ON THE MULTIPLY PERIODIC SCHRÖDINGER OPERATORS

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1. Introduction

This paper presents an approach, based on the results of the theory of the hyperelliptic Kleinian σ–function, to the problem of general interest:

$$\left(\sum_{i,j=1}^{g}\Lambda_{ij}(\mathbf{u})\partial_i\partial_j+\mathcal{U}(\mathbf{u})\right)\Phi(\mathbf{u};\alpha)=0, \tag{1.1}$$

where Λ_{ij} are some functions, $\mathcal{U}$ is a multiply periodic potential, Φ is a Bloch function. Actually, here we will deal with the case $\Lambda_{ij}=\delta_{ij}$ (the Kronecker δ).

At the base of our investigation lies the explicit construction and analytical properties of the hyperelliptic Baker function. The Baker function here is such an object, that has very much in common with the Baker–Akhiezer function, investigated very extensively by Krichever (see e.g.[6]). We use the explicit expression of the Baker function as the starting point to develop by elementary algebraic means the description of the set of second order linear differential operators such that (1.1) holds.

This construction has become possible due to recent investigations by V.M. Buchstaber and the authors on the hyperelliptic Kleinian σ–functions [4, 3, 2, 7].

The multi dimensional spectral problem for Bloch functions was discussed by Novikov and coworkers since the middle of the seventies (see e.g. [8]). In [5] the multi dimensional spectral problem for Schrödinger equation under the action of an external magnetic field was considered and the corresponding Bloch solutions were constructed for one value of the spectral parameter, which appeared to be a coordinate on a compact Riemann surface. Afterwards in [11] and [10] the *pure potential problem* (i.e. without external field) was solved for a fixed energy level for special hyperelliptic curves which permit involution with two stable points and are therefore a Prym variety. Novikov [8] put a question about the existence of multi dimensional potentials for which the Bloch variety is an algebraic variety of positive dimension. In the paper we consider only the case of one dimensional spectral variety. The discussion of the relation of the present paper's results to the general theory will be given elsewhere.

The paper is organized in the following way. We define the Bloch function and consider then the family of commuting operators for which it is an eigenfunction. By imposing the condition that the operators are realized as 2×2 traceless matrices polynomial in the spectral parameter and multiply periodic in space variables, we construct explicitly all the family of operators in terms of hyperelliptic abelian functions, the whole construction being based on the properties of the hyperelliptic Baker function. As the result we express the multiply periodic potential of Schrödinger equation in terms of operators entries.

2. Bloch function

2.1. **Notations.** A vector $\mathbf{u}$ belongs to g–dimensional space $\mathbb{C}^g$; Γ denotes a lattice in $\mathbb{C}^g$ generated by the set of $2g$ vectors $\mathbf{e}_m^T\omega, \mathbf{e}_n^T\omega'$, $m, n = 1, \dots, g$, over $\mathbb{Z}^{\not\in\eth}$, where $\mathbf{e}_k^T = (\delta_{1n}, \dots, \delta_{gn})$, and matrices ω and ω' are nondegenerate $g\times g$; $\mathcal{M}$ is a subset in $\mathbb{C}^g/\Gamma$; below α denotes a point in $\mathcal{M}$.

In this paper we shall call the function $\Phi : \mathbb{C}^{\eth} \times \mathcal{M} \to \mathbb{C}$ the *Bloch function*, if for any vector from the lattice, $\Omega(\mathbf{m}, \mathbf{n}) = \mathbf{m}^T\omega + \mathbf{n}^T\omega' \in \Gamma$, $\mathbf{m}, \mathbf{n} \in \mathbb{Z}$, Φ has the periodicity property $\forall\alpha \in \mathcal{M}$

$$\Phi(\mathbf{u} + \Omega(\mathbf{m}, \mathbf{n}); \alpha) = \xi_i(\alpha)\Phi(\mathbf{u}; \alpha), \quad \mathbf{m}, \mathbf{n} \in \mathbb{Z}. \tag{2.2}$$

The set $\mathcal{M}$ is called the *Bloch set*.

