# *Numerical solution of first kind Volterra integral equations*

Penny J Davies

University of Strathclyde

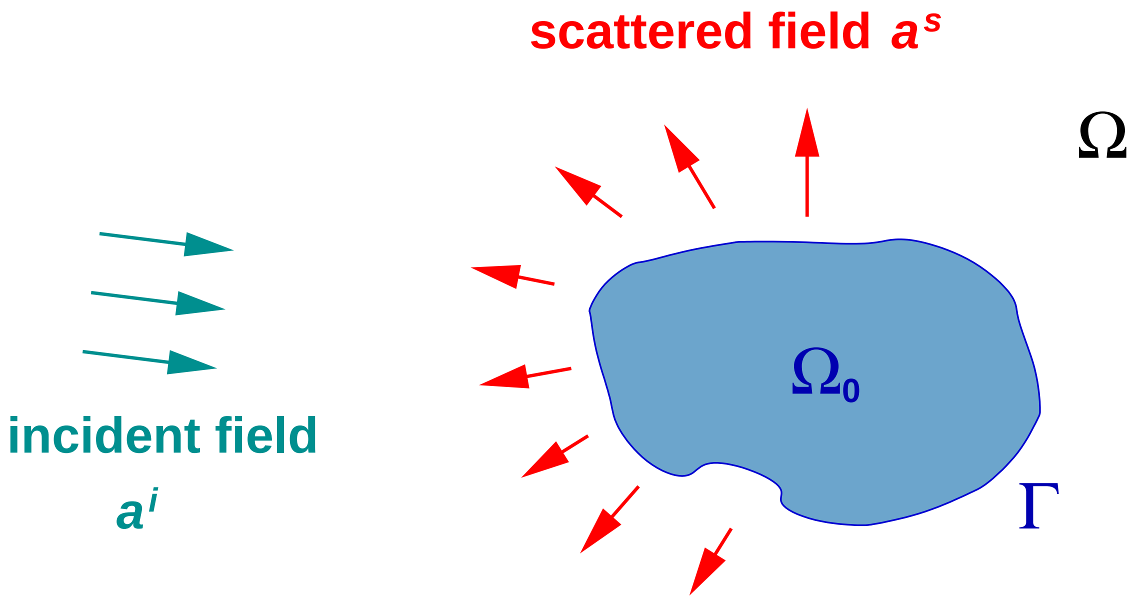
**Joint work with**

**Hermann Brunner (Memorial University of Newfoundland)**

**Dugald Duncan (Heriot–Watt University)**

# *Motivation: acoustic scattering*

Find the scattered field $a^s$, given the incident field $\boldsymbol{a^i}$:




**Assume:**

- time dependent problem with wave speed $c = 1$;
- $\boldsymbol{a^i}$ has not reached the obstacle by time $t = 0$.
- “sound soft” BC: $\boldsymbol{a^s} + \boldsymbol{a^i} = 0$ on $\Gamma$

# Boundary integral formulation

$$a^s(\boldsymbol{x},t) = \frac{1}{4\pi}\int_\Gamma \frac{u(\boldsymbol{x}',t-|\boldsymbol{x}'-\boldsymbol{x}|)}{|\boldsymbol{x}'-\boldsymbol{x}|}\,dS' \quad \boldsymbol{x}\in\Omega,\ t>0, \text{ where}$$

$$\frac{1}{4\pi}\int_\Gamma \frac{u(\boldsymbol{x}',t-|\boldsymbol{x}'-\boldsymbol{x}|)}{|\boldsymbol{x}'-\boldsymbol{x}|}\,dS' = -a^i(\boldsymbol{x},t) \quad \boldsymbol{x}\in\Gamma,\ t>0. \qquad (*)$$

**IC:** $u \equiv 0$ , $a^i \equiv 0$ on $\Gamma$ for all $t \leq 0$ (causality).

- **BIE:** solve $(*)$ for $u(\boldsymbol{x},t)$ given $\boldsymbol{a^i}$ on $\Gamma$
- Numerical approximation: time-stepping is often **unstable**
- Isolate time-stepping and analyse when $\boldsymbol{\Gamma}$ is an **infinite flat plate** (stability in this case necessary, but not sufficient)

# *Spatial Fourier transform:* $\Gamma = \mathbb{R}^2$

- BIE:

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{u(\boldsymbol{x}', t - |\boldsymbol{x}' - \boldsymbol{x}|)}{|\boldsymbol{x}' - \boldsymbol{x}|} \, dS' = a(\boldsymbol{x}, t) \qquad (*)$$

- Fourier transform in space at frequency $\boldsymbol{\omega} \in \mathbb{R}^2$:

$$\int_0^t J_0(|\boldsymbol{\omega}| R) \, \widehat{u}(\boldsymbol{\omega}, t - R) \, dR = \widehat{a}(\boldsymbol{\omega}, t)$$

($J_0$ is the zeroth order Bessel function of the first kind)

– this is a convolution Volterra integral equation (VIE)

- **Note:** in full numerical approx of $(*)$, can have $|\boldsymbol{\omega}| \, \Delta t = O(1)$, but following analysis is only for $|\boldsymbol{\omega}| = O(1)$

