that takes the values in a *bounded* set V.
A pair (u,v) of trajectory function u and control function v is said to be *admissible* if:
(i) The function $(x,t) \longrightarrow u(x,t)$ is a solution of (1), that is in $L^\infty(0,T;H_0^1(D)\cap L^P(D))$ and satisfies (2)- (4).
(ii) The control function is continuous in Q_T.
(iii) The terminal relationships

$$\begin{aligned} u(x,T) &= g_1(x),\ x \in D \\ u_t(x,T) &= g_2(x),\ x \in D \end{aligned} \tag{5}$$

are satisfied; $g_1(x)$ and $g_2(x)$ are given continuous functions on D_T.
The set of admissible pairs will be denoted by $\mathcal{F}$. We make a further point that, since the control set V is bounded, there is a constant M_V so that

$$\|v(x,t)\| \leq M_V,\ (x,t) \in \overline{Q}_T.$$

There is a set $A \subset I\!R$, so that

$$u(x,t) \in A,\ \forall (x,t) \in \overline{Q}_T. \tag{6}$$

The optimization problem associated with this equation is as follows:
Let f_0, f_1 be continuous, non-negative, real functions on $I\!R^{n+2}$. We assume that there is a constant $h > 0$ so that

$$f_0(u(x,t),x,t) \leq h|u|,\ (u,x,t) \in A \times \overline{Q}_T.$$

Then we wish to find a minimizing pair (u,v) in $\mathcal{F}$ for the functional

$$\begin{aligned} J = &\int_{Q_T} f_0(u(x,t),x,t)dxdt + \\ &\int_{Q_T} f_1(v(x,t),x,t)dxdt. \end{aligned} \tag{7}$$

2 Metamorphosis

The integral form of the problem (1)-(4) is

$$\int_{Q_T}(u\psi_{tt} - u \bigtriangleup \psi + |u|^\rho u\psi)dxdt - \int_{Q_T}(v\psi)dxdt$$
$$= \int_{D_T}(g_1\psi_t - g_2\psi)dx + \int_{D_0} u_1\psi dx, \tag{8}$$

where $\psi \in H_0^1(\overline{Q}_T)$.
By Riesz representation Theorem, an admissible pair (u, v) defines two unique Radon measure (μ, ν), on $\Omega \times \omega$, where $\Omega := A \times Q_T$ and $\omega := V \times Q_T$ such that

$$\int_{Q_T} F(u, x, t)dxdt = \int_\Omega F d\mu,$$

for all $F \in C(\Omega)$, and

$$\int_{Q_T} G(v, x, t)dxdt = \int_\omega G d\nu,$$

for all $G \in C(\omega)$. Thus (8) changes to:

$$\int_\Omega F_\psi d_\mu + \int_\omega G_\psi d\nu = \alpha_\psi,\ \forall\psi \in H_0^1(\overline{Q}_T), \tag{9}$$

where

$$F_\psi(u(x,t), x, t) := u\psi_{tt} - u \bigtriangleup \psi + |u|^\rho u\psi$$
$$G_\psi(v(x,t), x, t) := -v\psi$$

$$\alpha_\psi := \int_{D_T}(g_1\psi_t - g_2\psi)dx + \int_{D_0} u_1\psi dx. \tag{10}$$

Now, the minimization of the functional (7) over $\mathcal{F}$ is equivalent to the minimization of

$$I(\mu, \nu) = \mu(f_0) + \nu(f_1) \tag{11}$$

where,

$$\mu(f_0) := \int_\Omega f_0 d\mu$$

$$\nu(f_1) := \int_\omega f_1 d\nu$$

over the set of measures (μ, ν) corresponding to admissible pair (u, v) subject to:

$$\mu(F_\psi) + \nu(G_{\psi)} = \alpha_\psi, \ \forall \psi \in H_0^1(\overline{Q}_T). \tag{12}$$

Up to now, we have changed only the appearance of the problem; nothing else. Now we consider the extension of our problem; we consider the minimization (11) over the set Q of all pairs of measures $(\mu, \nu) \in \mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ satisfying (12). Measures in Q, where

$$Q = \{(\mu, \nu) : \ (\mu, \nu) \in \mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$$

$$\text{and satisfy (12)}\}$$

should have some extra properties which are deduced from the definition of admissible pairs (u, v). In fact if

$$\xi : \Omega \longrightarrow I\!R$$

depends only on (x, t), then

$$\mu(\xi) = \int_{Q_T} \xi(x, t) dx dt = a_\xi \qquad (13a)$$

the Lebesgue integral of ξ over Q_T. Also if a function

$$\zeta : \omega \longrightarrow I\!R$$

depends only on (x, t), then

$$\nu(\zeta) = \int_{Q_T} \zeta(x, t) dx dt = b_\zeta \qquad (13b)$$

Now we have defined a new optimization problem. We are going to analyse this problem and start the process of approximation, that is, of building a framework so as to construct admissible trajectory control pairs which are nearly minimizer of the functional (11) and satisfies the terminal conditions (5) as nearly as desired.