Denote $\partial_k = \partial/\partial u_k$, and the derivatives of $\Phi(\mathbf{u}; \alpha)$, $\Phi_i = \partial_i\Phi(\mathbf{u}; \alpha)$, $\quad \Phi_{ij} = \partial_i\partial_j\Phi(\mathbf{u}; \alpha)$ etc.

2.2. **A family of linear operators.** Let $\Psi^T = (\Phi, \Phi_g)$. Consider the family of 2×2 matrix operators,

$$L_0, L_1, \dots, L_g, \quad L_k = \begin{pmatrix} V_k & U_k \\ W_k & -V_k \end{pmatrix} \tag{2.3}$$

with entries, U_k, V_k, W_k, polynomial in x and multiply periodic functions in $\mathbf{u} \in \mathbb{C}^g$ with respect to the lattice Γ. Define the family (2.3) by the equalities

$$L_0\Phi = -\frac{1}{2}y\Psi, \quad L_k\Phi = \frac{\partial}{\partial u_k}\Psi, \; k = 1, \dots, g, \tag{2.4}$$

The compatibility conditions for the action of the operators L_k read as

$$[L_k, L_0] = \frac{\partial L_0}{\partial u_k}, \quad [L_k, L_i] = \frac{\partial L_i}{\partial u_k} - \frac{\partial L_k}{\partial u_i}. \tag{2.5}$$

It follows from (2.5) that $\det L_0$ is a constant with respect to the variables $\mathbf{u}$, so we can infer that the pair of quantities (y, x) defines a point on the hyperelliptic curve V (note that entries of L's are the polynomials in x by definition),

$$y^2 = V_0^2(x) + U_0(x)W_0(x), \tag{2.6}$$

of some genus $g = \deg U_0(x)$, for which we assume the form

$$y^2 = \sum_{i=0}^{2g+2} \lambda_i x^i = 4 \prod_{i=0}^{2g+2} (x - a_i) \tag{2.7}$$

with $a_i \neq a_j \in \mathbb{C}, i, j = 1, \dots, 2g+2$ and $\lambda_k = (-1)^k s_k(a_1, \dots, a_{2g+2})$, where s_i denotes an elementary symmetrical function of degree i of its arguments.

3. Baker function

3.1. **Definition of the Kleinian σ–function.** Equip the hyperelliptic curve (2.7) by the canonical basis of cycles $(A_1, \dots, A_g; B_1, \dots, B_g)$ in $H_1(V, \mathbb{Z})$ with the intersection matrix $I = \begin{pmatrix} 0 & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix}$, where $\mathbf{1}_g$ be the unit $g \times g$–matrix. Let $\{du_i\}_{i=1}^g$, $du_i = x^{i-1}dx/y, \quad i = 1, \dots, g$ are canonical holomorphic differentials in $\mathcal{H}^1(V, \mathbf{C})$. The $g \times g$–matrices of A and B–periods, $2\omega = \left(\oint_{A_k} du_l\right), \quad 2\omega' = \left(\oint_{B_k} du_l\right)$ are known to be nondegenerate. Under the action of the transformation $(2\omega)^{-1}$ in $\mathcal{H}^1(V, \mathbf{C})$ the vector $d\mathbf{u}^T = (du_1, \dots, du_g)$ maps to the vector of normalized holomorphic differentials $d\mathbf{v}^T = (dv_1, \dots, dv_g)$, i.e.the vector from $\mathcal{H}^1(V, \mathbb{C})$ which components are normalized by the conditions $\oint_{A_k} v_k = \delta_{kl}, k, l = 1 \dots, g$. It is known, that $g \times g$ matrix,$\tau = \left(\oint_{B_k} v_l\right) = \omega^{-1}\omega'$ belongs to the upper Siegel half space $\mathcal{S}_g$ of degree g, i.e. it is symmetric and has a positively defined imaginary part.

Let us introduce the standard abelian differential of the second kind

$$dr_j = \sum_{k=j}^{2g-j} (k - j + 1)\lambda_{k+j+1} \frac{x^k dx}{4y}, \quad i = 1, \dots, g \tag{3.8}$$

and denote their matrices of A and B–periods, $\eta = \left(-\oint_{A_k} dr_l\right)$, $\quad \eta' = \left(-\oint_{B_k} dr_l\right)$.

