Stabilization of convection-diffusion problems by Shishkin mesh simulation

Bosco García-Archilla

Universidad de Sevilla

SCMS 19th Annual Meeting, Edinburgh 2010
In honour of David F. Griffiths on his retirement

Outline

1) Convection-diffusion problems

    Numerical difficulties

    Shishkin meshes

2) The new stabilization technique

    Shishkin mesh simulation

    Numerical experiments

3) Conclusions

Convection-diffusion problems

$$\left.\begin{aligned} -\varepsilon\Delta u + b\cdot\nabla u &= f, &\quad &\text{in } \Omega,\\ u &= 0, & &\text{on } \partial\Omega, \end{aligned}\right\},$$

$$\Omega \subset \mathbb{R}^d, \quad d = 1,2,3, \qquad b \in \mathbb{R}^d$$

Convection-diffusion problems

$$\left.\begin{aligned} -\varepsilon\Delta u + b\cdot\nabla u &= f, && \text{in } \Omega,\\ u &= 0, && \text{on } \partial\Omega, \end{aligned}\right\},$$

$$(b = b(x), \quad x \in \Omega)$$

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$$(\varepsilon \ll \|b\|_{L^\infty})$$

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Case $\varepsilon = 0$

$$\left.\begin{aligned} b\cdot\nabla u = f, &\qquad \text{in } \Omega,\\ u = 0, &\qquad \text{on } \partial\Omega_-, \end{aligned}\right\},$$

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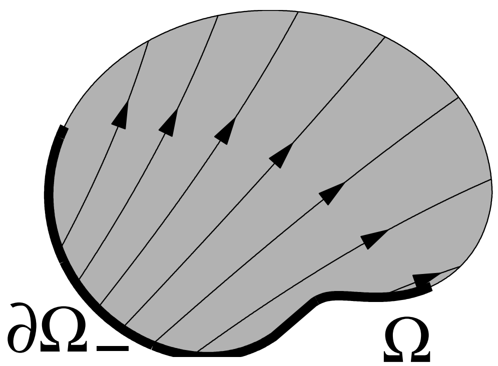

characteristic curves : $\left\{\begin{array}{l} \dfrac{d}{ds}x(s) = b(x(s)), \\ \\ x(0)) = x_0 \in \partial\Omega_-. \end{array}\right.$

Convection-diffusion problems

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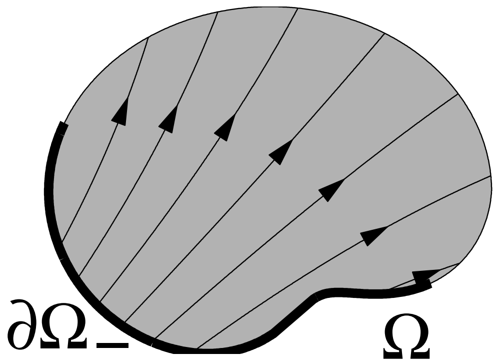

$$\varepsilon = 0: \qquad \left\{\begin{array}{l} \dfrac{d}{ds}u(x(s)) = f(x(s)), \\ \\ u(x(0))) = 0. \end{array}\right.$$

(u evolves along the characteristics)

Convection-diffusion problems

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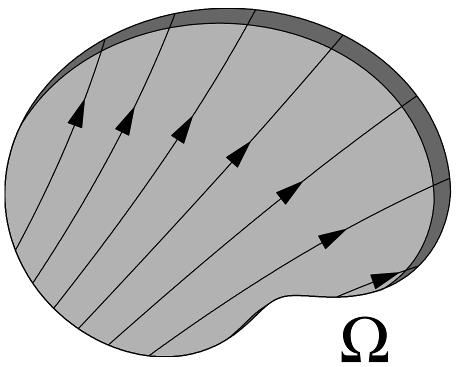


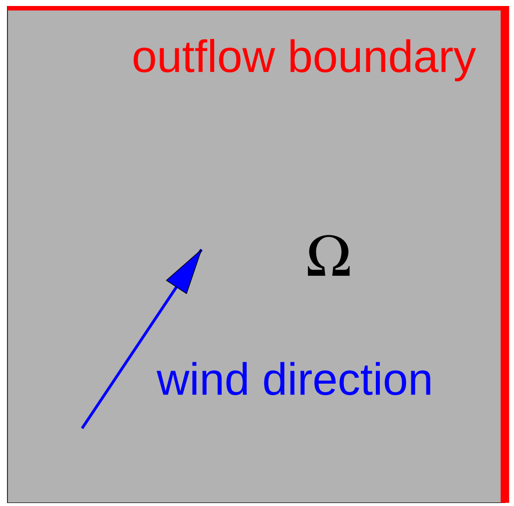
$\varepsilon > 0$: boundary layer

Example

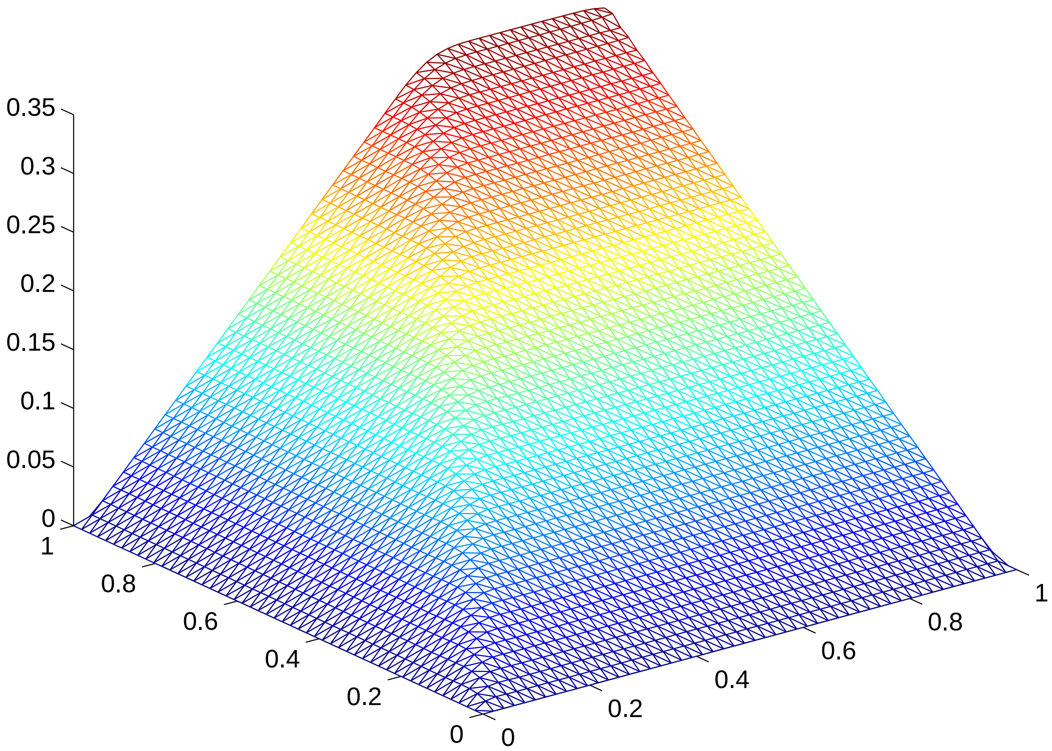
$$\left.\begin{aligned} -\varepsilon \Delta u + b \cdot \nabla u = f, \qquad & \text{in } \Omega, \\ u = 0, \qquad & \text{on } \partial\Omega, \end{aligned}\right\}$$

where

$$\Omega = [0,1] \times [0,1], \qquad b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \qquad \text{and} \quad f(x) = 1.$$