# The problem . . .

. . . is to solve the first kind Volterra integral equation (VIE)

$$\int_0^t K(t-s)\,u(s)\,ds = a(t)\,, \quad t \in (0,T] \qquad (*)$$

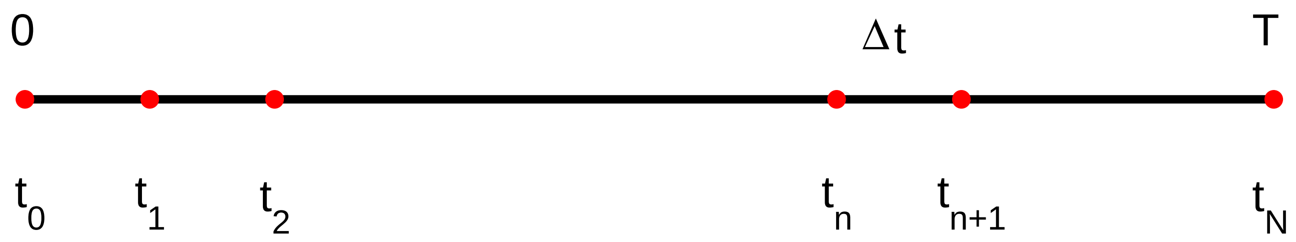
for $u$, given smooth $a$, $K$ with $a(0) = 0$ and $K(0) = 1$

## The talk. . .

- . . . explain the connection between collocation, Galerkin and quadrature–Galerkin approx of $(*)$
- describe a new choice of basis functions for collocation – enables a more complete convergence analysis
- look at the implications for the scattering BIE

# Discontinuous polynomial approx

- Split interval $[0, T]$ into $N$ chunks of size $\Delta t$. Set $t_n = n\,\Delta t$:




- Let $\{\phi_k\}_{k=0}^{m}$ be polynomial basis functions of degree $\leq m$
- Approx: $\boldsymbol{U(t) \approx u(t)}$ is $m$ deg poly in each $(t_n, t_{n+1})$:

$$U(t_n + s\Delta t) = \sum_{k=0}^{m} U_k^n\,\phi_k(s)\,, \quad s \in [0, 1]$$

- Exact solution: $$u(t_n + s\Delta t) = \sum_{k=0}^{m} u_k^n\,\phi_k(s) + R_n(s)$$

# Approximation error

- Approx error is: $\boldsymbol{e(t) = U(t) - u(t)}$, or

$$e(t_n + s\Delta t) = \sum_{k=0}^{m} \boldsymbol{\varepsilon}_k^n \, \phi_k(s) - R_n(s)$$

where $\boldsymbol{\varepsilon}_k^n = U_k^n - u_k^n$

- “Residual” is

$$r(t) = \int_0^t K(t-\tau)\, U(\tau)\, d\tau - a(t) = \int_0^t K(t-\tau)\, e(\tau)\, d\tau$$

# *Three methods. . .*

"Residual" $r(t)$:

$$r(t) = \int_0^t K(t-\tau)\, U(\tau)\, d\tau - a(t) = \int_0^t K(t-\tau)\, e(\tau)\, d\tau$$

- **Collocation:** $r(t_j^n) = 0$ at a discrete set of points $\{t_j^n\}$
- **Galerkin:**
$$\int_0^T r(t)\, \phi_j^n(t)\, dt = 0 \quad \text{for each } j,\, n$$
- **Quadrature Galerkin:** obtained by approximating the above set of integral equations

# Form of approximation scheme

- Approx scheme (**all cases**) has the convolution sum form

$$\sum_{\ell=0}^{n} M^{\ell} \boldsymbol{U}^{n-\ell} = \boldsymbol{a}^{n} \quad \text{for } n = 0 : N-1$$

with matrices $M^{\ell} \in \mathbb{R}^{(m+1)\times(m+1)}$

- When $M^0$ is nonsingular, solve by **time-marching**:

$$M^{0} \boldsymbol{U}^{n} = \boldsymbol{a}^{n} - \sum_{\ell=0}^{n-1} M^{n-\ell} \boldsymbol{U}^{\ell}, \quad n = 0 : N-1$$

# Approximation error equation

The approx error is $e(t_n + s\Delta t) = \sum_{k=0}^{m} \boldsymbol{\varepsilon}_k^n \, \phi_k(s) - R_n(s)$ where:

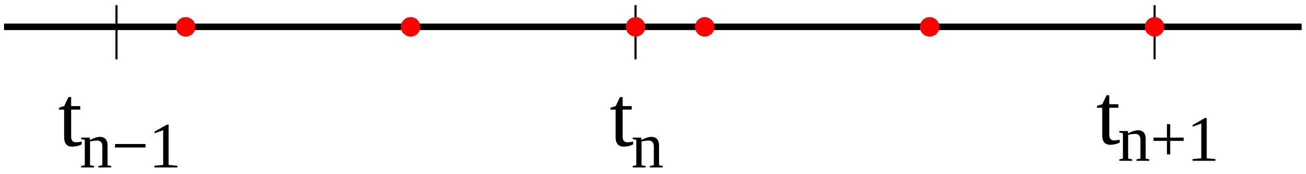
- $R_n$ is the "remainder" in the representation of the exact solution $u(t_n + s\Delta t)$ by the polynomial basis functions
- the coefficient error vector $\boldsymbol{\varepsilon}^n$ satisfies the matrix convolution sum equation

$$\sum_{\ell=0}^{n} M^{\ell} \, \boldsymbol{\varepsilon}^{n-\ell} = \boldsymbol{\beta}^n \quad \text{for } n = 0 : N-1$$