$2g\times 2g$–matrix $\mathcal{G} = \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}$ belongs to PSp_{2g}, i.e.

$$\mathcal{G}\begin{pmatrix} 0 & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix}\mathcal{G}^T = -\frac{\pi i}{2}\begin{pmatrix} 0 & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix}. \tag{3.9}$$

Let denote by T^g the torus $\mathbb{C}^g/\mathbf{1}_g \oplus \tau$ which is the Jacobian of (2.7).

Let $\mathcal{D}\in V$ be the divisor, $\mathcal{D}=\mathcal{A}-\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are the divisors of the same degree, $\mathcal{A}=\{a_1,\ldots,a_g\}, \mathcal{B}=\{b_1,\ldots,b_g\}\in V$. The Abel map $\mathfrak{A}(\mathcal{D}):\mathcal{D}\to J(V)$ put into the correspondence with the divisor $\mathcal{D}$ the point $\zeta=(\zeta_1\ldots,\zeta_g)\in J(V)$ according to the rule

$$\mathcal{D}\Leftrightarrow\zeta=\int_{\mathcal{B}}^{\mathcal{A}} d\mathbf{v} = \left(\sum_{k=1}^{g}\int_{b_k}^{a_k} dv_1,\ldots,\sum_{k=1}^{g}\int_{b_k}^{a_k} dv_g\right).$$

Let $\mathcal{D}=\{x_1,\ldots,x_g\}-\{gx_0\}$ be a divisor and $\mathbf{u}^T=(u_1,\ldots,u_g)$ its image under the action of the Abel map in $J(V)$ being defined by the formula

$$u_k=\sum_{j=1}^{g}\int_{x_0}^{x_j}\frac{x^{k-1}dx}{y},\quad k=1,\ldots,g, \tag{3.10}$$

where $x_0\in V$ is an arbitrary point.

Definition *Hyperelliptic Kleinian σ-function is defined by the formula*

$$\sigma(\mathbf{u}) = C\mathrm{e}^{\mathbf{u}^T a\mathbf{u}}\theta((2\omega)^{-1}\mathbf{u}-\mathbf{K}_{x_0}|\tau), \tag{3.11}$$

where $a=(2\omega)^{-1}\eta$ be a symmetric $g\times g$–matrix,

$$\theta(\mathbf{v}|\tau)=\sum_{\mathbf{m}\in\mathbb{Z}^g}\exp\pi i\left\{\mathbf{m}^T\tau\mathbf{m}+2\mathbf{v}^T\mathbf{m}\right\}$$

is the standard theta function on $T^g\times\{\tau=\omega^{-1}\omega'\}$, $\mathbf{K}_{x_0}$ is the vector of Riemann constants with the basis point x_0, C is some constant.

At $g=1$ the function $\sigma(u)$ is the standard Weierstrass σ–function.

3.2. **Jacobi inversion problem.** ζ and $\wp$–functions are defined as $\zeta_i=\partial_i\ln\sigma(\mathbf{u})$, $\wp_{\mathrm{ij}}=-\partial_{\mathrm{i}}\partial_{\mathrm{j}}\ln\sigma(\mathbf{u})$, $\wp_{ijk}=-\partial_i\partial_i\partial_k\ln\sigma(\mathbf{u})$, etc.

Theorem 3.1. *Jacobi inversion problem is solved as follows: put into the correspondence to the point $(u\in T^g)\to(\wp(u),\wp'(u))\to(x_k,y_k)\in$*

V, $k=1,\dots g$, where $x_k, k=1,\dots,g$ is the set of the roots of the equation

$$x^g - \sum_{k=1}^{g} \wp_{ig}(u)x^{k-1} = 0 \tag{3.12}$$

and y_n is given for every x_n by the formula

$$y_n = \sum_{k=1}^{g} \wp_{kng}(u)x_k^{l-1} \tag{3.13}$$

We note, that any $g+1$ functions $\wp_{ij}(\mathbf{u})$ are algebraically dependent, hence the point $\wp_{ij} \in \mathbb{C}^{g(g+1)/2}$ belongs to $g(g-1)/2$–dimensional variety, which is hyperelliptic Kummer variety which is an intersection of quadrics. An explicit description of this variety in terms of $\wp$–coordinates is given in [4].

