Ladder Theorems and Length Scales in Solutions of a Generalized Diffusion Model

December 12, 1995

M.V.Bartuccelli   *S.A.Gourley*
Department of Mathematical and Computing Sciences,
University of Surrey, Guildford GU2 5XH, England

Abstract

Length scales involved in dissipative partial differential equations are arguably one of the most important dynamical concepts for properly understanding the spatio-temporal patterns of dissipative flows. Here, a set of length scales for the solutions of the dissipative partial differential equation

$$u_t = -\alpha\nabla^4 u - \beta\nabla^2 u + \gamma\nabla^2 u^{2q+1} + \lambda u - \delta u^2,$$

on periodic boundary conditions and in various spatial dimensions have been investigated. This model, in the case $q = 1$, was first studied in the context of population dynamics by Cohen and Murray [1]. Our length scales are based on ratios of norms, which involve a set of differential inequalities proved for the above equation. Lower bounds are derived for the time averages of these length scales.

SUBMITTED TO “EUROPEAN JOURNAL OF APPLIED MATHEMATICS”

1 Introduction

An interesting and important question in the study of the behaviour of solutions of dissipative partial differential equations, is whether it can be shown that a set of well defined length scales are obtainable directly from the solutions of the equations themselves. Length scales are one of the most important dynamical ingredient for understanding the spatio-temporal patterns of the solutions of any dissipative PDE. In particular they give information about the smallest features in the flows.

The most famous length scale is the Kolmogorov length scale, obtained by Kolmogorov in 1941 for incompressible fluid flows using scaling properties of the Navier-Stokes equations and few other assumptions like homogeneity and isotropy for the probability distribution function of differences of velocities [2]. The Kolmogorov length scale is also known as the dissipation length scale, and it is defined as the length scale below which the dynamics is completely dominated (smoothed) by dissipation. Any numerical scheme must 'resolve' these dissipation length scales in order to get an accurate representation of the flow.

In the theory of global attractors in dissipative PDE's one measure the dissipation length scale using upper bounds on the Lyapunov dimension of the universal attractor [3, 4]. This upper bound $\mathcal{N}$, is usually interpreted as the number of degrees of freedom (or relevant modes) in the PDE under investigation. On a domain $[0, L]^d$ of volume L^d (d spatial dimensions), $\mathcal{N}$ is related to the minimum scale l by $\mathcal{N} = (L/l)^d$. In other words, $\mathcal{N}$ represents the number of eddies of volume l^d which are contained in the box of volume L^d. The only problem that exists with this approach is whether computing a lower bound on l from an upper bound of the Lyapunov dimension is always appropriate. In general this identification is not quite so simple as one might hope. We have illustrated this problem for the Complex Ginzburg-Landau equation on periodic boundary conditions [5, 7], where we showed that for this equation when $dq \geq 2$, large fluctuations can occur in certain regions of the (μ, ν) plane but not in the rest of this plane. These large fluctuations in the case $d = 2$ and $q = 1$ occur on scales $l \geq R^{-|\nu|}$, whereas in the rest of the plane, where only weak fluctuations can occur, $l \geq R^{-1}$. The Lyapunov dimension of the attractor in both regions was found to be uniform giving $l \geq R^{-1}$.

In some sense this result is not surprising as the method of computing the evolution of volume elements of the universal attractor through which one finds an upper bound on the Lyapunov dimension, requires a time average on low norms of the solutions. It contains no information on higher norms, in which the energy may be appreciable during the time interval of a large fluctuation.

It therefore appears necessary to find an alternative definition of a length scale which takes account of the possibility of large fluctuations in the solutions of any dissipative PDE.

This paper is dedicated to the problem of defining the smallest length scales for an equation modelling population dynamics.
Our length scales will be defined in much the same way they were defined in [6, 7, 8].

However, here we do not introduce any additional term in the ladder (see below), and we show that we can obtain lower bounds for the time averages of these length scales.

The equation we study has the following form:

$$u_t = -\alpha \nabla^4 u - \beta \nabla^2 u + \gamma \nabla^2 u^{2q+1} + \lambda u - \delta u^2, \tag{1}$$

where $u = u(x,t) \geq 0$, $t \geq 0$ and $x \in \Omega = [0, L]^d$ with periodic boundary conditions, and in d spatial dimensions. In (1) u represents the population density of a single animal species. The parameter $\lambda > 0$ is considered as a control parameter, and actually it is a measure of the birth rate. The other parameters are also real and positive and $q > 0$ determines the order of nonlinearity.

This model, in the case $q = 1$, was first studied in the context of population dynamics by Cohen and Murray [1].
They proposed the model in an attempt to consider whether spatial structure could be maintained by means of a diffusive mechanism more general than Fickian diffusion. The detailed derivation of the diffusion terms is given in [1]. Here we simply state that the first term has a stabilizing effect,

whereas the Laplacian term is stabilizing or destabilizing according as β is positive or negative. This system has a spatially uniform steady state solution given by $u = (\lambda/\delta)$, and it has been shown in [1] that it is stable to small spatial disturbances if β is negative. However if $\beta > 0$ then spatially structured solutions may occur; it is this the main reason why we take $\beta > 0$.