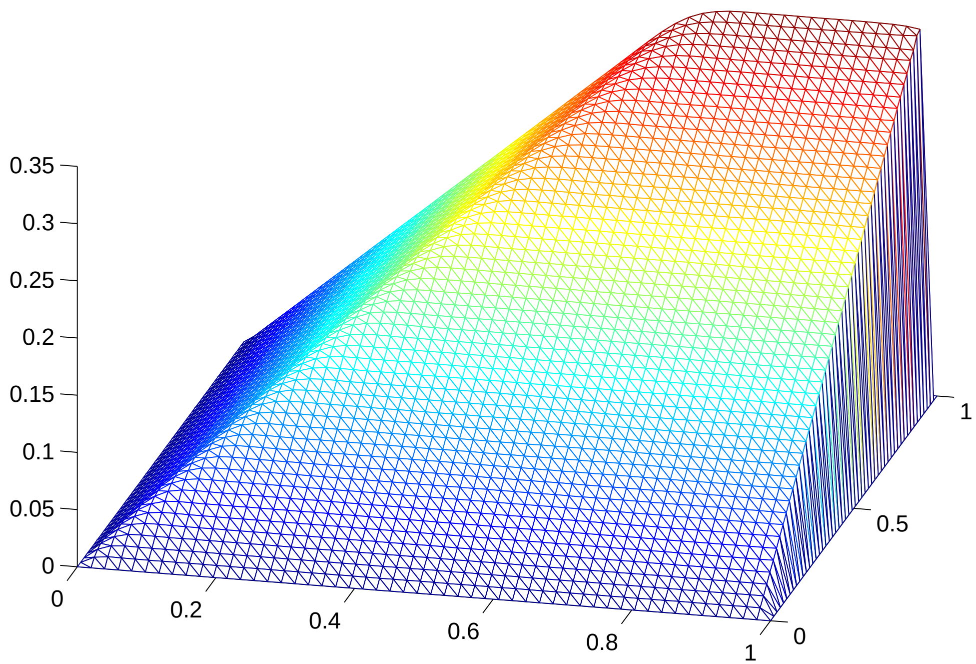


Example



true solution ($\varepsilon = 0.001$).

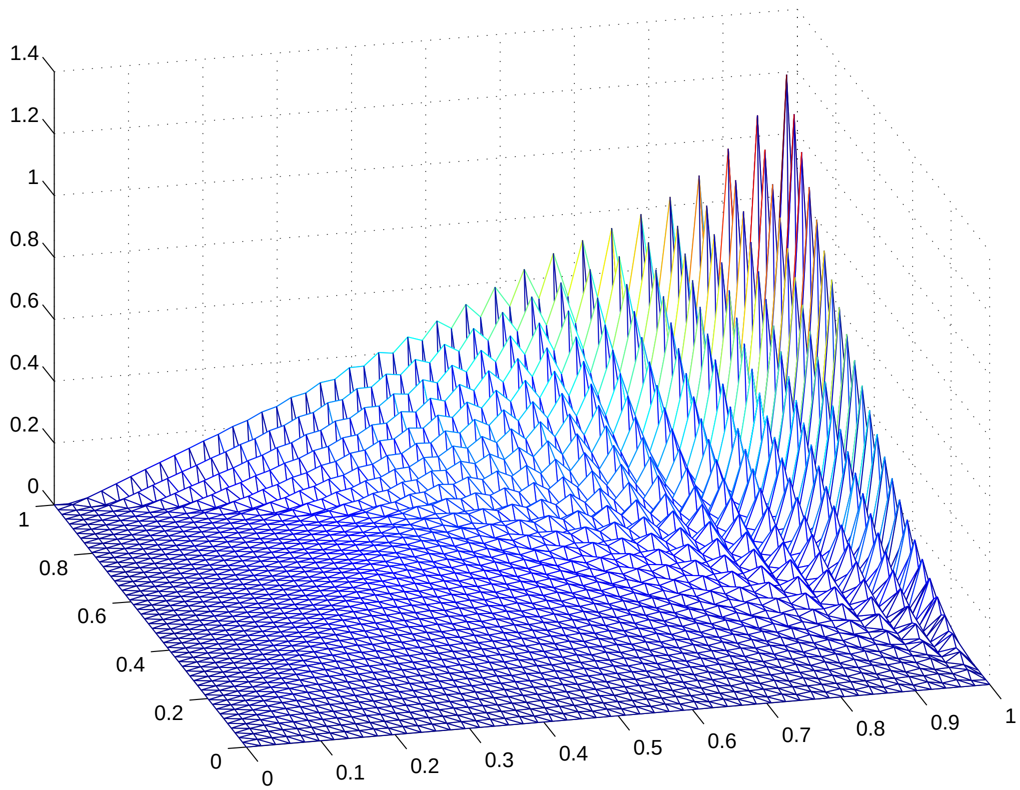
Example



true solution from a different point of view

(notice the boundary layer at $x = 1$)

Example



piecewise linear finite-element approximation ($h = 0.0283$)

Possible remedies:

- Upwinding
- Strongly consistent stabilized methods (streamline diffusion, GALS, etc)
- Bubble functions
- Shishkin meshes

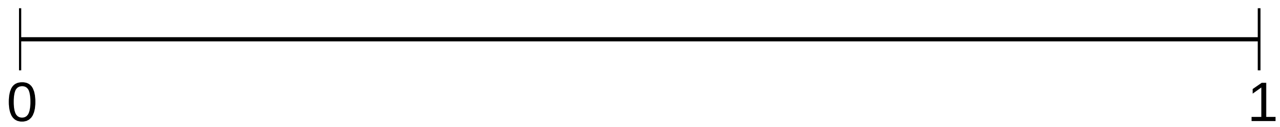
⋮

(See e. g., M. Stynes, *Acta Numerica*, **14** (2005), 445–508.)

Shishkin mesh

$$\left.\begin{array}{r}-\varepsilon u_{xx} + b u_x = f, \\ u(0) = u(1) = 0.\end{array}\right\}$$

N subdivisions



Shishkin mesh

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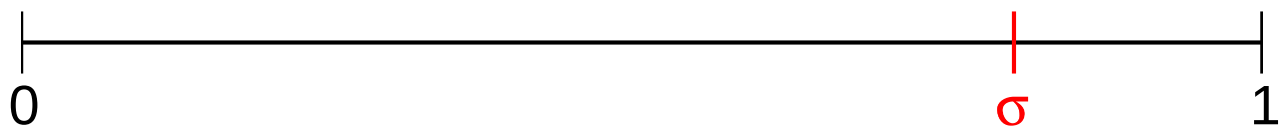
$$1 - \sigma = O(\varepsilon \log(N))$$



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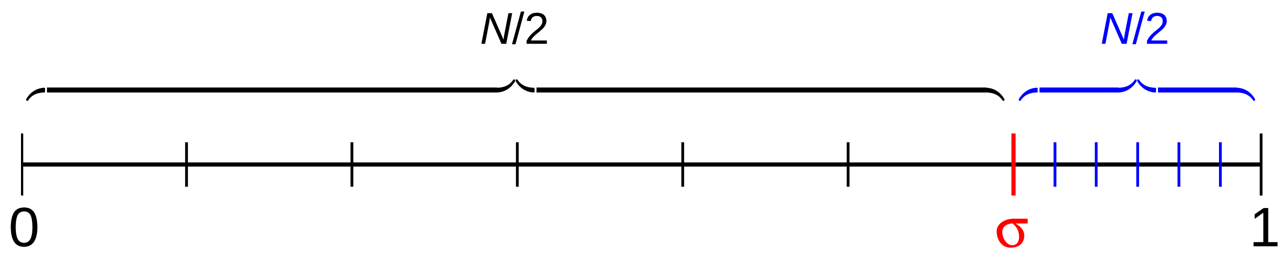

