EULER ON EULER'S METHOD

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Dedicated with gratitude to DF Griffiths on his retirement,
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• We all know that science and technology routinely need to solve differential equations . . .

• . . . a task usually undertaken by numerical methods . . .

• . . . as first suggested by L. Euler.

However . . .

• . . . we usually read that, even though Euler was brilliant at manipulating symbols, his understanding of the subtler points of analysis was not something to write home about, e.g. he was at a total loss when it comes to convergent series. In short: he would not pass an undergraduate analysis exam today.

- Would Euler pass an exam on numerical methods?

or for that matter . . .

- Would Euler pass an exam on Euler's method?

• In 2007, on the occasion of Euler's anniversary, I was asked to deliver a speech on Euler' method . . .

•. . . and I decided to read the original . . .

1768 **(E342)** *Institutionum Calculi Integralis, Volumen Primum.* Sec. II: *De integratione aequationum differentialum.* Cap. VII: *De integratione aequationum differentialium per approximationem.* [Institutions of integral calculus, first volume. Second section: On the integration of differential equations: Chapter 7: On the integration of differential equations by approximation.]

Problema 85: Proposita aequatione differentiali quacumque, ejus integrale completum vero proxime assignare. [Given any differential equations, assign its complete integral very approximately.]

• $\frac{\partial y}{\partial x} = V$, V *functione quacumque ipsarum* [an arbitrary function of] x et y; a, b initial values of x, y.

• If $x = a+\omega$ (ω *particula minima* [tiny part]), y is hardly different from b and V remains constant.

• Therefore, for the *exigua mutatione* [wee change] from a to x, $\frac{\partial y}{\partial x} = A$ (A value of V at a, b) and integration yields $y = b + (x-a)A$.

• Thus we advance from a, b to $a+\omega$, $b+A\omega$. Iterating the idea 'by very small intervals' we reach x, y *'quantumvis remotos'* [arbitrarily remote].

• Euler does not provide any graphical interpretation.

• But notes that we would reach ‘better clarity for our eyes’ by means of the table:

$$\begin{array}{lllllll} x & \qquad & a, & a', & a'', & a^{IV}, & \ldots \\ y & & b, & b', & b'', & b^{IV}, & \ldots \\ V & & A, & A', & A'', & A^{IV}, & \ldots \end{array}$$

• Euler provides some discussion as follows:

• *Corollarium 2:* Euler notes that the smaller the x-intervals are taken the more accurate the values of y. And this not only for the errors perpetrated 'singularly (which are very much smaller)' but also after the 'the multitude piles up'.

• *Corollarium 3:* Euler notes that the errors will be smaller if V (which the method assumes locally constant) changes by only small amounts.

• *Scholion 1:* Euler tells us that when we reach a singularity of V we should Taylor expand to find the next value of y and then return to the numerical method.

• *Scholion 2:* he compares the new method with the method of solution by series ‘given in many places’ (in fact it goes back to Newton, who is not credited). He notes that the series method may not work due to convergence difficulties if the series is to be evaluated at a point x far away from a. The suggested method is universally applicable.

• An example is provided where the series converges (as seen by today’s analysts) but is of no numerical use due to the slow convergence.

• . . . but Euler was aware of how to take the best of both worlds and combine his idea of time-stepping with (Taylor) series . . .

Problema 86 *'Methodum praecedentem, aequationes differentiales proxime integrandi, magis perficere, ut minus a veritate aberret'.* [Perfect greatly the method above . . . so that it deviates less from the truth.]

- Solution: Replace

$$y(x+h) \approx y(x) + hy'(x)$$

by

$$y(x+h) \approx y(x) + hy'(x) + \frac{1}{2}h^2y''(x) + \cdots + \frac{1}{n!}h^n y^{(n)}(x).$$

and find the numerical values of the derivatives by differentiation of the given equation.

$$
\begin{aligned}
y' &= f, \\
y'' &= \frac{d}{dx} f = f_x + f_y y' = f_x + f_y f, \\
y''' &= \frac{d}{dx}[f_x + f_y f] = f_{xx} + 2 f_{xy} f + f_{yy} f^2 + f_y (f_x + f_y f), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

Of course this is what textbooks now call Taylor's method of order n and should be referred to as Euler's method of order n.

• In the *Corollarium 3* he discourages the use of too long time-steps, even if many coefficients in the series are available. And particularly so if we see that the coefficients grow as we compute more of them.

- Euler fully works out examples. On of them has

$$y' = x^2 + y^2$$

and he successively finds:

$$\begin{aligned} y'' &= 2x + 2x^2y + 2y^3, \\ y''' &= 2 + 4xy + 2x^4 + 8x^2y^2 + 6y^4, \\ y^{(iv)} &= 4y + 12x^3 + 20xy^2 + 16x^4y + 40x^2y^3 + 24y^5 \\ y^{(v)} &= 40x^2 + 24y^2 + 104x^3y + 120xy^3 + 16x^6 + 156x^4y^2 \\ &\quad +240x^2y^4 + 120y^6. \end{aligned}$$

• Several authors in last 50 years have observed that de Taylor/Euler method may be highly efficient provided it is implemented in a clever way so as to avoid differentiation of the given differential equation. This works as follows:

• Given values a of x and b of y at the beginning of the step, the next value of y, corresponding to $x = a + h$, will be

$$y = b + h\alpha + h^2\beta + h^3\gamma + h^4\delta + h^5\epsilon + h^6\zeta + \cdots$$

and we have to compute α, β, ... Differentiation leads to

$$y' = \alpha + 2h\beta + 3h^2\gamma + 4h^3\delta + 5h^4\epsilon + 6h^5\zeta + \cdots ,$$

and this has to coincide with

$$\begin{aligned} x^2 + y^2 &= (a+h)^2 + (b + h\alpha + h^2\beta + h^3\gamma + h^4\delta + \cdots)^2 \\ &= (a^2 + b^2) + h(2\alpha b + 2a) + h^2(2\beta b + \alpha^2 + 1) + \cdots \end{aligned}$$

- Equating like powers of h:

$$\begin{aligned}\alpha &= aa + bb,\\ 2\beta &= 2\alpha b + 2a,\\ 3\gamma &= 2\beta b + \alpha\alpha + 1,\\ 4\delta &= 2\gamma b + 2\alpha\beta,\\ 5\epsilon &= 2\delta b + 2\alpha\gamma + \beta\beta,\\ 6\zeta &= 2\epsilon b + 2\alpha\delta + 2\beta\gamma,\end{aligned}$$

and the coefficients may be found recursively without any need to differentiate!

- Neat . . .

- . . . but not new . . .

- . . . the formulae are from §663: *Quoniam totum negotium ad inventionem horum coefficientium redit, observo eosdem sine differentiatione inveniri posse.* [Because all the business boils down to the computation of the coefficients, I observe that they may be found without differentiation!]

EVALUATION:

• NUMERICAL MATHS: Would LE pass an exam on Euler's method? Did he understand the difference between local and global error? Did he understand the structure of the local error? Was he aware of the competition between computational work and accuracy?

• ANALYSIS: Was his grasping of series limited to formal manipulations? Did he possess the notion of initial value problem? Did he know that, except at singular points, an IVP has a unique solution?

• TEACHING: How would you assess his expository style — worked out concrete examples, first case $n = 1$ only then general situation,. . . —