In what follows we treat this model as an example of a dissipative nonlinear partial differential equation which can be used to analyse general features of a certain class of systems. More precisely, we shall be interested in proving regularity and global existence in various spatial dimensions; furthermore the length scales involved in its solutions will also be investigated.

The material below is organised as follows: section 2 contains the derivation of a ladder theorem for our system. In section 3, the estimates for the bottom rung of the ladder are obtained. Section 4 investigates the temporal averages of various dynamical quantities and section 5 will be devoted to the length scale analysis. Finally, we will outline what conclusions may be drawn in section 6.

2 Ladder Theorem for (1)

Define the set of quantities in d spatial dimensions as follows:

$$G_p := \|u\|_{2p}^{2p} = \int |u|^{2p}\,\mathrm{d}x, \tag{2}$$

$$J_n := \|\nabla^n u\|_2^2 = \int |\nabla^n u|^2\,\mathrm{d}x. \tag{3}$$

As it was shown in [6, 5, 7], the time-dependent quantities above contain the most important dynamical information for any dissipative partial differential equation. We know that for large p the G_p can be bounded by interpolation between J_n and G_{p^*}, with $p^* \leq p$. Thus the fundamental importance of the G_p lies for small p where they will have a decisive role for estimating the "bottom rung" of the ladder. Hence in what follow we concentrate on the J_n component of the ladder. However the evolution equation for the G_p will be important in order to estimate the bottom rung.

Theorem 1 *For (1), the J_n obey the following evolution equation:*

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}J_n}{\mathrm{d}t} &\leq \lambda\, J_n + \beta\, J_{n+1} - \frac{\alpha}{2}\, J_{n+2} + c(\delta,n) J_{n+1}^{\psi}\, G_1^{\chi} + c\,\frac{\gamma^2}{\alpha}\, J_{n+1}^{\eta}\, G_p^{\theta}, && (4)\\
n &\geq 1, \quad \frac{qd}{2} < p \leq 2q+1, && (5)\\
\psi &= \frac{4n+d}{4(n+1)}, && (6)\\
\chi &= \frac{2n+6-d}{4(n+1)}, && (7)\\
\eta &= \frac{2pn+2qd+d-pd}{2pn+2p-dp+d}, && (8)\\
\theta &= \frac{4qn+4q+2-2qd}{2pn+2p-dp+d}. && (9)\\
& && (10)
\end{aligned}$$

Remarks:

Naturally, the J_n component of the ladder shows that it is crucial to have estimates on the bottom rung, which is G_p, with $p > dq/2$.

In the $dq < 2$ case the bottom rung is G_1. If $dq \geq 2$ then we need an higher G_p.

Proof:
Differentiating the J_n with respect to time and inserting the time derivative into the equation we get:

$$\frac{1}{2}\frac{\mathrm{d}J_n}{\mathrm{d}t} = \lambda J_n + \beta J_{n+1} - \alpha J_{n+2} + \gamma \int (\nabla^n u)(\nabla^{n+2} u^{2q+1})\,\mathrm{d}x - \delta \int (\nabla^n u)(\nabla^n u^2)\,\mathrm{d}x. \quad (11)$$

In order to estimate the first nonlinear term we first integrate twice by parts and then we apply Cauchy-Schwarz inequality to get

$$\int (\nabla^n u)(\nabla^{n+2} u^{2q+1})\,\mathrm{d}x \le J_{n+2}^{\frac{1}{2}} \left(\int (|\nabla^n (u^{2q+1})|)^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

Performing a Leibniz-expansion on $\nabla^n(u^{2q+1})$ and then a $(2\,q+1)$-fold Hölder inequality we obtain:

$$\begin{aligned}
\int (\nabla^n u)(\nabla^{n+2} u^{2q+1})\,\mathrm{d}x &\le c \sum_{\substack{\phi \in N^{2q+1} \\ |\phi| = n}} J_{n+2}^{\frac{1}{2}} \left(\prod_{j=1}^{2q+1} \|\nabla^{\phi_j} u\|_{2(2q+1)} \right) \\
&\le \text{(by interpolation on each factor)} \\
&\le c\, J_{n+2}^{\frac{1}{2}} \|\nabla^{n+1} u\|_2^{\sum_{j=1}^{2q+1} a_j} \cdot \|u\|_{2p}^{\sum_{j=1}^{2q+1}(1-a_j)} \\
&= c\, J_{n+2}^{\frac{1}{2}} J_{n+1}^{\rho}\, G_p^{\sigma}.
\end{aligned}$$

where $0 < \rho < 1$ and $\sigma > 0$ are given by

$$\rho = \frac{2pn + (q+1)d - pd + 2p}{2(2pn + 2p - dp + d)}, \qquad \frac{qd}{2} < p \le 2q+1;$$

$$\sigma = \frac{2qn + 2q + 1 - qd}{2pn + 2p - dp + d}.$$

Applying Young's inequality to the last term and absorbing one of the terms into the $-\alpha J_{n+2}$ we get the third and the last terms in the theorem. To obtain the fourth term of the theorem a similar argument is used. This gives the end of the proof for the J_n.