(matrices $M^{\ell}$ are same as for the approx scheme, $\boldsymbol{\beta}^n$ can be bounded by a power of $\Delta t$)

# Collocation (Brunner, 1970s)

- **Equations:** $r(t_n + c_j \Delta t) = 0$, for $0 < c_0 < \dots c_m \leq 1$




- **Global convergence** iff $|\rho_m| \leq 1$, where $\rho_m := (-1)^{m+1} \prod_{j=0}^{m} \left( \frac{1-c_j}{c_j} \right)$

$$\max_{[0,T]} |e(t)| \leq C \begin{cases} \Delta t^{m+1} & \text{if } \rho_m \in [-1,1) \\ \Delta t^m & \text{if } \rho_m = 1 \end{cases}$$

- **Local superconvergence.** If $\rho_m = 0$ (i.e. $c_m = 1$), then

$$|e(t_n + s_k \Delta t)| \leq C\, \Delta t^{m+2} \qquad \text{for } \{s_k\}_{k=0}^{m}$$

# Galerkin (BrDaDu):

- **Equations:** $\int_0^1 r(t_n + s\Delta t)\,\phi_j(s)\,ds = 0$

- **Global convergence**

$$\max_{[0,T]} |e(t)| \le C \begin{cases} \Delta t^m & \text{when } m \text{ is } \textbf{odd} \\ \Delta t^{m+1} & \text{when } m \text{ is } \textbf{even} \end{cases}$$

- **Local superconvergence**

**odd $\boldsymbol{m}$:** $|e(t_n + s_r\,\Delta t)| \le C\,\Delta t^{m+1}$ (where $P'_{m+1}(2s_r - 1) = 0$)

**even $\boldsymbol{m}$:** $|e(t_n + s_r\,\Delta t)| \le C\,\Delta t^{m+2}$ (where $P'_{m+2}(2s_r - 1) = 0$) **iff**

> the exact solution $u$ satisfies $u_0^{(m+1)} = 0$

# Quadrature Galerkin

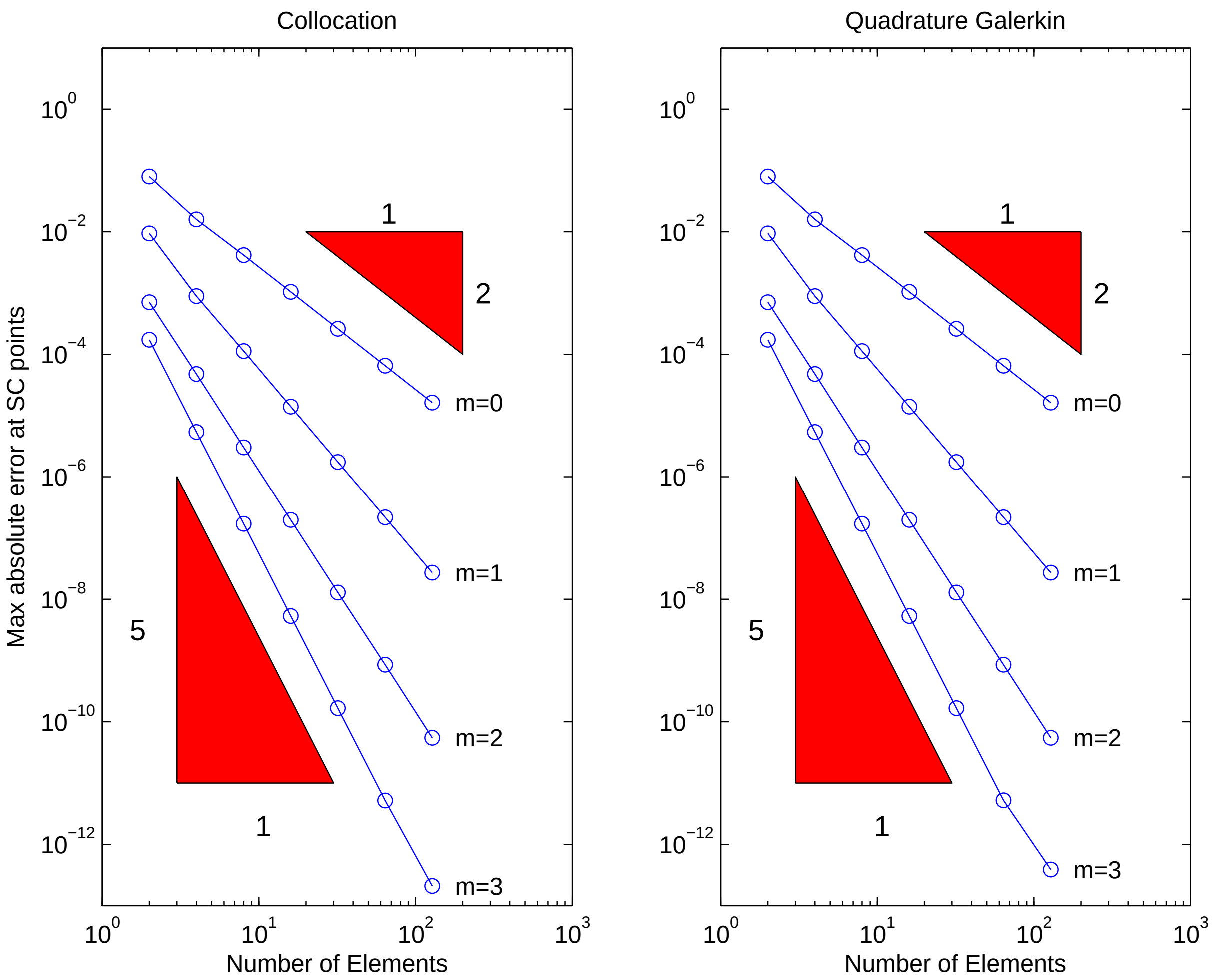
- $q+1$ **point quadrature rule:** $\int_0^1 f(s)\,ds \approx Q[f(s);s] = \sum_{i=0}^{q} \omega_i\, f(d_i)$
- **Equations:** $Q[r(t_n + s\Delta t)\,\phi_j(s);s] = 0$, for $j = 0:m$, i.e.