3 Approximation

We back to original problem to determine under what conditions there is a pair of optimal measure (μ, ν) for the function

$$(\mu, \nu) \longrightarrow \mu(f_0) + \nu(f_1)$$

in the set Q defined by

$$\mu(F_\psi) + \nu(G_\psi) = \alpha_\psi,\ \forall \psi \in H_0^1(\overline{Q}_T)$$

$$\mu(\xi) = a_\xi$$

$$\nu(\zeta) = b_\zeta \tag{14}$$

We try to find a topology on the space $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ so that Q is compact in this topology and the function $(\mu, \nu) \longrightarrow \mu(f_0) + \nu(f_1)$ is continuous. This topology can be the $weak^* - topology$ on

$$\mathcal{S} = \Big\{(\mu, \nu) : (\mu, \nu) \in \mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)\Big\}.$$

Now we can prove:
Proposition 1: The set Q of measures (μ, ν) defined as those measures in $\mathcal{S}$ satisfying in (14) is compact in the topology induced by the weak*-topology on $\mathcal{S}$.

Proposition 2: There exists an optimal measure (μ^*, ν^*) in the set Q that minimizes

$$I(\mu, \nu) = \mu(f_0) + \nu(f_1)$$

The proof of the following proposition is much like that of Theorem 7.1 in (Kamyad, and Rubio, and Wilson [2]) and Proposition II in (Rubio [6]).
Proposition 3: The set $Q_1 \subset Q$ of measures associated with (u, v) which are piecewise constant functions on Ω and ω respectively and satisfy (14) is weakly*-dense in Q.
Up to here we have developed an infinite-dimensional program by considering minimization of

$$I(\mu, \nu) = \mu(f_0) + \nu(f_1)$$

over the set Q. Now we are going to consider the minimization of $I(\mu,\nu)$, not over the set Q, but over a subset of Q defined by requiring that only a finite number of constraints in (14) be satisfied. This will be achieved by choosing a countable set of functions

$$\Psi = \{\psi_i : \psi_i \in H_0^1(\overline{Q}_T),\ i = 1, 2, ...\}$$

that is *total* (fundamental) in $H_0^1(\overline{Q}_T)$. (see Kreyszig [3]). We also take sets of functions $\{\xi_j, j = 1, 2, ...\}$ and $\{\zeta_k, k = 1, 2, ...\}$ which are total in the respective subspaces of $C(\Omega)$ and $C(\omega)$. We shall write

$$F_i := F_{\psi_i},\ G_i := G_{\psi_i},\ \alpha_i = \alpha_{\psi_i},\ i = 1, 2, ...$$

$$a_j := a_{\xi_j},\ j = 1, 2, ...,\ b_k = b_{\zeta_k},\ k = 1, 2, ...$$

Then we have the important following proposition for our first result of approximation.

Proposition 4: Let M_1, M_2, M_3 be positive integers. Consider the problem of minimizing

$$(\mu,\nu) \longrightarrow \mu(f_0) + \nu(f_1)$$

over the set $Q(M_1, M_2, M_3)$ of measures $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ satisfying

$$\mu(F_i) + \nu(G_i) = \alpha_i,\ i = 1, 2, ..., M_1$$

$$\mu(\xi_j) = a_j,\ j = 1, 2, ..., M_2$$

$$\nu(\zeta_k) = b_k,\ k = 1, 2, ..., M_3$$

then as $M_1, M_2, M_3 \longrightarrow \infty$

$$\eta(M_1, M_2, M_3) = \inf_{Q(M_1,M_2,M_3)} [\mu(f_0) + \nu(f_1)]$$

tends to

$$\eta = \inf_Q [\mu(f_0) + \nu(f_1)].$$

By this proposition, we have limited the number of constraints in the original problem while the underlying space is still infinite-dimensional. In fact the problem is:

$$\text{Minimize}\ \ I(\mu,\nu) = \mu(f_0) + \nu(f_1) \qquad (15a)$$

over the set $Q(M_1, M_2, M_3)$ subject to:

$$\mu(F_i) + \nu(G_i) = \alpha_i,\ i = 1, 2, ..., M_1$$

$$\mu(\xi_j) = a_j,\ j = 1, 2, ..., M_2$$
$$\nu(\zeta_k) = b_k,\ k = 1, 2, ..., M_3. \qquad (15b)$$

Now we consider the construction of the suboptimal pairs of trajectories and controls for (7). We first obtain optimal measure (μ^*, ν^*) from problem like (3). There are several methods to find the numerical solution such problems, we used discretization method. Then we obtain a weak* approximation to (μ^*, ν^*) by set of two piecewise-constant functions (u, v) by means of results given in Proposition 3.