Shishkin mesh (2-D)

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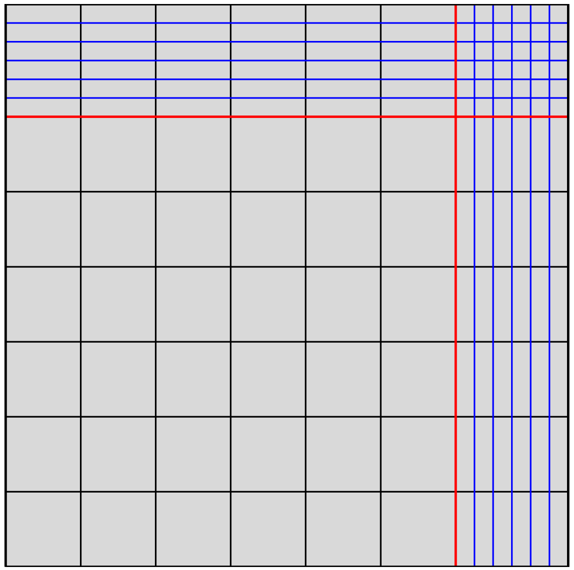
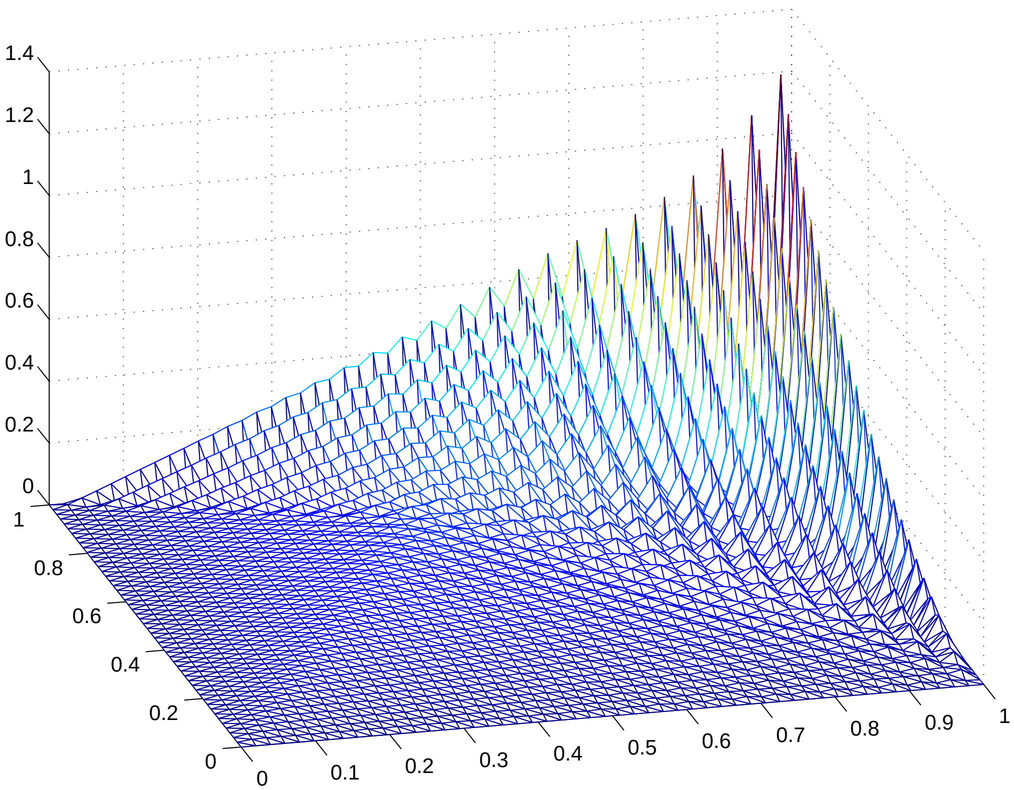
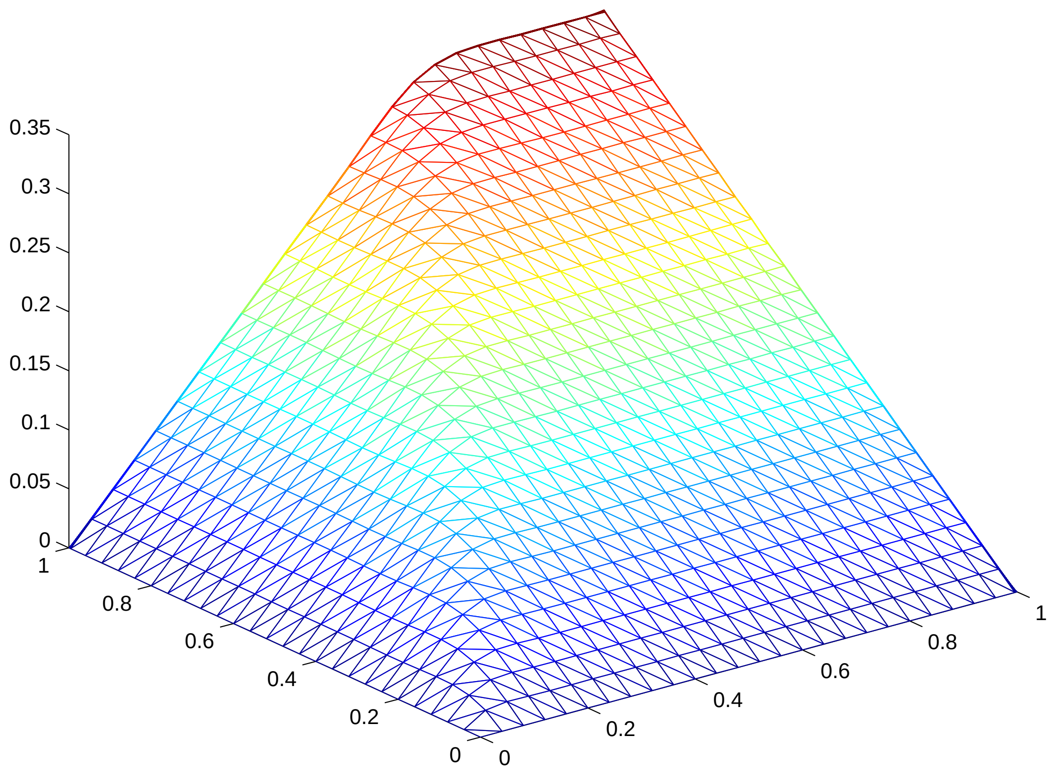




piecewise linear finite-element approximation on a uniform mesh



piecewise linear finite-element approximation on a Shishkin mesh

Shishkin mesh

But . . .

But . . .

Difficult to construct if curved boundaries.