$$[\phi_k(d_i)] \begin{pmatrix} \omega_0 & & \\ & \ddots & \\ & & \omega_q \end{pmatrix} \begin{pmatrix} r(t_n + d_0\Delta t) \\ \vdots \\ r(t_n + d_q\Delta t) \end{pmatrix} = \mathbf{0}$$

- **Result:** If $q = m$, all the nodes $d_i$ are distinct and none of the weights $\omega_i$ are zero, then the quadrature Galerkin approx is identical to collocation using the nodes $d_i$, **whatever the choice of weights**.
- Shall return to $q > m$ after a looking at collocation superconvergence for $\rho_m \neq 0$.

# Collocation & QG

Quadrature weights are **randomly generated** (inconsistent)



# Collocation revisited

- VIE: $\displaystyle\int_0^t K(t-s)\,u(s)\,ds = a(t)$, $a$ and $K$ are smooth, $K(0)=1$
- The approx error is $\displaystyle e(t_n + s\Delta t) = \sum_{k=0}^{n} \boldsymbol{\varepsilon}_k^n\, \phi_k(s) - R_n(s)$, where

$$\sum_{\ell=0}^{n} M^{\ell} \boldsymbol{\varepsilon}^{n-\ell} = \boldsymbol{\beta}^n \qquad (*)^n$$

- Results for general smooth $K$ follow from $K \equiv 1$ case (after lots of tedious manipulation)
- Set $\boldsymbol{K} \equiv \mathbf{1}$: then $(*)_j^{n+1} - (*)_m^n$ (Brunner) gives

$$M^0\, \boldsymbol{\varepsilon}^{n+1} = M\, \boldsymbol{\varepsilon}^n + \boldsymbol{b}^n$$

# Collocation: $K \equiv 1$

- **Error equation:** $M^0\,\boldsymbol{\varepsilon}^{n+1} = M\,\boldsymbol{\varepsilon}^n + \boldsymbol{b}^n$
- Matrix $(M^0)^{-1}\,M$ has only one nonzero eigenvalue, and it is $\rho_m$ (Brunner) – analysis very technical because $(M^0)^{-1}$ isn't known
- Life is easier with new polynomial basis functions – chosen to make $M^0 = I$
- **Matrix components:** $M^0_{j,k} = \int_0^{c_j} \phi_k(s)\,ds$
- Let $c_{-1} = 0$ and $\{L_k(s)\}_{k=-1}^m$ be Lagrange polynomials of degree $m+1$ based at $c_k$ for $k = -1 : m$
- Set $\phi_k(s) = L_k'(s)$ for $k = 0 : m$. Then $M^0_{j,k} = L_k(c_j) - L_k(c_{-1}) = \delta_{j,k}$
- Makes it possible to investigate superconvergence when $\rho_m \neq 0$