3.3. **Definition of the Baker function.** Let $\mathbf{u}\times\alpha$ is an arbitrary points in $J(V)\times\mathcal{M}$. The Baker function of genus g (see [1], p.421) is defined by the formula

$$\Phi(\mathbf{u};\alpha) = \frac{\sigma(\alpha-\mathbf{u})}{\sigma(\alpha)\sigma(\mathbf{u})}\exp\{\zeta^{\mathrm{T}}(\alpha)\mathbf{u}\}, \tag{3.14}$$

where σ is the canonical σ-function $\zeta^T(\alpha)\mathbf{u} = \sum_{i=1}^{g}\zeta_i(\alpha)u_i$.

The Baker function has the following periodicity property,

$$\begin{aligned} \Phi(\mathbf{u};\alpha+\Omega(\mathbf{m},\mathbf{m}')) &= \Phi(\mathbf{u};\alpha), \\ \Phi(\mathbf{u}+\Omega(\mathbf{m},\mathbf{m}');\alpha) &= e^{\zeta^T(\alpha)\Omega(\mathbf{m},\mathbf{m}')-\eta^T(\mathbf{m},\mathbf{m}')\alpha}\Phi(\mathbf{u};\alpha) \end{aligned} \tag{3.15}$$

In the case, when $\mathbf{u}$ is the image of nonspecial divisor on V and $\alpha = \int_{x_0}^{x}\mathbf{v}$, then the Baker function is a function on the curve V and represent the so called Baker–Akhiezer function which studies was originated by Krichever [6]. Hence in the case of the consideration the Bloch variety $\mathcal{M}$ is one dimensional and presents the image of the curve under the action of the Abel map.

4. Construction of the family of linear operators

4.1. ζ–**formula.** The construction is based on the following relation for the Kleinian ζ–function (see, e.g. [1])

Theorem 4.1. *Let $V=(y,x)$ be the hyperelliptic curve equipped with given homology basis. Let $x_0, x_1\dots, x_g$ are arbitrary points on V $a_1,\dots,a_g$ are the left edges of the gaps and a_0 is arbitrary point. Then the following principal relation for the hyperelliptic ζ–functions is valid*

$$-\quad \zeta_j\left(\sum_{k=0}^{g}\int_{a_k}^{x_k}d\mathbf{u}\right)=\sum_{k=0}^{g}\int_{a_k}^{x_k}dr_j-\mathcal{Z}_j(\mathbf{x},\mathbf{y}), \tag{4.16}$$

where

$$\mathcal{Z}_j(\mathbf{x},\mathbf{y})=\sum_{k=0}^{g}\frac{y_k}{2\mathcal{R}'(x_k)}\mathcal{S}_{g-j}(x_k\mid x_0,x_1,\ldots,\hat{x_k},\ldots,x_g), \tag{4.17}$$

and $\mathcal{R}(z)=(z-x)\mathcal{P}(z)$,

$$\mathcal{S}_m(x\mid t_1,\ldots,t_n)=\sum_{l=0}^{m}(-1)^l x^{g-l}s_l(t_1,\ldots,t_n)$$

and $s_0=1,\ s_l(t_1,\ldots,t_n),\ \ l=1,\ldots$ *are elementary symmetric functions of degree* l *of the variables* $t_1,\ldots,t_n$

$$-\quad \zeta_j\left(\sum_{k=1}^{g}\int_{a_k}^{x_k}d\mathbf{u}\right)=\sum_{k=1}^{g}\int_{a_k}^{x_k}dr_j+\frac{1}{2}\wp_{g,g,j+1} \tag{4.18}$$

where

The formula (4.16) can be interpreted as an addition theorem of the form *point* (x_0) + *divisor* $(x_1+\ldots+x_g)$; in the case $g=1$ it presents the standard addition law for the Wierstrass ζ–functions. Tending in (4.16) $x\to a_0$ and then $a_0\to\infty$ we derive another important formula