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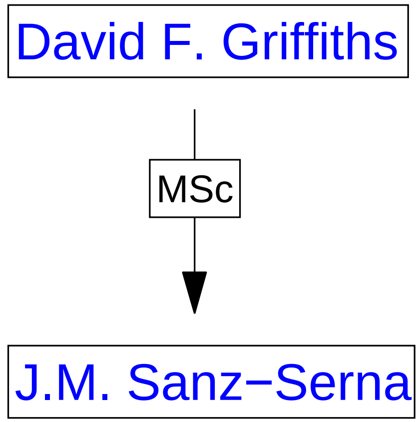
Also, linear systems difficult to solve, Linß & Stynes (2001)

THE NEW STABILIZATION TECHNIQUE

Shishkin mesh simulation

David F. Griffiths


David F. Griffiths
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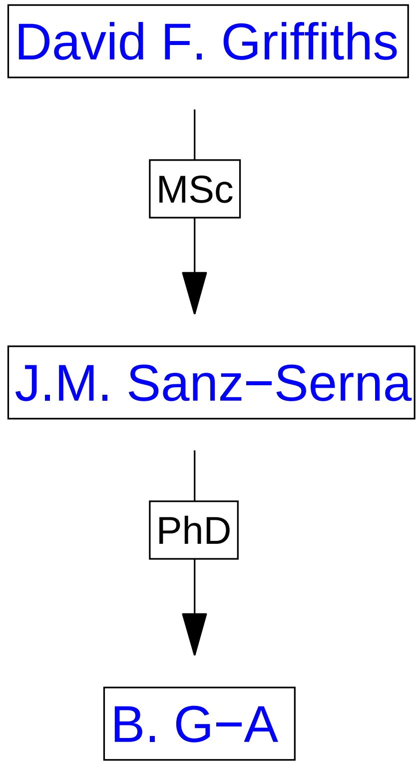
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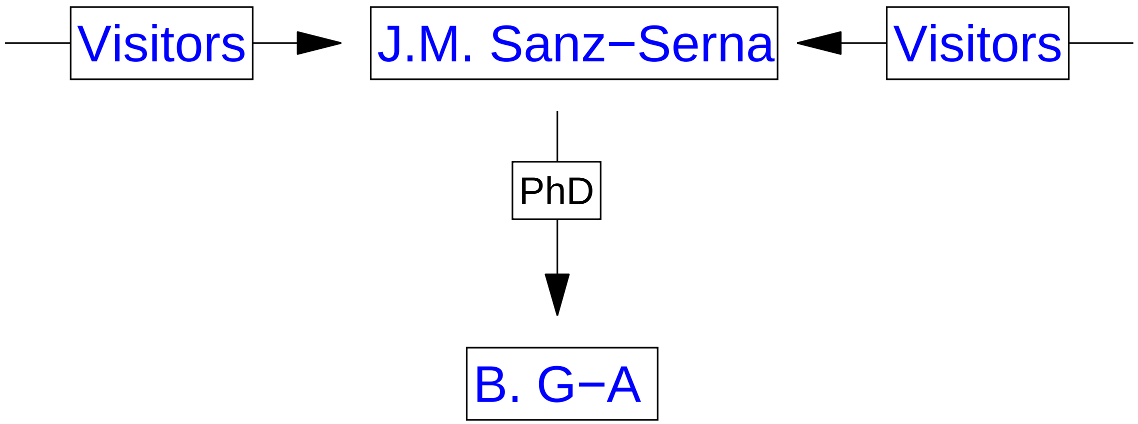
Visitors
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Consider the Shishkin mesh in $[0, 1]$,

$$0, h, 2h, \ldots (N-1)h, \sigma, \sigma + h', \ldots, \sigma + (N-1)h', 1.$$

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Let $\underline{u}_h^s$ the vector on *interior* nodal values of the Galerkin appr.

$$\underline{u}_h^s = \left[u_1, \ldots, u_{N-1}, u_N, u_{N+1}, \ldots, u_{2N-1}\right]^T.$$

obtained by solving the linear system $\qquad S_h \underline{u}_h^s = \underline{f}_h^s,$

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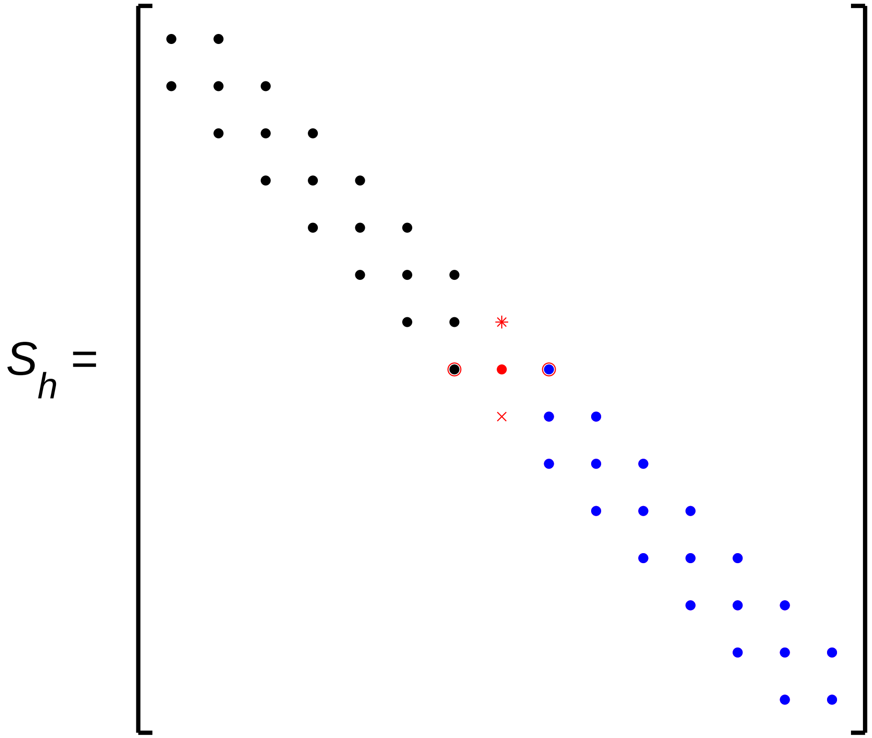
obtained by solving the linear system $\qquad S_h \underline{u}_h^s = \underline{f}_h^s,$

and le us write $\qquad \underline{u}_h^s = \begin{bmatrix} \underline{u}_h^c \\ u_N \\ \underline{u}_{h'}^o \end{bmatrix}, \qquad \underline{f}_h^s = \begin{bmatrix} \underline{f}_h^c \\ f_N \\ \underline{f}_{h'}^o \end{bmatrix}.$

The estructure of the matrix S_h



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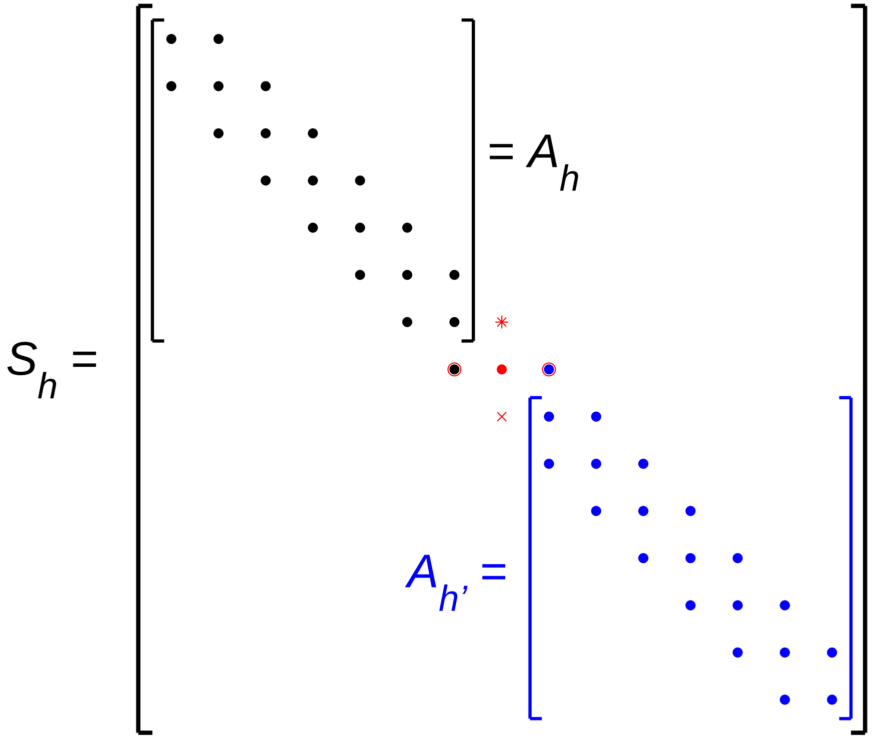