# Collocation: superconvergence

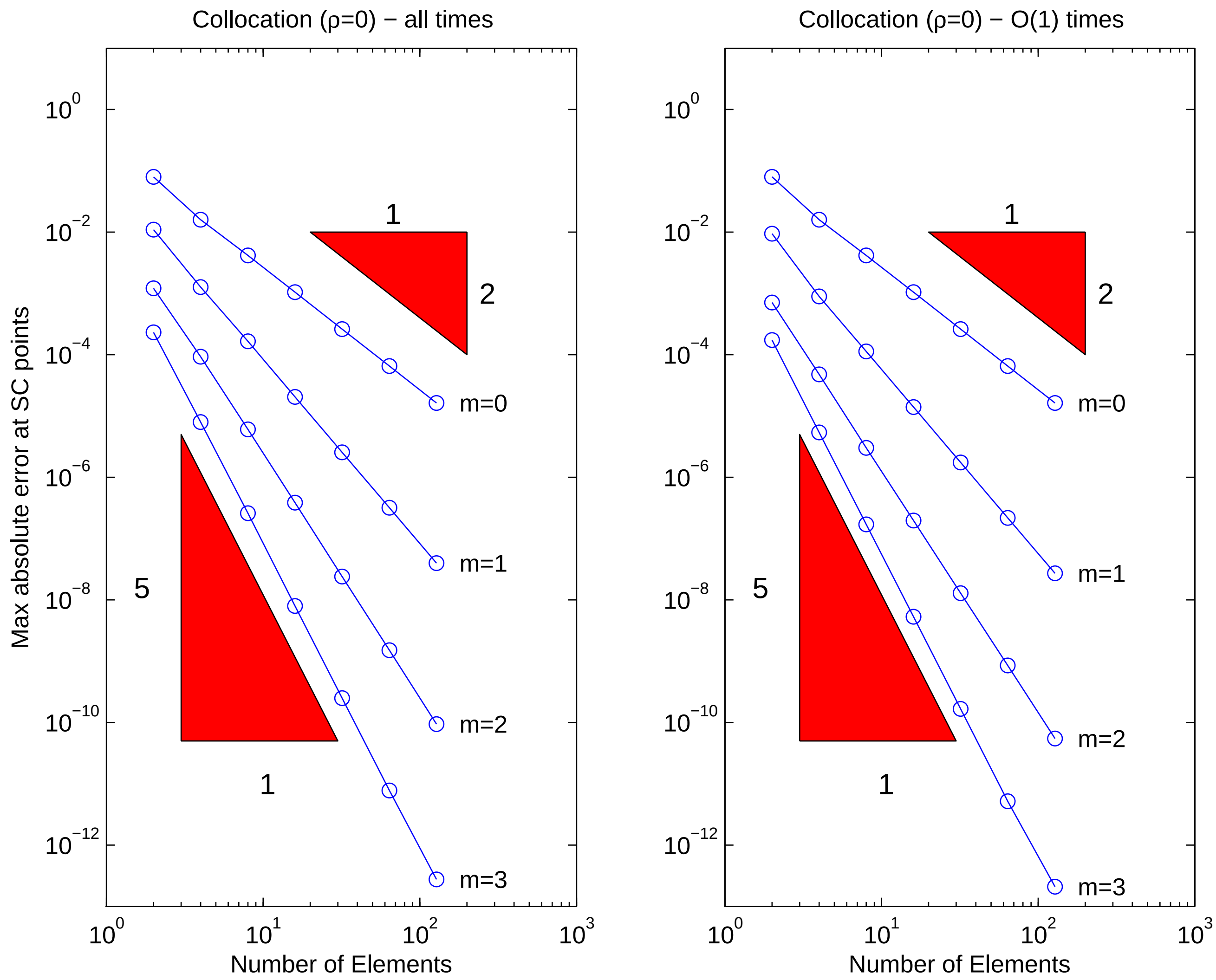
**Notation:** $\overline{P}_k(s) = P_k(2s-1)$: Legendre polys on $(0,1)$

- **Error when $\rho_m = 1$:** $e(t_n + s\Delta t) = \Delta t^m\, \overline{P}_m(s) + O(\Delta t^{m+1})$
- **Error when $\rho_m \in [-1,1)$:**

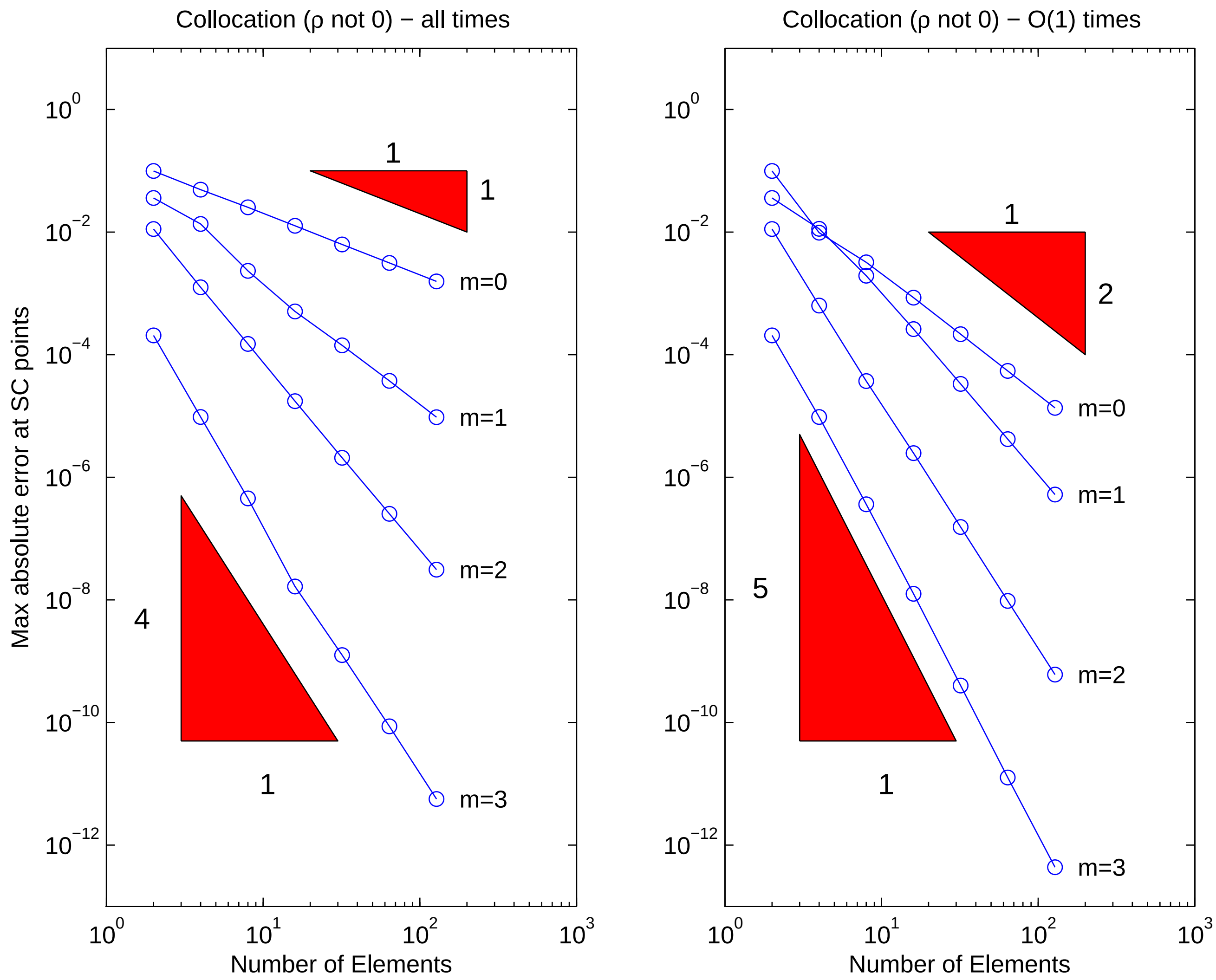
$$e(t_n + s\Delta t) = \Delta t^{m+1} \left[ \overline{P}_{m+1}(s) - \rho_m^{n+1}\, \overline{P}_m(s)\, u_0^{(m+1)} \right] + O(\Delta t^{m+2})$$