The dynamical content of the J_n ladder will be heavily used to obtain information on various quantities, such as the $\|u\|_\infty^2$ of the solutions of (1). The connection between the $\|u\|_\infty^2$ and the J_n is obtained using the Gagliardo-Nirenberg inequality,namely

$$\|u\|_\infty^2 \le c \left[\left(J_n^a G_p^{\frac{1-a}{p}} + G_1 \right) \right], \quad (12)$$

provided $0 \le a < 1$. From now on, with c we denote various constants, which may depend upon the domain Ω and various indices, but not upon the parameters or the solutions of the equation: here we are mainly interested in how the results will ultimately depend on the various parameter of the equation. More precisely we shall try to be sharp on their exponents.
¿From (12) one can see that having control on the J_n and on G_p gives the possibility to control the $\|u\|_\infty^2$, thus giving regularity for the solutions.
All we have to do then is to get some form of dynamical control on the J_n and on G_p. Let us start with the J_n.
Looking at (4) we can see that the dominant terms are the J_{n+2} and the last. Here we are interested only in the dominant behaviour of the solutions, and so we neglect the contribution of the lower

order terms. Under this approximation we look for an "absorbing ball" for the J_n as follows: first we use the interpolation

$$J_n \leq J_s^{\frac{n-r}{s-r}} J_r^{\frac{s-n}{s-r}},$$

where $r \leq n \leq s$, and we take $s = n+2$ and $r = 0$. Moreover after few manipulations the ladder can be cast into a form which to leading orders reads

$$\frac{1}{2}\frac{\mathrm{d}J_n}{\mathrm{d}t} \leq -\frac{\alpha}{8}\frac{J_n^{1+2/n}}{J_0^{2/n}} + c\,\frac{\gamma^4}{\alpha^3}\,J_n^{\frac{2pn+2qd+d-pd}{2pn+4p-dp+d-2qd}}\,G_p^{\frac{8qn+8q+4-4qd}{2pn+4p-dp+d-2qd}} + lower \quad order \quad terms, \tag{13}$$

where as above $\frac{dq}{2} < p \leq 2q+1$.

Noting that $J_0 \equiv G_1$, a standard absorbing ball argument finally gives for large n

$$\overline{J}_n \leq c\,(\frac{\gamma}{\alpha})^{2n}\,\overline{G}_p^{\frac{2qn}{2p-dq}}\,\overline{G}_1 + \text{lower order terms.} \tag{14}$$

For the L^∞ norm we have

$$\overline{\|u\|_\infty^2} \leq c\,\overline{J}_n^{\mu}\overline{G}_p^{\frac{1-\mu}{p}} \tag{15}$$

where $\mu = \frac{d}{2pn+d-pd}$, $\qquad n > \frac{d}{2}$.
Using (14) we finally obtain for large n

$$\overline{\|u\|_\infty^2} \leq c\,(\frac{\gamma}{\alpha})^{d/p}\,\overline{G}_p^{\frac{2}{2p-qd}} + \text{lower order terms.} \tag{16}$$

Inspecting (16) we see that we have to get "good" estimates for the bottom rung $\overline{G}_p$, with $\frac{dq}{2} < p \leq 2q+1$. As we anticipated earlier if $\frac{dq}{2} < 1$, the bottom rung is $\overline{G}_1$; if $\frac{dq}{2} > 1$, we need an higher $\overline{G}_p$.

3 Estimates for the Bottom Rung

We give the general evolution equation for the G_p, and then we try to find some sharper estimates for particular values of p. Differentiating the G_p with respect to time we see that for equation (1), they obey the following evolution equation:

$$\begin{aligned}\frac{1}{2p}\frac{\mathrm{d}G_p}{\mathrm{d}t} &= \lambda\,G_p - \delta\,G_{(2p+1)/2} - \alpha\int u^{2p-1}\nabla^4 u\,\mathrm{d}x + \beta\int(2p-1)u^{2p-2}(\nabla u)^2\,\mathrm{d}x \\ &- \gamma\,(2p-1)(2q+1)\int(\nabla u)^2(u^{2q})(u^{2p-2})\,\mathrm{d}x.\end{aligned} \tag{17}$$

Because the solutions of our equation are positive functions it follows that the last term of (17) is negative definite. Thus, if we are able to control the third and fourth terms in the right hand side, we can get an absorbing set for G_p. We start with the third term:

$$\begin{aligned}\int u^{2p-1}\nabla^4 u\,\mathrm{d}x &= \text{integrating by parts twice} \\ &= \int \nabla^2 u\,[(2p-1)u^{2p-2}\,\nabla^2 u + (2p-1)(2p-2)u^{2p-3}\,(\nabla u)^2\,]\mathrm{d}x.\end{aligned}$$

The first term is positive and it will be used below to control the β term;
Let I denote the second term:
$I = (2p-1)(2p-2)\int \nabla^2 u\,u^{2p-3}\,(\nabla u)^2\,\mathrm{d}x.$
Integrating by parts we get
$I = -2I - (2p-1)(2p-2)(2p-3)\int u^{2p-4}\,(\nabla u)^4\,\mathrm{d}x,$

so that $I \geq 0$ provided

$$(2p-1)(2p-2)(2p-3) \leq 0; \quad Thus \quad 1 \leq p \leq 3/2. \tag{18}$$

We now analyze the β term: integrating by parts gives

$$\beta \int (2p-1) u^{2p-2} (\nabla u)^2 = -\beta \int u^{2p-1} (\nabla^2 u). \tag{19}$$