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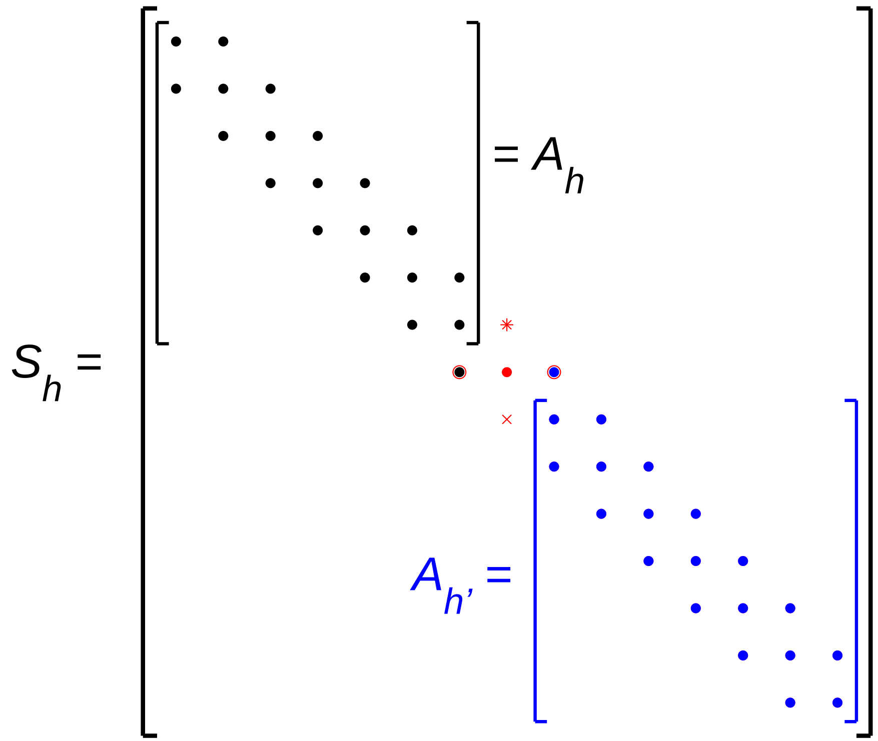
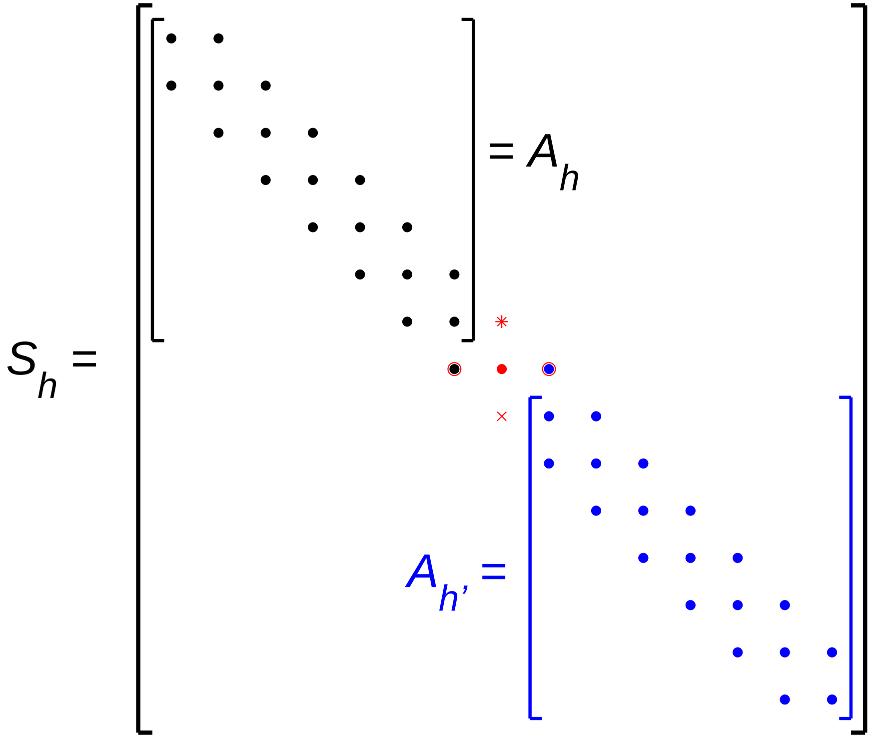


$$S_h \underline{u}_h^s = f_h^s$$

The estructure of the matrix S_h



$$\underline{u}_h^s = \begin{bmatrix} \underline{u}_h^c \\ u_N \\ \underline{u}_{h'}^o \end{bmatrix}, \qquad \underline{f}_h^s = \begin{bmatrix} \underline{f}_h^c \\ f_N \\ \underline{f}_{h'}^o \end{bmatrix},$$

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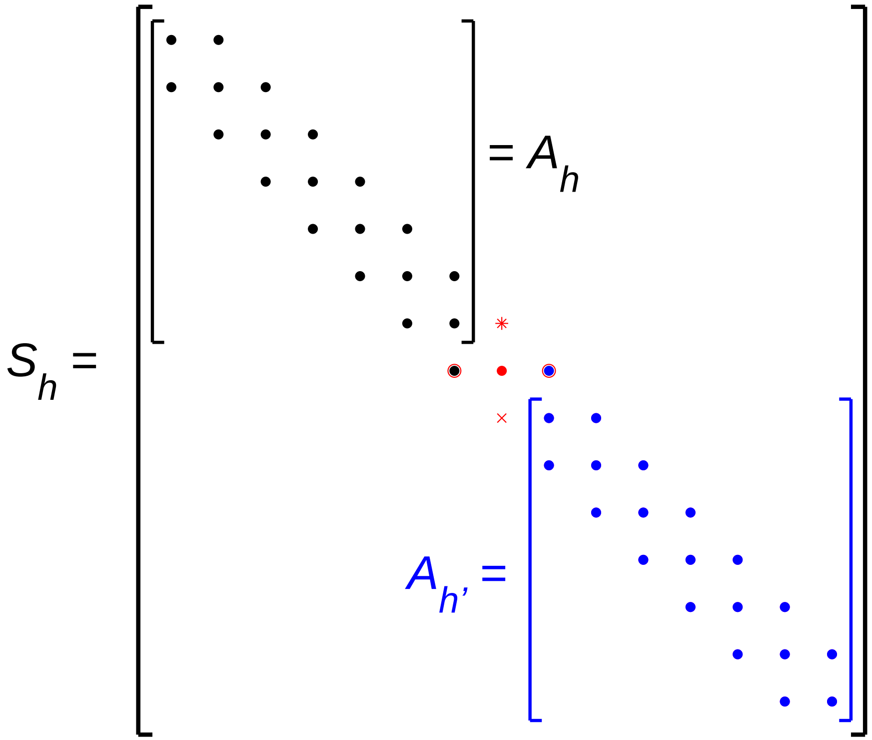