A number of important relations follows from this formula, in particular,

$$\begin{aligned}
\wp_{gggi} &= (6\wp_{gg}+\lambda_{2g})\wp_{gi}+6\wp_{g,i-1}-2\wp_{g-1,i}+\frac{1}{2}\delta_{gi}\lambda_{2g-1}, && (4.19)\\
\wp_{ggi}\wp_{ggk} &= 4\wp_{gg}\wp_{gi}\wp_{gk}-2(\wp_{gi}\wp_{g-1,k}+\wp_{g,k}\wp_{g-1,i})\\
&+ 4(\wp_{gk}\wp_{g,i-1}+\wp_{gi}\wp_{g,k-1})+4\wp_{k-1,i-1}-2(\wp_{k,i-2}+\wp_{i,k-2})\\
&+ \lambda_{2g}\wp_{gk}\wp_{gi}+\frac{\lambda_{2g-1}}{2}(\delta_{ig}\wp_{kg}+\delta_{kg}\wp_{ig})+c_{(i,k)}, && (4.20)
\end{aligned}$$

where $c_{(i,k)}=\lambda_{2i-2}\delta_{ik}+\frac{1}{2}(\lambda_{2i-1}\delta_{k,i+1}+\lambda_{2k-1}\delta_{i,k+1})$.

The first of them describes the KdV flows; the second one describes the Jacobian (see [4]). Both sets of the equations, as well the equation defining the Kummer variety, can be written in a matrix form [4].

4.2. **Umbral derivatives.** Introduce the "umbral derivative" (see e.g. [9]), D_j acting on a polynomial $P(x)=\sum_{k=0}^n p_k x^k$ of degree n by the formula

$$D_j P(x)=\left(\frac{\sum_{k=0}^n p_k x^k}{x^j}\right)_+=\sum_{k=j}^n p_k x^{j-k}, \tag{4.21}$$

where + in subscript means the positive part of Laurent series. Then

Proposition 4.1. *The term $\mathcal{Z}_j(\mathbf{x},\mathbf{y})$ in (4.16) can be written in the form.*

$$\mathcal{Z}_j(\mathbf{x},\mathbf{y})=\frac{1}{2\mathcal{P}(x;\mathbf{u})}\left(yD_j+\partial_j\right)\mathcal{P}(x;\mathbf{u})+\frac{1}{2}\wp_{g,g,j+1}(\mathbf{u}), \tag{4.22}$$

where D_j is the umbral derivative of the order j and $\partial_j=\partial/\partial u_j$

Proof. We rewrite $\mathcal{Z}_j(\mathbf{x},\mathbf{y})$ in the form

$$\begin{aligned}
\mathcal{Z}_j(\mathbf{x},\mathbf{y}) &= \frac{y}{2\mathcal{P}(x;\mathbf{u})}\mathcal{S}_{g-j}(x|x_1,\dots,x_g)\\
&+ \sum_{k=1}^g \frac{y_k}{2P'(x_k)(x_k-x)}\mathcal{S}_{g-j}(x_k|x_1,\dots,x_g)\\
&+ \sum_{k=1}^g \frac{y_k}{2P'(x_k)}\mathcal{S}_{g-j-1}(x_k|x_1,\dots,\hat{x}_k,\dots,x_g)\\
&= \frac{y}{2\mathcal{P}(x;\mathbf{u})}\mathcal{S}_{g-j}(x|x_1,\dots,x_g)\\
&+ \sum_{k=1}^g \frac{y_k}{2P'(x_k)(x_k-x)}\mathcal{S}_{g-j}(x_k|x_1,\dots,x_g)+\frac{1}{2}\wp_{g,g,j+1}(\mathbf{u}),
\end{aligned}$$

where (3.12,3.13) and (4.18) were used. According to the definition of the umbral derivative $D_j\mathcal{P}(x;\mathbf{u})=\mathcal{S}_{g-j}(x|x_1,\dots,x_g)$. To calculate the second sum we note, that $\mathcal{S}_{g-j}(x_k|x_1,\dots,x_g)=(-1)^{g-j}s_{g-j}(x_1,\dots,\hat{x}_k,\dots,x_g)$. Hence using this equation an also