Equation (17) now reads

$$\begin{aligned} \frac{1}{2p}\frac{\mathrm{d}G_p}{\mathrm{d}t} &= \lambda\, G_p - \delta\, G_{(2p+1)/2} - \alpha\,(2p-1) \int u^{2p-2} (\nabla^2 u)^2 \,\mathrm{d}x - \alpha\, I - \beta \int u^{2p-1} (\nabla^2 u)\,\mathrm{d}x \\ &- \gamma\,(2p-1)(2q+1) \int (\nabla u)^2 (u^{2q})(u^{2p-2})\,\mathrm{d}x. \end{aligned} \tag{20}$$

If $1 \leq p \leq 3/2$ we can neglect the αI term. Now

$$\begin{aligned} &-\alpha\,(2p-1) \int u^{2p-2} (\nabla^2 u)^2 \,\mathrm{d}x - \beta \int u^{2p-1} (\nabla^2 u)\,\mathrm{d}x \\ &= -\alpha\,(2p-1) \int u^{2p-2} \left[\left(\nabla^2 u + \frac{\beta\, u}{2(2p-1)\alpha} \right)^2 - \frac{\beta^2\, u^2}{4(2p-1)^2 \alpha^2} \right] \mathrm{d}x \\ &\leq \frac{\beta^2}{4(2p-1)\alpha}\, G_p. \end{aligned} \tag{21}$$

Also by Poincaré inequality:

$$G_{p+q} = \int u^{2(p+q)} \leq L^2 \int (\nabla (u^{p+q}))^2 = L^2 (p+q)^2 \int (\nabla u)^2 (u^{2q})(u^{2p-2})\,\mathrm{d}x. \tag{22}$$

Therefore using these results we finally get for $1 \leq p \leq 3/2$

$$\frac{1}{2p}\frac{\mathrm{d}G_p}{\mathrm{d}t} \leq \left(\lambda + \frac{\beta^2}{4(2p-1)\alpha} \right) G_p - \delta\, G_{(2p+1)/2} - \gamma^*\, G_{(p+q)}, \tag{23}$$

where

$$\gamma^* = \frac{\gamma L^{-2}\,(2p-1)(2q+1)}{(p+q)^2}.$$

Note that the result (23) can alternatively be obtained using Cauchy-Schwarz's inequality as follows: take $-\beta \int u^{2p-1} (\nabla^2 u)$ and Cauchy-Schwarz it using $u^{p-1}(\nabla^2 u)$ and u^p. Substituting into (20) after some manipulation and Young's inequality we get again (23). Note also that the γ^* term is the dominant one if $q > 1/2$.

Applying a standard Hölder's inequality on the δ and γ terms (23) becomes:

$$\frac{1}{2p}\frac{\mathrm{d}G_p}{\mathrm{d}t} \leq \left(\lambda + \frac{\beta^2}{4(2p-1)\alpha} \right) G_p - \delta\, \frac{G_p^{\frac{2p+1}{2p}}}{\|1\|_{2p}} - \gamma^*\, \frac{G_p^{\frac{p+q}{p}}}{\|1\|_{2p}}. \tag{24}$$

For $q \leq 1/2$, taking the δ term as the dominant term and using a standard absorbing set argument we obtain

$$\overline{G}_p \leq \left(\frac{\Omega\, \lambda^*}{\delta} \right)^{2p}, \qquad 1 \leq p \leq 3/2. \tag{25}$$

where Ω is the domain of integration and $\lambda^* = \lambda + \frac{\beta^2}{4(2p-1)\alpha}$.

For $q > 1/2$ taking the γ term as the dominant term we get

$$\overline{G}_p \leq \left(\frac{\Omega\,\lambda^*}{\gamma^*}\right)^{\frac{p}{q}}, \qquad 1 \leq p \leq 3/2. \tag{26}$$

In order to get the estimates for the $\overline{\|u\|_\infty^2}$ norm, all we have to do is to insert the bottom rung estimates found above into (16). We obtain:

$$\overline{\|u\|_\infty^2} \leq c\,(\frac{\gamma}{\alpha})^{d/p} \left(\frac{\Omega\,\lambda^*}{\delta}\right)^{\frac{4p}{2p-qd}}, \qquad\qquad q \leq 1/2; \tag{27}$$

For $q > 1/2$ we have instead

$$\overline{\|u\|_\infty^2} \leq c\,(\frac{\gamma}{\alpha})^{d/p} \left(\frac{\Omega\,\lambda^*}{\gamma^*}\right)^{\frac{2p}{q(2p-qd)}}, \tag{28}$$

where $\frac{dq}{2} < p \leq 2q+1$. Inspecting the $\overline{\|u\|_\infty^2}$ norm, we realize that we have global existence for $2p - dq > 0$; in particular in the case $q = 1$ we get global existence for $d = 1$ and $d = 2$. The case $q = 1, \;\; d = 3$ is critical.