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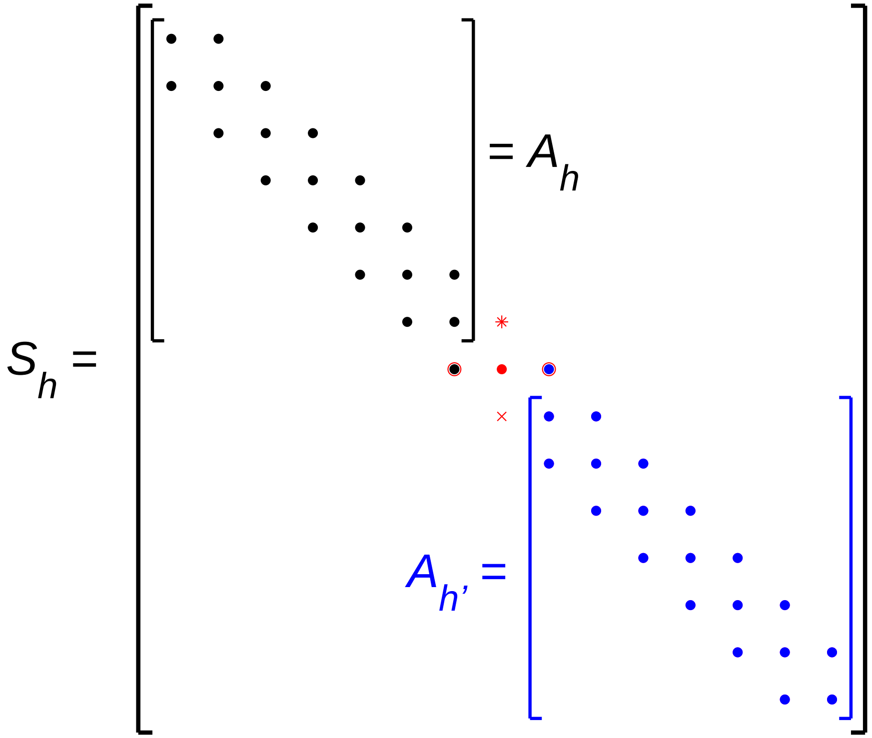
The estructure of the matrix S_h



$$
\begin{aligned}
A_h \underline{u}_h^c \quad &+ \quad \beta u_N \underline{e}_{N-1} & &= \underline{f}_h^c, \\
\boxed{s_{N-1,N}}\, u_{N-1} \quad &+ \quad s_{N,N} u_N \quad + \quad \boxed{s_{N,N+1}}\, u_{N+1} & &= f_N, \\
&\quad \gamma u_N \underline{e}_1 \quad + \quad A_{h'} \underline{u}_{h'}^o & &= \underline{f}_{h'}^o.
\end{aligned}
$$

Let us summarize.

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a Shishkin mesh, $\quad 0, h, 2h, \ldots (N-1)h, \sigma, \sigma + h', \ldots, \sigma + (N-1)h', 1.$

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the Galerkin appr. $\underline{u}_h^s = [u_1, \ldots, u_{N-1}, u_N, u_{N+1}, \ldots, u_{2N-1}]^T$

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solution of the linear system $S_h \underline{u}_h^s = \underline{f}_h^s,$

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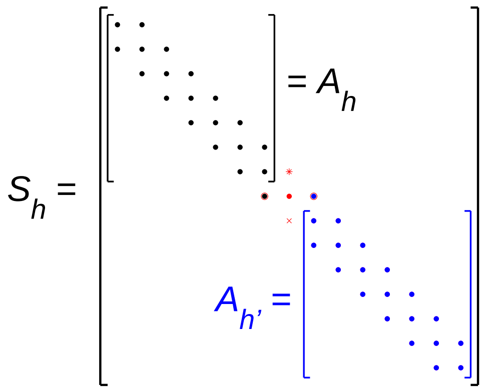
solution of the linear system $S_h \underline{u}_h^s = \underline{f}_h^s,$

with matrix



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solution of the linear system $S_h \underline{u}_h^s = \underline{f}_h^s,$

so that writing $\underline{u}_h^s = \begin{bmatrix} \underline{u}_h^c \\ u_N \\ \underline{u}_{h'}^o \end{bmatrix}, \quad \underline{f}_h^s = \begin{bmatrix} \underline{f}_h^c \\ f_N \\ \underline{f}_{h'}^o \end{bmatrix}, \quad \begin{matrix} \underline{e}_{N-1} \\ \\ \underline{e}_1 \end{matrix}$ coord. vects.

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solution of the linear system $S_h \underline{u}_h^s = \underline{f}_h,$

$$
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A_h \underline{u}_h^c & + & \beta u_N \underline{e}_{N-1} & & & = & \underline{f}_h^c, \\
\boxed{s_{N-1,N}} u_{N-1} & + & s_{N,N} u_N & + & \boxed{s_{N,N+1}} u_{N+1} & = & f_N, \\
 & & \gamma u_N \underline{e}_1 & + & A_{h'} \underline{u}_{h'}^o & = & \underline{f}_{h'}^o.
\end{array}
$$

The key idea

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The key idea

$$A_h \underline{u}_h^c \quad + \quad \underbrace{\beta u_N}_{\alpha^*} \underline{e}_{N-1} \quad = \quad \underline{f}_h^c,$$

If we had α^*, we could forget about the rest.

The key idea

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If we had α^*, we could forget about the rest.

(i.e., we would solve $\quad A_h \underline{u}_h^c = \underline{f}_h^c - \underbrace{\beta u_N}_{\alpha^*} \underline{e}_{N-1}$)

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If we had α^*, we could forget about the rest.

So let us approximate it

The key idea

$$A_h \underline{u}_h^c \qquad + \quad \underbrace{\beta u_N}_{\alpha^*} \underline{e}_{N-1} \qquad\qquad = \quad \underline{f}_h^c,$$

If we had α^*, we could forget about the rest.

So let us approximate it

(i.e., find $\alpha \approx \alpha^*$)

The key idea

$$A_h \underline{u}_h^c \qquad + \quad \underbrace{\beta u_N}_{\alpha^*} \underline{e}_{N-1} \qquad\qquad = \quad \underline{f}_h^c,$$

If we had α^*, we could forget about the rest.

So let us approximate it

How?

The key idea

$$A_h \underline{u}_h^c \qquad + \quad \underbrace{\beta u_N}_{\alpha^*} \underline{e}_{N-1} \qquad\qquad = \quad \underline{f}_h^c,$$

If we had α^*, we could forget about the rest.

So let us approximate it

By demanding that $\underline{u}_h^\alpha = A_h^{-1} \underline{f}_h^c - \alpha A_h^{-1} \underline{e}_{N-1}$ be “oscillation-free”

The key idea

$$A_h \underline{u}_h^c \qquad + \quad \underbrace{\beta u_N}_{\alpha^*} \underline{e}_{N-1} \qquad\qquad = \quad \underline{f}_h^c,$$

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So let us approximate it

By demanding that $\underline{u}_h^\alpha = A_h^{-1} \underline{f}_h^c - \alpha A_h^{-1} \underline{e}_{N-1}$ be "oscillation-free"

or $$\min_{\alpha \in \mathbb{R}} \left\| A_h^{-1} \underline{f}_h^c - \alpha A_h^{-1} \underline{e}_{N-1} \right\|_M$$

Some details

$$\underline{u}_h^{\alpha} = A_h^{-1}\underline{f}_h^{c} - \alpha A_h^{-1}\underline{e}_{N-1},$$

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$$\underline{u}_h^{\alpha} = \underbrace{A_h^{-1}\underline{f}_h^{c}}_{\underline{u}_h} - \alpha\, A_h^{-1}\underline{e}_{N-1},$$

u_h (oscillation-ridden) Galerkin approximation of

$$\left.\begin{array}{c} -\varepsilon u_{xx} + b u_x = f, \\ u(0) = u(\sigma) = 0, \end{array}\right\}$$