$$\frac{\partial x_k}{\partial u_j}=\frac{y_k s_{g-j}(x_1,\dots,\hat{x}_k,\dots,x_g)}{\prod_{i\neq j}(x_i-x_j)},$$

we have

$$\begin{aligned}
&\sum_{k=1}^g \frac{y_k}{2P'(x_k)(x_k-x)}\mathcal{S}_{g-j}(x_k|x_1,\dots,x_g)\\
=\ &\frac{(-1)^{j+1}}{2\mathcal{P}(x;\mathbf{u})}\sum_{k=1}^g\sum_{l=1}^g(-1)^{l-1}x^{g-l}\frac{\partial x_k}{\partial u_{g-l+1}}s_{g-j}(x_1,\dots,\hat{x}_k,\dots,x_g)
\end{aligned}$$

$$= \frac{(-1)^{j+1}}{2\mathcal{P}(x;\mathbf{u})}\sum_{l=1}^{g} x^{g-l}\frac{\partial}{\partial u_{g-l}} s_{g-j+1}(x_1,\dots,x_g)$$
$$= \frac{1}{2\mathcal{P}(x;\mathbf{u})}\sum_{l=1}^{g} x^{g-l}\wp_{g,l,j}(\mathbf{u}) = \frac{1}{2\mathcal{P}(x;\mathbf{u})}\frac{\partial}{\partial u_j}\mathcal{P}(x;\mathbf{u})$$

Hence the proposition is proved.

Introduce the functions

$$\psi_i = \partial_i\Phi(\mathbf{u};\alpha) = -\zeta_i(\alpha-\mathbf{u}) - \zeta_i(\mathbf{u}) + \zeta_i(\alpha), \quad \psi_{ij} = \partial_j\psi_i, \dots$$

It follows from (4.16) and (4.22) that

$$\psi_i = \frac{(yD_i + \partial_i)\mathcal{P}(\mathbf{u};\alpha)}{\in\mathcal{P}(\mathbf{u};\alpha)}, \tag{4.23}$$

and, in particular,

$$\psi_g = \frac{y + \partial_g\mathcal{P}(\S;\mathbf{u})}{\in\mathcal{P}(\mathbf{u};\alpha)}, \tag{4.24}$$

4.3. **Main theorem.** Below we give the theorem which gives the explicit construction for the family of operators of the Section 2.2.

Theorem 4.2. *Let Φ is Baker function, $\Phi = \Phi(\mathbf{u};\alpha)$ with $\alpha = \mathfrak{A}(x - a)$ and $\mathbf{u} = \mathfrak{A}(x_1 + \ldots + x_g - a_1 - \ldots - a_g)$. Then the family of operators (2.3) satisfying the compatibility conditions (2.5) can be found with the entries of the matrices L_k being polynomials on x over the field of meromorphic hyperelliptic functions and having the form*

$$U_k = D_k\mathcal{P}(x;\mathbf{u}) = x_{g-i} - \sum_{k=1}^{g-i}\wp_{g,i+k}x^{k-1}, \tag{4.25}$$

$$V_k = -\frac{1}{2}\partial_g U_k = \frac{1}{2}\sum_{k=1}^{g-i}\wp_{g,g,i+k}x^{k-1}, \tag{4.26}$$

$$\begin{aligned} W_k &= -\frac{1}{2}\partial_g^2 U_k + U_k W_g \\ &= x^{g-k+1} + \left(\wp_{gg} + \frac{\lambda_{2g}}{4}\right)x^{g-k} \\ &+ \sum_{i=1}^{g-k}\left[\left(\wp_{gg} + \frac{\lambda_{2g}}{4}\right)\wp_{g,i+k} + 2\wp_{g,i+k-1} - \wp_{g-1,i+k}\right. \\ &+ \left.\delta_{g,i+k}\frac{\lambda_{2g-1}}{4}\right]x^{k-1} + \wp_{gk}. \end{aligned} \tag{4.27}$$