4 Temporal Averages

In this section we analyze temporal estimates which will give information on various dynamical quantities describing population dynamics. We start with the time average of G_p. From (23) we know that for $1 \leq p \leq 3/2$ we have:

$$\frac{1}{2p}\frac{\mathrm{d}G_p}{\mathrm{d}t} \leq \lambda^*\,G_p -\; \delta\,G_{(2p+1)/2} -\; \gamma^*\,G_{(p+q)}, \tag{29}$$

Time averaging (29) we obtain:

$$< G_{(2p+1)/2} > \leq (\frac{\lambda^*}{\delta}) \; < G_p > -\frac{\gamma^*}{\delta} \; < G_{(p+q)} >, \qquad q \leq 1/2. \tag{30}$$

Therefore

$$< G_p > \leq (\frac{\lambda^*}{\delta})^2 \qquad q \leq 1/2. \tag{31}$$

For $q \geq 1/2$ the dominant term is the γ^* term and so we obtain:

$$< G_{(p+q)} > \leq (\frac{\lambda^*}{\gamma^*}) \; < G_p > -\frac{\delta}{\gamma^*} \; < G_{(2p+1)/2} >, \qquad q \geq 1/2. \tag{32}$$

Thus

$$< G_p > \leq (\frac{\lambda^*}{\gamma^*})^{2p} \qquad q \geq 1/2. \tag{33}$$

For $p > 3/2$ we have to take into account the contribution of the αI term. Thus (29) reads:

$$\frac{1}{2p}\frac{\mathrm{d}G_p}{\mathrm{d}t} \leq \lambda^*\,G_p - \delta\,G_{(2p+1)/2} + c\,\alpha \int u^{2p-4}\,(\nabla u)^4\,\mathrm{d}x \;-\; \gamma^*\,G_{(p+q)}. \tag{34}$$

In order to keep the algebra as simple as possible, from now on we assume that $q \geq 1/2$, and so we neglect the contribution of the δ term. Thus time averaging (34) we get

$$< G_{(p+q)} > \leq (\frac{\lambda^*}{\gamma^*}) \; < G_p > +c\,\frac{\alpha}{\gamma^*} \left\langle \left(\int u^{2p-4}\,(\nabla\,u)^4 \right) \right\rangle. \tag{35}$$

Assuming $p \geq 2$, by a Hölder-type inequality we obtain:

$$< G_{(p+q)} > \leq (\frac{\lambda^*}{\gamma^*}) < G_p > + c\,\frac{\alpha}{\gamma^*} < G_{p+q}^{\frac{p-2}{p+q}} \|\nabla u\|_4^{\frac{4p+4q}{q+2}} > . \tag{36}$$

Höldering again (36) we get

$$< G_{(p+q)} > \leq (\frac{\lambda^*}{\gamma^*}) < G_p > + c\,\frac{\alpha}{\gamma^*} < G_{p+q} >^{\frac{p-2}{p+q}} < \|\nabla u\|_{\frac{4p+4q}{q+2}}^{\frac{4p+4q}{q+2}} >^{\frac{q+2}{p+q}} . \tag{37}$$

by a Young-type inequality we obtain:

$$< G_{(p+q)} > \leq (\frac{\lambda^*}{\gamma^*}) < G_p > + \frac{p-2}{p+q} < G_{(p+q)} > + \frac{q+2}{p+q} \left[(c\,\frac{\alpha}{\gamma^*})^{\frac{p+q}{q+2}} \right] < \|\nabla u\|_{\frac{4p+4q}{q+2}}^{\frac{4p+4q}{q+2}} > . \tag{38}$$

Therefore, we finally obtain:

$$< G_{(p+q)} > \leq (\frac{p+q}{q+2})(\frac{\lambda^*}{\gamma^*}) < G_p > + \left(c\,\frac{\alpha}{\gamma^*} \right)^{\frac{p+q}{q+2}} < \|\nabla u\|_{\frac{4p+4q}{q+2}}^{\frac{4p+4q}{q+2}} > . \tag{39}$$

Now using an interpolation inequality we have:

$$\|\nabla u\|_{\frac{4p+4q}{q+2}}^{\frac{4p+4q}{q+2}} \leq c\, J_n^{\frac{(2p+2q)a}{q+2}}\, G_{p+q}^{\frac{2(1-a)}{q+2}} , \tag{40}$$

where $n > 1$, and the exponent a is given by

$$\frac{1}{n} \leq a = \frac{4p+4q-dq}{2d+4n(p+q)-2qd(p+q)} < 1, \quad 1 < n \leq \frac{2(p+q-1)}{q}. \tag{41}$$

The condition on n given by (41) guarantees that $\frac{2(1-a)}{q+2} < 1$. Thus

$$< \|\nabla u\|_{\frac{4p+4q}{q+2}}^{\frac{4p+4q}{q+2}} > \leq c < J_n^{\frac{(2p+2q)a}{q+2a}} >^{\frac{q+2a}{q+2}} < G_{p+q} >^{\frac{2(1-a)}{q+2}} . \tag{42}$$