Some details

$$\underline{u}_h^{\alpha} = \underbrace{A_h^{-1}\underline{f}_h^c}_{\underline{u}_h} - \alpha \underbrace{A_h^{-1}\underline{e}_{N-1}}_{\underline{q}_h},$$

u_h (oscillation-ridden) Galerkin approximation of

$$\left.\begin{aligned} -\varepsilon u_{xx} + b u_x = f, \\ u(0) = u(\sigma) = 0, \end{aligned}\right\}$$

q_h is the Galerkin approximation of

$$\left.\begin{aligned} -\varepsilon q_{xx} + b q_x = \delta_{\sigma - h}, \\ q(0) = 0, \quad q(\sigma) = 0, \end{aligned}\right\}$$

Some details

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4) The new approximation is $v_h = u_h - \alpha q_h.$

Example 1

$$\left.\begin{array}{l} -\varepsilon u_{xx} + u_x = f, \\ u(0) = u(1) = 0. \end{array}\right\}$$

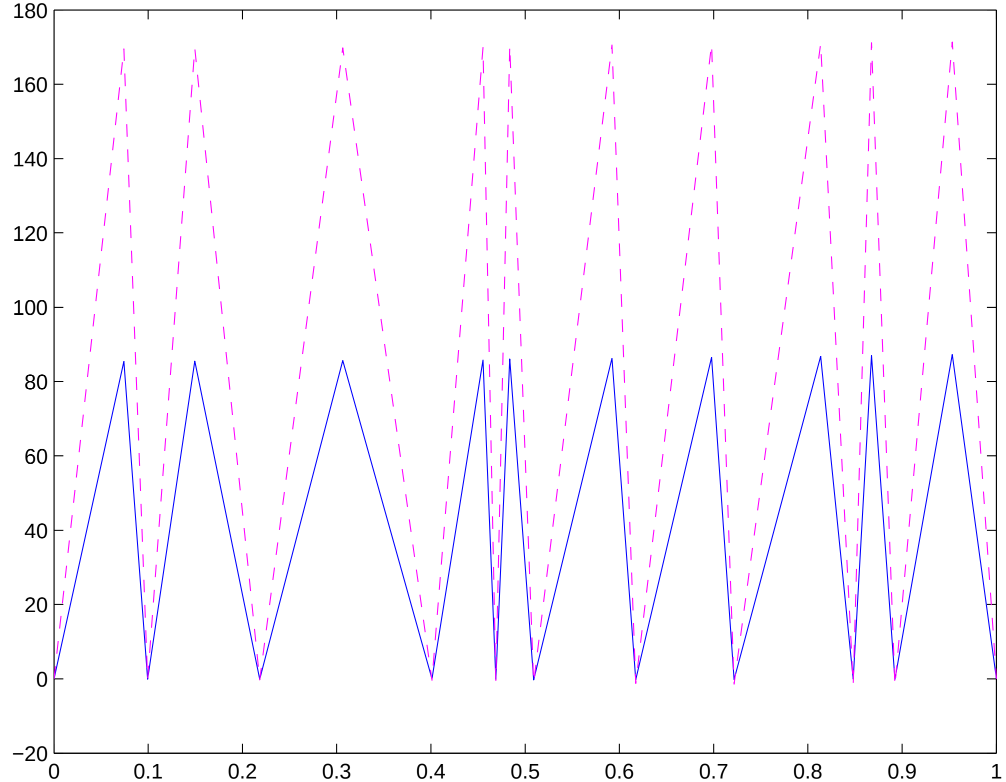
with solution

$$u(x) = x^2 - \frac{e^{(x-1)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}.$$

Results for

$$\varepsilon = 0.00001$$

Example 1



Galerkin approximation u_h (blue) and q_h (magenta)

on a random mesh with $N = 20$ elements

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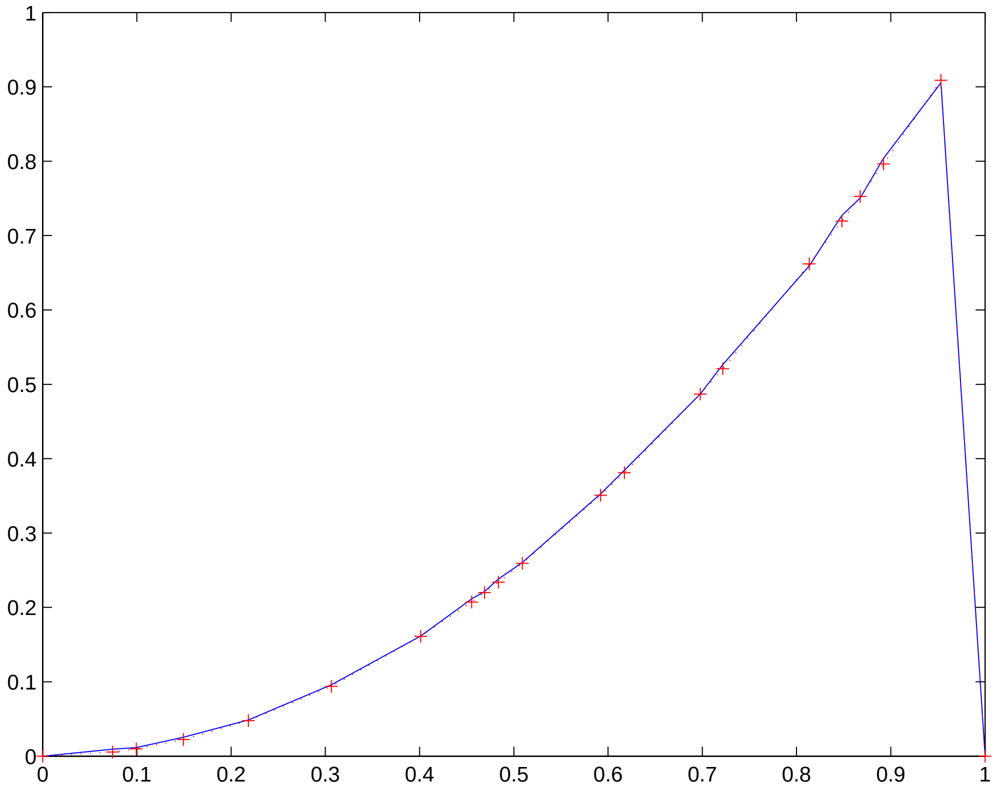

New approximation $v_h = u_h - \alpha q_h$ (blue)