- **Superconvergence results:**
  - if $\rho_m = 1$ then $e(t_n + \tilde{s}_r \Delta t) = O(\Delta t^{m+1})$
  - if $\rho_m = 0$ then $e(t_n + s_r \Delta t) = O(\Delta t^{m+2})$ (Brunner)
  - if $\rho_m \in [-1,0) \cup (0,1)$ and $u_0^{(m+1)} = 0$ then $e(t_n + s_r\Delta t) = O(\Delta t^{m+2})$
  - if $0 < |\rho_m| < 1$ and $u_0^{(m+1)} \neq 0$ then get $O(\Delta t^{m+2})$ superconvergence at **late times**

# Collocation: $\rho_m = 0$, $u_0^{(m+1)} \neq 0$



# Collocation: $0 < |\rho_m| < 1$, $u_0^{(m+1)} \neq 0$



# **Quadrature Galerkin:** $q \geq m$

- $q+1$ **point quadrature rule:** $\int_0^1 f(s)\,ds \approx Q[f(s);s] = \sum_{i=0}^{q} \omega_i\, f(d_i)$

- **If $q = m$:** convergence is as for collocation and is **independent** of the weights $\omega_i$

- **Take $q \geq m$** and suppose that $Q$ has degree of precision $\geq 2m+1$ (Gauss quadrature with $q = m$ has dop $= 2m+1$)

- **Basis functions:** set $\phi_k(s) = P_k(2s-1)$ – Legendre polys on $(0,1)$

- Analysis very similar to that in BrDaDu for Galerkin

# QG convergence

- Set $Q_m = Q[\phi_m \cdot \phi_{m+2}]$. (Note $Q_m = 0$ if dop $> 2m+1$)
- **Global convergence**

$$\max_{[0,T]} |e(t)| \le C \begin{cases} \Delta t^m & \text{when } m \text{ is } \textbf{odd} \\ \Delta t^{m+1} & \text{when } m \text{ is } \textbf{even} \end{cases}$$