Splitting (42) with Young's inequality and inserting the result into (39) we finally obtain:

$$< G_{(p+q)} > \leq (\frac{p+q}{q+2})(\frac{\lambda^*}{\gamma^*}) < G_p > + \left(c\,\frac{\alpha}{\gamma^*} \right)^{\frac{p+q}{q+2a}} \left\langle J_n^{\frac{(p+q)2a}{q+2a}} \right\rangle . \tag{43}$$

Thus we need to time average the J_n as well. Taking $p = 1$ in (17) and time averaging we get:

$$< J_2 > \leq \frac{1}{\alpha}(\beta < J_1 > + \lambda < G_1 >) - \frac{\delta}{\alpha} G_{3/2} - \frac{\gamma^*}{\alpha} < G_{1+q} > . \tag{44}$$

Using $< J_1 > \leq < J_2^{1/2} G_1^{1/2} > \leq \frac{1}{2} < J_2 > + \frac{1}{2} < G_1 >$ one sees that

$$< J_2 > \leq \left(\frac{\beta^2}{\alpha^2} + \frac{2\lambda}{\alpha} \right) < G_1 > - \frac{\delta}{\alpha} < G_{3/2} > - \frac{\gamma^*}{\alpha} < G_{1+q} > . \tag{45}$$

For $n > 1$ we do not know how to obtain a sharp time average, and so the higher J_n will not be considered. The time average of the L^∞-norm is done in the following way:

$$< \|u\|_\infty^{4q} > \leq c < \left(\frac{J_n}{J_0} \right)^{2q\nu} J_0^{2q\nu}\, G_p^{\frac{2q(1-\nu)}{p}} >, \tag{46}$$

where

$$\nu = \frac{d}{2pn + d - dp}. \tag{47}$$

By a Hölder-type inequality we obtain:

$$< \|u\|_\infty^{4q} > \leq c < \left(\frac{J_n}{J_0}\right)^{\frac{2}{n}} >^b < J_0^{\frac{2q\nu}{1-b}} G_p^{\frac{2q(1-\nu)}{(1-b)p}} >^{1-b}, \qquad b = qn\nu, \qquad p > qd/2. \tag{48}$$

In order to handle the $< \left(\frac{J_n}{J_0}\right)^{\frac{2}{n}} >^b$ term we make use of the following version of the ladder:

$$\frac{1}{2}\frac{\mathrm{d}J_n}{\mathrm{d}t} \leq \lambda J_n + \beta J_{n+1} - \alpha J_{n+2} + c\,\delta\, J_{n+1}^{\psi} G_0^{\chi} + c\,\gamma\, \|u\|_\infty^{2q} J_{n+1}. \tag{49}$$

Making the same manipulations done to obtain (13) we can see that to leading orders (49) becomes:

$$\frac{1}{2}\frac{\mathrm{d}J_n}{\mathrm{d}t} \leq -\frac{\alpha}{8}\frac{J_n^{1+2/n}}{J_0^{2/n}} + c\,\frac{\gamma^2}{\alpha}\|u\|_\infty^{4q} J_n. \tag{50}$$

Dividing by J_n and taking the time average we get:

$$\langle\frac{\dot{J}_n}{J_n}\rangle \leq -\frac{\alpha}{4} < \left(\frac{J_n}{J_0}\right)^{\frac{2}{n}} > + c\,\frac{\gamma^2}{\alpha} < \|u\|_\infty^{4q} > . \tag{51}$$

We now investigate the time average $\langle \dot{J}_n / J_n \rangle$ and we argue that it is bounded for any regular solution of our PDE.

First we note that J_n is bounded as long as $1 \leq p \leq 3/2$ (this follows directly from the ladder). Thus the only problems in the evaluation of $\langle \dot{J}_n / J_n \rangle$ may arise in the behaviour of the solutions of the equation which are close to zero. This is because the actual value of $\langle \dot{J}_n / J_n \rangle$ is

$$\lim_{t \to +\infty} \frac{1}{t}\left(log(\frac{J_n(t)}{J_n(0)}) \right). \tag{52}$$

Thus the behaviour of the solutions close to the zero solution has to be analyzed; in particular their asymptotic decay rate.

Looking at (52), one can see that for any smooth initial condition it follows that the limit as $t \to +\infty$ exists and is finite if the asymptotic behaviour of the solutions are characterized by a decay rate which is not faster than exponential.

Now it is well known [9, 10] that for a certain class of reaction-diffusion systems there are theorems which guarantee that solutions cannot tend to zero faster than they do in the corresponding system linearised about the zero solution, i.e. exponentially. This kind of behaviour is even true for the Navier-Stokes Equations [11]. For our equation we have not been able to rigorously prove such a linearization result. However, a linear analysis of the system about the zero solution does suggest that solutions cannot tend to zero faster than exponentially. On this assumption it follows that $\langle \dot{J}_n / J_n \rangle$ does remain bounded.