Proof. Consider the equations (2.3) At $k = g$ we have

$$V_g + U_g\psi_g = \psi_g, \quad W_g - V_g\psi_g = \psi_{gg} + \psi_g^2. \tag{4.28}$$

Taking into the account that $U_g = 1, \quad V_g = 0$ we obtain in this case only the equation

$$W_g = \psi_{gg} + \psi_g^2. \tag{4.29}$$

and direct calculation involving equalities (4.19,product3) comes to the expression

$$W_g = x + 2\wp_{gg}(\mathbf{u}) + \frac{\lambda_{2g}}{4} \tag{4.30}$$

Let further $0 < k < g$, then

$$V_k + U_k\psi_g = \psi_k, \tag{4.31}$$

$$W_k - V_k\psi_g = \psi_{gk} + \psi_g\psi_k. \tag{4.32}$$

Differentiating (4.31) by u_g, adding the result to the product (4.31) with ψ_g and subtracting (4.32) we obtain

$$\left(\frac{\partial U_k}{\partial u_g} + 2V_k\right)\psi_g + \frac{\partial V_k}{\partial u_g} + U_kW_g - W_k = 0 \tag{4.33}$$

This equation is satisfied because of (4.25). Hence it remains to prove (4.31). Really, using (4.23) we write

$$\begin{aligned}\psi_k - U_k\psi_g = {}& -\frac{1}{2}\wp_{g,g,k+1}(\mathbf{u}) + \sum_{l=1}^{g}\frac{y_l}{2P'(x_l)(x_l - x)} \\ & \times\left(\mathcal{S}_{g-k}(x_l|x, x_1, \ldots, x_l, \ldots, x_g) - \mathcal{S}_{g-k}(x|x_1, \ldots, x_g)\right)\end{aligned}$$

Using the equalities

$$\begin{aligned}&\mathcal{S}_{g-k}(x_l|x, x_1, \ldots, x_l, \ldots, x_g) - \mathcal{S}_{g-k}(x|x_1, \ldots, x_g) \\ &= (x_l - x)\left(\mathcal{S}_{g-k-1}(x_l|x, x_1, \ldots, x_l, \ldots, x_g) + \mathcal{S}_{g-k-1}(x|x_1, \ldots, x_g)\right)\end{aligned}$$

and

$$\wp_{g,g,k+1}(\mathbf{u}) = (-1)^k\sum_{l=1}^{g}\frac{y_l}{2P'(x_l)}s_{g-k-1}(x_k|x_1, \ldots, \hat{x}_l, \ldots, x_g) \tag{4.34}$$

we prove (4.31).

At $k = 0$ we have

$$V_0 + U_0\psi_g = -\frac{1}{2}y, \quad W_0 - V_0\psi_g = -\frac{1}{2}y$$

The first from these equation coincides with that given in (4.26) in virtue of the form of ψ_g; the second ones comes to the formula

$$W_g = \frac{y^2 - \mathcal{P}_g^2(x;\mathbf{u}) + 2\mathcal{P}(x;\mathbf{u})\mathcal{P}_{gg}(x;\mathbf{u})}{4\mathcal{P}^2(x;\mathbf{u})}, \tag{4.35}$$

which validity holds because of (4.19,4.20).

4.4. **Multidimensional Schrödinger equation.** In this section we apply the given above theorem to construct a solution for the problem formulated in the introduction. Really one can see that the definition of the operators L_k yields the second order differential equation for Φ for any $k, l = 1, \dots, g$

$$\begin{aligned} \frac{\Phi_{lk}}{\Phi} &= \frac{1}{2}\left(\frac{\partial U_k}{\partial u_l} + \frac{\partial U_l}{\partial u_k}\right)\frac{\Phi_g}{\Phi} \\ &+ V_l V_k + \frac{1}{2}\left(\frac{\partial V_k}{\partial u_l} + \frac{\partial V_l}{\partial u_k} + U_k W_l + W_k U_k\right) \end{aligned} \tag{4.36}$$

with the spectral parameters (y, x) being the coordinates of the curve (2.7).