- **Local superconvergence**

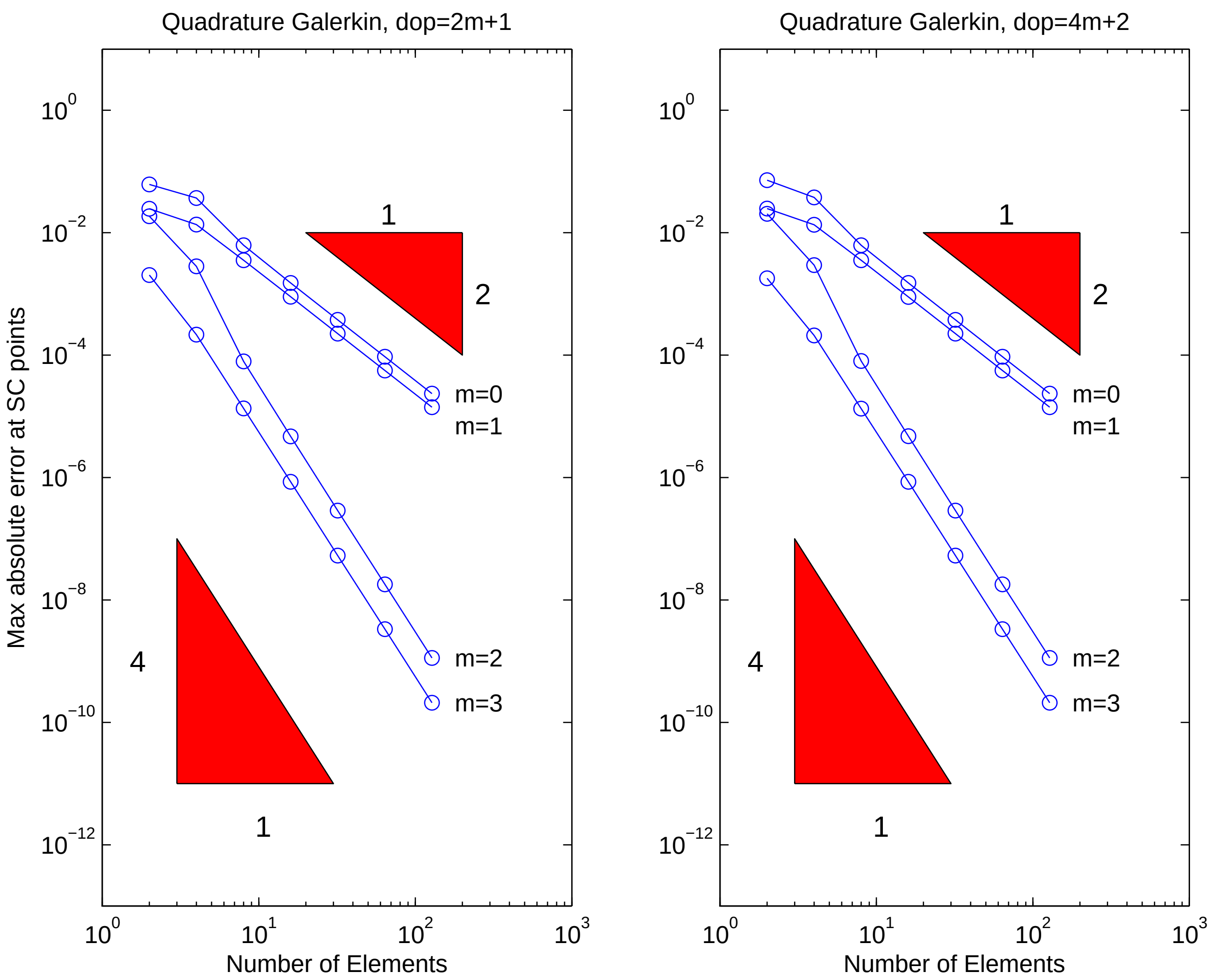
**odd $m$:** $|e(t_n + s_r\,\Delta t)| \le C\,\Delta t^{m+1}$ (where $P'_{m+1}(2s_r - 1) = 0$)

Note: if $Q_m = 1/(2m+1)$ then $e(t)$ is globally $O(\Delta t^{m+1})$

**even $m$:** $|e(t_n + s_r\,\Delta t)| \le C\,\Delta t^{m+2}$ **iff** $u$ satisfies $u_0^{(m+1)} = 0$ ($s_r$ are the roots of $P'_{m+2} - (2m+1)\,Q_m\,P'_m = 0$)

# QG results: $q > m$

## $q = 2m + 1$; at superconvergence points $(u_0^{(m+1)} = 0)$



# Summary: (smooth) VIE

- **Galerkin.** If $m$ is even, then there is local superconvergence of $O(\Delta t^{m+2})$, but only if $u_0^{(m+1)} = 0$
- **Collocation.** If $\rho_m = 0$ then there is local superconvergence of $O(\Delta t^{m+2})$. If $0 < |\rho| < 1$ then there is local superconvergence of $O(\Delta t^{m+2})$:
  - if $u_0^{(m+1)} = 0$
  - at late times, whatever $u_0^{(m+1)}$ is
- **$q-$point Quadrature Galerkin.** If $q = m$, same as collocation. If $q \geq m$ and dop is at least $2m+1$, similar to Galerkin

# Implications for scattering BIE?

Acoustic scattering BIE:

$$\frac{1}{2\pi}\int_\Gamma \frac{u(\boldsymbol{x}', t - |\boldsymbol{x}' - \boldsymbol{x}|)}{|\boldsymbol{x}' - \boldsymbol{x}|}\, dS' = a(\boldsymbol{x}, t)$$

**Two problems:**

- **High frequency:** the Fourier transformed BIE has kernel $K(t) = J_0(\omega\, t)$ where can have $\omega \approx 1/\Delta t$. But VIE convergence analysis needs $\omega\Delta t \to 0$ as $\Delta t \to 0$

- The "method of lines" approximation of the BIE gives a system of VIEs with **non-smooth** kernels.

  **Example.** $\Gamma = [-1, 1]^2$ with piecewise constant spatial approximation using $\Delta x = 2$ (one element), collocated in space at midpoint $\boldsymbol{x}_0 = (0, 0)$.