Therefore, to leading orders we obtain:

$$< \left(\frac{J_n}{J_0}\right)^{\frac{2}{n}} > \leq c\,(\frac{\gamma}{\alpha})^2 < \|u\|_\infty^{4q} > . \tag{53}$$

Inserting (53) into (48) and doing some algebra we finally get for large values of n:

$$< \|u\|_\infty^{4q} > \leq c \left(\frac{\gamma}{\alpha}\right)^{\frac{2qd}{2p-dq}} < G_p^{\frac{4q}{2p-dq}} >; \qquad p > dq/2. \tag{54}$$

In (54) we have to insert low values of p, as we know that $< G_p >$ is not too large. For $1 \leq p \leq 3/2$ we know the $< G_{\frac{2p+1}{2}} >$ or the $< G_{p+q} >$ estimates; for larger p we can use the estimates (43).

Note that if we substitute $p = qd$ in (54) we obtain the elegant set of estimates:

$$< \|u\|_\infty^{4q} > \leq c \left(\frac{\gamma}{\alpha}\right)^2 < G_{qd}^{\frac{4}{d}} >, \qquad p > dq/2. \tag{55}$$

Other routes are possible, but they give essentially the same results. Here we have tried to avoid choices which would give a very messy algebra, thus obscuring the essence of these methods. For example in the case $dq = 1$, using another version of the ladder one could get slightly sharper results, at the cost of having to change (55), which is nice and compact.

In the table below we compare the time asymptotic upper bounds and the time averaged bounds, for the case $q = 1$.

Dimension	$< \|u\|_\infty^{4q} >$	$\overline{\|u\|}_\infty^{4q}$
$d = 1$	$c\,(\frac{\gamma}{\alpha})^2\,(\frac{\lambda^*}{\gamma^*})^4$	$c\,(\frac{\gamma}{\alpha})^2\,(\frac{\lambda^*}{\gamma^*})^4$
$d = 2$	$c\,(\frac{\gamma}{\alpha})^2\,(\frac{\lambda^*}{\alpha})\,(\frac{\lambda^*}{\gamma^*})^4$	$c\,(\frac{\gamma}{\alpha})^{8/3}\,(\frac{\lambda^*}{\gamma^*})^6$
$d = 3$	$c\,(\frac{\gamma}{\alpha})^2\,<\,G_3^{\frac{4}{3}}\,>$	$c\,(\frac{\gamma}{\alpha})^{6/p}\,\overline{G}_p^{\frac{4}{2p-3}}, \qquad p > 3/2$

5 Length Scales

In this section we shall investigate the length scales which appear in our flow. As we mentioned in the introduction, information on the length scales is crucial in order to have an accurate picture of the dynamics involved in the solutions of any dissipative flows.

A set of length scales is normally obtained through a set of 'relevant modes', which are defined using the Fourier expansion of the solutions of the equation [6].

Length scales are then defined using the classical formula $\mathcal{N} = (L/l)^d$, where d is the spatial dimension, L^d the volume of the domain and l the so called dissipation length scale.

The final result reads:

$$L^{-d} \leq \frac{\|u\|_\infty^2}{J_0} \leq \left(\frac{J_n}{J_0}\right)^{\frac{d}{2n}} \equiv l_n^{-d}. \tag{56}$$

The above formula involves the J_n, and thus the ladder. Thus we have been able to 'link' our definition of a set of length scales to the dynamics of the solutions of the equation.

Note that the quantity $\frac{\|u\|_\infty^2}{J_0}$ gives a natural lower bound for our inverse length scales l_n^{-d}, and it is frequently used in real experiments, where it is known as the *crest factor*.

We have now to investigate the dynamics content of our length scales. First we note that

$$(length)^{-1} := \left(\frac{J_n}{J_0}\right)^{\frac{1}{2n}}. \tag{57}$$

is a function of time; thus it is natural to say that it defines a dynamic length scale. Its dynamic evolution is in general not easy to obtain; so a useful thing to do is to try to define a length scale which is time independent. This is normally achieved taking time averages or time asymptotic bounds.

We start with the time average estimates of our length scales. To do that we have to use the dynamical content of the ladder (50):

$$\frac{1}{2}\frac{\mathrm{d}J_n}{\mathrm{d}t} \leq -\frac{\alpha}{8}\,\frac{J_n^{1+2/n}}{J_0^{2/n}} + c\,\frac{\gamma^2}{\alpha}\,\|u\|_\infty^{4q}\,J_n. \tag{58}$$

Dividing by J_n and taking the time average we obtain again (53) and (54), as $\langle \frac{\dot{J}_n}{J_n}\rangle$ vanishes. Thus the final result reads:

$$\begin{aligned}\langle \ell^{-1}\rangle &= \langle \left(\frac{J_n}{J_0}\right)^{\frac{1}{2n}} \rangle \leq c\langle \left(\frac{J_n}{J_0}\right)^{\frac{2}{n}} \rangle^{1/4} \\ &\leq c\,\langle\, \|u\|_\infty^{4q}\,\rangle^{1/4} \leq c\,\left(\frac{\gamma}{\alpha}\right)^{\frac{qd}{2(2p-dq)}}\,\langle\, G_p^{\frac{4q}{2p-dq}}\,\rangle^{1/4}; \qquad p > dq/2. \end{aligned} \tag{59}$$

Putting as above $p = qd$ we get

$$\langle \ell^{-1}\rangle \leq c\,\left(\frac{\gamma}{\alpha}\right)^{\frac{1}{2}}\,\langle\, G_{qd}^{\frac{4}{d}}\,\rangle^{1/4}; \qquad p > dq/2. \tag{60}$$

Comparing (54) and (60) one can see that the $\langle\, \|u\|_\infty^{4q}\,\rangle^{1/4}$ and $\langle \ell^{-1}\rangle$ scale in the same way.