For the case $i, j = g$ we obtain the know result of the finite–gap integration method.

Lemma 4.1. *The following formula is valid*

$$\frac{\Phi_{gg}(\mathbf{u};\alpha)}{\Phi(\mathbf{u};\alpha)} = 2\wp_{gg}(\mathbf{u}) + x + \frac{1}{4}\lambda_{2g}$$

Proof. We calculate that

$$\begin{aligned} \psi_g(\mathbf{u};\alpha) &= \frac{1}{2\mathcal{P}(\S;\mathbf{u})}(y + \mathcal{P}_g(x;\mathbf{u})) \\ \psi_{gg}(\mathbf{u};\alpha) &= \frac{1}{2\mathcal{P}^{\in}(\S;\mathbf{u})}(\mathcal{P}_{gg}(x;\mathbf{u})\mathcal{P}(x;\mathbf{u}) - (y + \mathcal{P}_g(x;\mathbf{u})\mathcal{P}_g(x;\mathbf{u}) \end{aligned} \tag{4.37}$$

The required relation follows from the expression for three and four indexed functions.

For other cases the expression (4.36) can be simplified: this result will be published elsewhere. Below we consider only the example.

4.5. **Example: genus two.** Consider the case of genus two. The Kummer variety is described by the equation $\det \mathfrak{K} = 0$, where

$$\mathfrak{K} = \begin{pmatrix} -\lambda_0 & \frac{1}{2}\lambda_1 & 2\wp_{11} & -2\wp_{12} \\ \frac{1}{2}\lambda_1 & -(\lambda_2 + 4\wp_{11}) & \frac{1}{2}\lambda_3 + 2\wp_{12} & 2\wp_{22} \\ 2\wp_{11} & \frac{1}{2}\lambda_3 + 2\wp_{12} & -(\lambda_4 + 4\wp_{22}) & 2 \\ -2\wp_{12} & 2\wp_{22} & 2 & 0 \end{pmatrix}. \tag{4.38}$$

defines the Kummer surface, which is quartic in $\mathbb{C}^3$.

The following theorem is valid

Theorem 4.3. *The following formulae are valid*

$$\left(-\partial_2^2 + x + 2\wp_{22} + \frac{\lambda_4}{4}\right)\Phi(\mathbf{u};\alpha) = 0, \tag{4.39}$$

$$\begin{aligned} &\left(-\partial_2\partial_1 - \frac{1}{2}\partial_2\wp_{222} + x^2\right. \\ + &\left.\left(\wp_{22} + \frac{\lambda_4}{4}\right)x + \wp_{22}\left(\wp_{22} + \frac{\lambda_4}{4}\right) + 2\wp_{12} + \frac{\lambda_3}{4}\right)\Phi(\mathbf{u};\alpha) = 0, \end{aligned} \tag{4.40}$$

$$\begin{aligned} &\left(-\partial_1^2 - \wp_{122}\partial_2 + x^3 + \frac{\lambda_4}{4}x^2 + \left(2\wp_{12} + \frac{\lambda_3}{4}\right)x\right. \\ + &\left.\ 2\wp_{12}\left(\wp_{22} + \frac{\lambda_4}{4}\right) + \frac{\lambda_2}{4}\right)\Phi(\mathbf{u};\alpha) = 0. \end{aligned} \tag{4.41}$$

Proof is straightforward: use (4.36), (4.25–4.27) and the relations from (4.19) and (4.20) which read:

$$\begin{aligned} \wp_{2222} &= 6\wp_{22}^2 + \frac{1}{2}\lambda_3 + \lambda_4\wp_{22} + 4\wp_{12}, \\ \wp_{1222} &= 6\wp_{22}\wp_{12} + \lambda_4\wp_{12} - 2\wp_{11} \\ \wp_{222}^2 &= 4\wp_{11} + \lambda_3\wp_{22} + 4\wp_{22}^3 + 4\wp_{12}\wp_{22} + \lambda_4\wp_{22}^2 + \lambda_2 - 4\lambda_0\wp_{22}\wp_{11} + 4\wp_{11}^3 \end{aligned}$$

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