# Simple BIE example

**BIE:** $$\frac{1}{2\pi}\int_{[-1,1]^2}\frac{u(\boldsymbol{x}',t-|\boldsymbol{x}'-\boldsymbol{x}|)}{|\boldsymbol{x}'-\boldsymbol{x}|}\,d\boldsymbol{x}' = a(\boldsymbol{x},t)$$

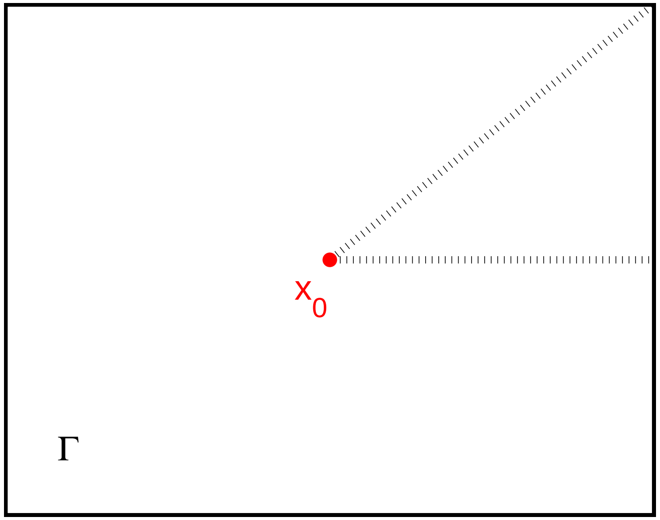
## Spatial approximation:

- One spatial element ($\Delta x = 2$)
- Piecewise constant approximation: $u(\boldsymbol{x},t) \approx U(t)$ where $U$ is **causal**, i.e. $U(t) = 0$ for $t < 0$
- Collocate at $\boldsymbol{x} = \boldsymbol{x}_0 = (0,0)$

$\Rightarrow$ **Approx BIE:** $$\frac{1}{2\pi}\int_{[-1,1]^2}\frac{U(t-|\boldsymbol{x}'-\boldsymbol{x}_0|)}{|\boldsymbol{x}'-\boldsymbol{x}_0|}\,d\boldsymbol{x}' = a(\boldsymbol{x}_0,t)$$

# Convert to VIE

**Approx BIE:** $$\frac{1}{2\pi}\int_{[-1,1]^2}\frac{U(t-|\boldsymbol{x}'-\boldsymbol{x}_0|)}{|\boldsymbol{x}'-\boldsymbol{x}_0|}\,d\boldsymbol{x}' = a(\boldsymbol{x}_0,t)$$



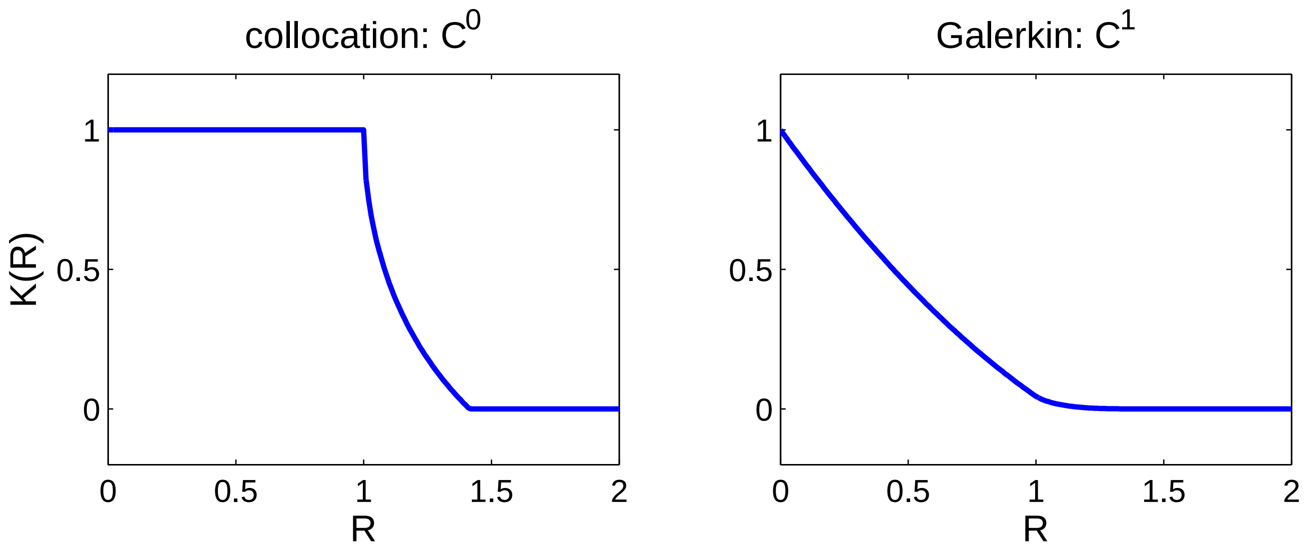

- Change variables: $\boldsymbol{x}' = \boldsymbol{x}_0 + R\,(\cos\theta,\sin\theta)$ and integrate in $\theta$

# Approx BIE as VIE

- Change of variables gives $\displaystyle\int_0^t K(R)\,U(t-R)\,dR = a(\boldsymbol{x}_0, t)$ with

$$K(R) = \begin{cases} 1 & 0 < R \le 1 \\ 1 - \frac{4}{\pi}\cos^{-1}(1/R) & 1 < R \le \sqrt{2} \end{cases}$$

- Kernel is continuous but **not smooth**



# Realistic discretised BIE:

- Discretising the BIE in space on a realistic mesh gives a matrix VIE of the form

$$\int_0^t \mathbb{K}(R)\,\boldsymbol{U}(t-R)\,dR = \boldsymbol{a}(t)$$

where components $\mathbb{K}_{i,j}$ have the same smoothness as $K$ on the previous slide

- **Current work:** we're looking at methods for equations/systems like these. . .