For the time asymptotic upper bounds we run into the same difficulty encountered in the CGL equation [7]. One can formally solve the problem as done in there, and obtain time asymptotic estimates which scale like the estimates (16).

6 Conclusions

This paper concerns the problem of length scales in the solutions of a fourth order dissipative partial differential equation. The equation had previously been proposed in order to address the question of whether stable spatially structured solutions are possible in a model for a single species. Length scales have been previously investigated in other dissipative partial differential equations such as the Navier-Stokes and the Complex Ginzburg-Landau equations, using the so-called ladder methods. In a population model, the smallest length scale is a measure of the smallest scale in which spatial fluctuations of a population occur.

We have obtained a set of ladder inequalities for our equation, and we have been able to use these inequalities to prove regularity of solutions in one and two spatial dimension for the case $q = 1$ (which is the case studied in [1]).

Qualitatively we have found that large values of α or δ give large length scales, whilst large values of γ or λ will give smaller length scales.

Cohen and Murray [1] performed a linear stability analysis of the system (1) at its spatially uniform steady state solution $u \equiv \lambda/\delta$, and determined when this steady state goes linearly unstable. They also derived an expression for the critical wave number at bifurcation. In slightly different notation, they chose λ as bifurcation parameter and found that bifurcation from the uniform state occurs when

$$\frac{3\gamma}{\delta^2}\lambda^2 + 2\sqrt{\alpha\lambda} - \beta = 0$$

and that the critical wavenumber is

$$k_c = \left(\frac{\lambda}{\alpha}\right)^{1/4} \tag{61}$$

giving the spatial structure $\exp \pm ik_c x$. We take as a measure of the length scale of the bifurcating solution the quantity $k_c^{-1} = (\alpha/\lambda)^{1/4}$. This is precisely the inverse of the quantity which is obtained when one applies our formula $\left(\frac{J_n}{J_0}\right)^{\frac{1}{2n}}$ to $\exp \pm(ik_c x)$.

When α is large, equation (61) yields, approximately,

$$\lambda \sim \frac{\beta^2}{4\alpha}$$

and therefore $k_c^{-1} = \sqrt{2\alpha/\beta}$. Hence our length scales analysis yields in particular the expected result, when applied to the particular solutions found in Cohen and Murray [1]. However, the important point about the analysis described in this paper is that it applies for all possible solutions of the equation, and not just those solutions near to the uniform steady state solution.

Acknowledgements

We should like to thank John Toland and Philip Aston for discussions concerning the estimates we have used in the bottom rung, and Peter Constantin for pointing out reference [11] and very helpful comments.

References

[1] Cohen D. S. and Murray J. D.; (1981) *A Generalized Diffusion Model for Growth and Dispersal in a Population*, J. Math. Biol., Vol. **12**, 237–249.

[2] Kolmogorov A. N.; (1941) The Local Structure of Turbulence in Incompressible Viscous Fluids for Very Large Reynolds Numbers, Dokl. Akad. Nauk. USSR **30**, 299-304.

[3] Constantin, P., Foias, C., and Temam R.; (1985) *Attractors Representing Turbulent Flows*, Memoirs of AMS, Providence, Rhode Island.

[4] Constantin, P., Foias, C., and Temam R.; (1988) *On the Dimension of the Attractors in Two-Dimensional Turbulence*, Physica D **44**, 284–296.

[5] Bartuccelli, M.V., Constantin, P., Doering, C.R., Gibbon, J.D. and Gisselfält, M.; (1990) *On the Possibility of Soft and Hard Turbulence in the Complex Ginzburg-Landau Equation*, Physica D **44**, 421–444.

[6] Bartuccelli, M.V., Doering, C.R., Gibbon, J.D., Malham, S.J.A; (1993) *Length Scales in Solutions of the Navier-Stokes Equations*, Nonlinearity **6** 549-568.

[7] Bartuccelli, M.V., Gibbon, J.D. and Oliver, M.; (1995) *Length Scales in Solutions of the Complex Ginzburg-Landau Equation*, Physica D, In Press.

[8] Gourley, S.A. and Bartuccelli, M.V; (1995) *Length scales in solutions of a scalar reaction–diffusion equation with delay*, Physics Letters A Volume **202**, No. 1, 79-87.

[9] Casten, R.G., Holland, C.J.; (1977) SIAM J. Appl. Math. **33** 353.

[10] Kielhöfer, H; (1976) J. Diff. Eqns. **22**, 193.

[11] Foias, C. and Saut, J.C; (1984) *Asymptotic Behaviour, as $t + \infty$ of Solutions of Navier-Stokes Equations and Nonlinear Spectral Manifolds*, Indiana University Mathematics Journal **33**, 459–477.