The function v obtained in this way can serve as control function for a weak solution of the problem (1)-(4). We denote this solution by u_v, where $u_v \in L^\infty(0, T, V_1)$, and

$$V_1 = H_0^1(D) \bigcap L^P(D).$$

(See Lion [4]). This pair (u_v, v) of trajectory and control function is not exactly admissible, i.e., while M_1, M_2, M_3 are not large enough, may (u_v, v) not belong to $\mathcal{F}$. We have the following important theorem

Theorem 1 Let (u_v, v) be the pair constructed as explained above. Then under the appropriate conditions of the approximations involved,

(i) The trajectory u_v as the weak solution of (1)-(4) satisfies the constraints (9) for every $\psi \in H_0^1(\overline{Q}_T)$.
(ii) As $M_1, M_2, M_3 \longrightarrow \infty$

$$\|u_v(\cdot, T) - g_1\|_{L_2} \longrightarrow 0$$

$$\|u_{vt}(\cdot, T) - g_2\|_{L_2} \longrightarrow 0$$

where g_1 and g_2 are described as (5),
(iii)

$$J(u_v, v) \longrightarrow \inf_Q I(\mu, \nu) = \eta,$$

when M_1, M_2, M_3 tend to ∞. $J(u_v, v)$ is the value of J at the pair (u_v, v).

4 Numerical Example

We consider the following nonlinear wave equation

$$u_{tt} - \triangle u + u^3 = v,$$

for the real function $u = u(x,t)$, $(x,t) \in Q_T$, where

$$u = 0 \quad \text{on } \Gamma_T,$$
$$u(x,0) = u_0(x) = 0, \quad x \in D,$$
$$u_t(x,0) = u_1(x) = 0, \quad x \in D,$$

and where $D := (0,1)$, $x \in [0,1]$, $\partial D = \{0,1\}$, $T = 1$, $t \in (0,1)$, $Q_T := (0,1)^2$, $\Gamma_T := \partial D \times (0,1)$, $D_0 = (0,1) \times \{0\}$, $D_T = (0,1) \times \{1\}$.
We assume the terminal relationship be as follows:

$$u(x,T) = g_1(x) = \sin(\pi x), \; x \in D,$$
$$u_t(x,T) = g_2(x) = 0, \; x \in D$$

Also we assume $v \in V = [-4,4]$, $u \in A = [0,4]$, $f_0 = u^2$, $f_1 = 0$, where $v \in V$ and $u \in A$, thus $\Omega := A \times Q_T = [0,1] \times (0,1)^2$, and $\omega := V \times Q_T = [-4,4] \times (0,1)^2$.
In this example we chose $\psi(x,t) = t^p \sin(xl\pi)$. Six such functions ψ were chosen, with values of $p = 1,2$ and $l = 1,2,3$.
The 64 functions ξ were chosen as the square $[0,1] \times [0,1]$ was divided into 64 equal squares, the functions ξ being the characteristic function of the individual squares. Similarly, the 64 functions of ζ were chosen. Thus the total numbers of constraints are $m = 6 + 64 + 64 = 134$. We divide the space Ω to 512 subspace and the space of ω to 512 subspace, thus the number of variables in the discretization form is 1024. In this way we found the following problem:

$$Minimize \quad \sum_{i=1}^{512} \alpha_i u_i^2 + \sum_{j=1}^{512} \beta_j v_j^2$$

subject to:

$$\alpha_i \geq 0, \quad \beta_j \geq 0,$$
$$\sum_{j=1}^{512} \alpha_j F_i(Z_j) + \sum_{j=1}^{512} \beta_j G_i(z_j) = a_i, \; i = 1,2,...,6$$

$$\sum_{j=1}^{512} \alpha_j \xi_j(Z_j) = \frac{1}{64},\ i = 1, 2, ..., 64$$

$$\sum_{j=1}^{512} \beta_j \zeta_j(z_j) = \frac{1}{64},\ i = 1, 2, ..., 64.$$

This finite-dimensional linear program were solved by a modified simplex method, taking a total of 269 iterations. The objective function took the value 1.419222×10^{-1}. Then the nearly optimal control v constructed: we used the method in (Farahi, and Rubio, and Wilson [1]). The graph of the resulting control v is shown in Fig. 1.

Now we use the control function v obtained above to find $u(x,T)$; the final

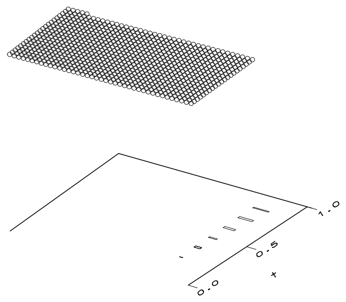

Figure 1: The nearly optimal control $v(x,t)$

value of the solution of the system (1)-(4). We use an implicit difference method (see Mitchell [5]). The graph of $U(x,T)$; the final value of the solution, produced by this scheme is shown in Fig. 2.

Conclusion

We have developed a method for the transformation of nonlinear wave equations into a finite-dimensional linear programming form, so that, by using this new problem we can estimate the optimal control.

We expect this approach to be useful for the treatment of other optimal control problems associated with linear or nonlinear hyperbolic differential equations.

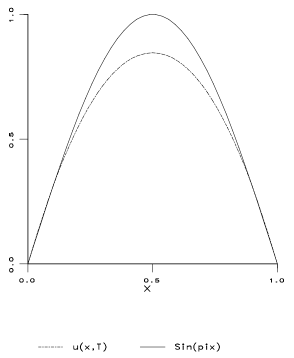

Figure 2: Functions $u(x,T)$ and $\sin(\pi x)$

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