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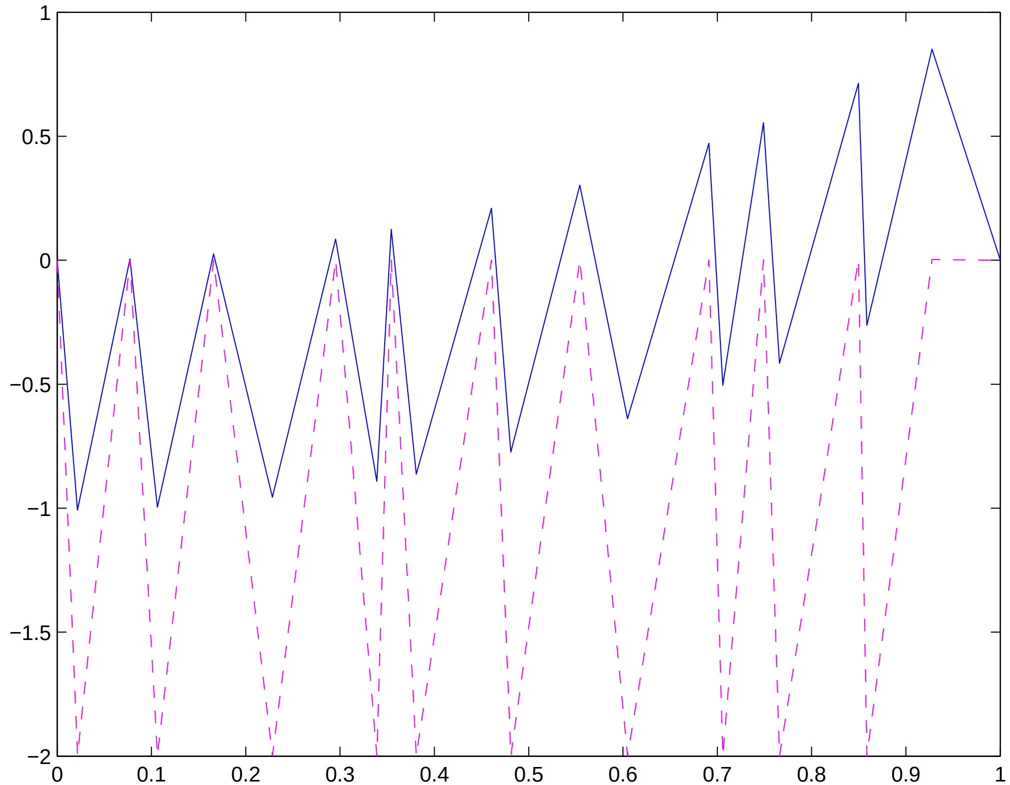

Galerkin approximation u_h (blue) and q_h (magenta)

on a random mesh with $N = 21$ elements

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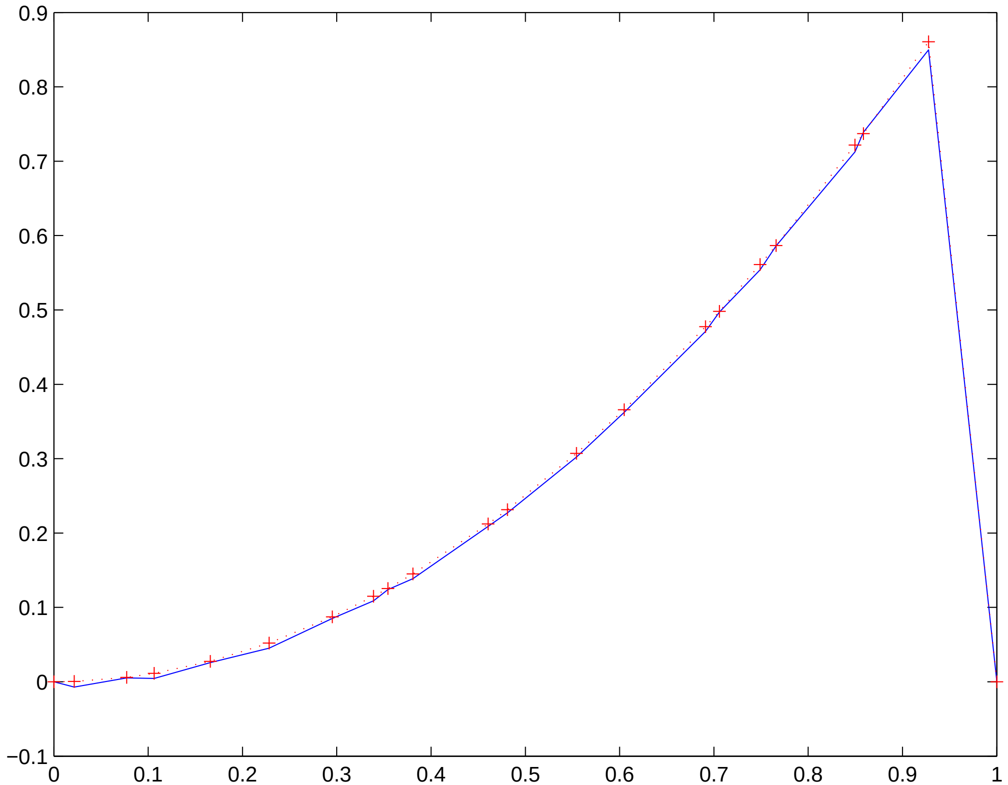

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Example 2

$$\left.\begin{aligned} -\varepsilon\Delta u + b\cdot\nabla u = f, &\qquad \text{in } \Omega,\\ u = 0, &\qquad \text{on } \partial\Omega, \end{aligned}\right\}$$

where

$$\Omega = [0,1]\times[0,1], \qquad b = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

with solution

$$u(x) = \left(x - \frac{e^{2(x-1)/\varepsilon} - e^{-2/\varepsilon}}{1 - e^{-2/\varepsilon}}\right)\left(y^2 - \frac{e^{3(y-1)/\varepsilon} - e^{-2/\varepsilon}}{1 - e^{-3/\varepsilon}}\right).$$

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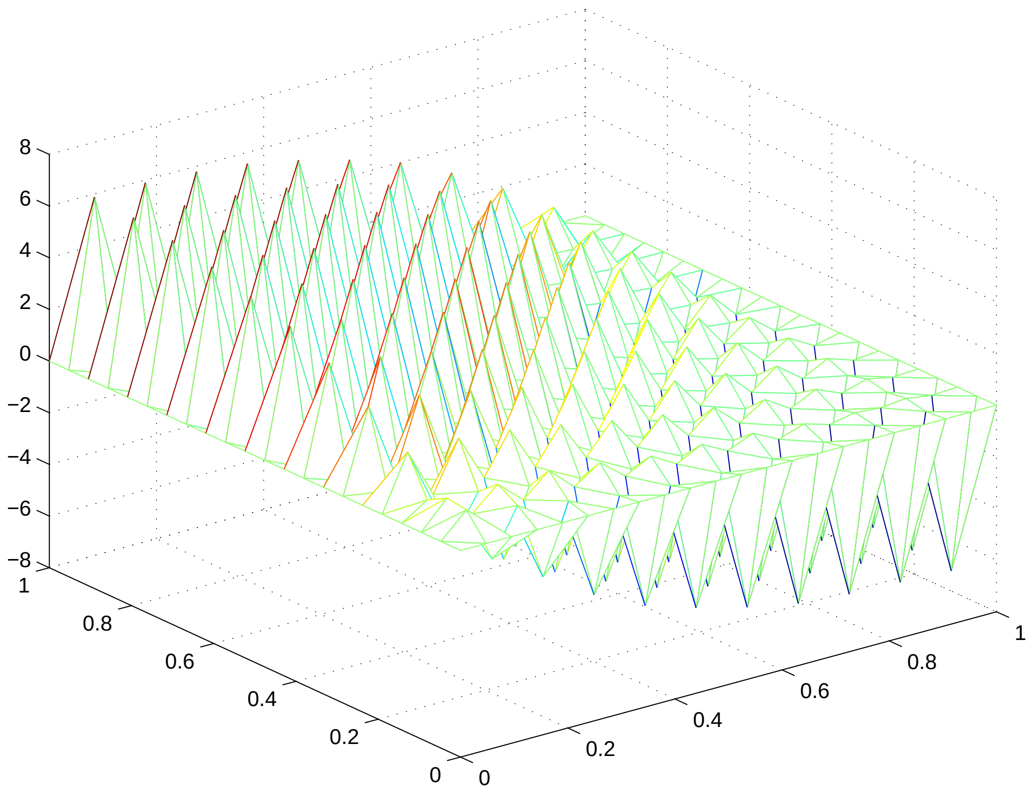
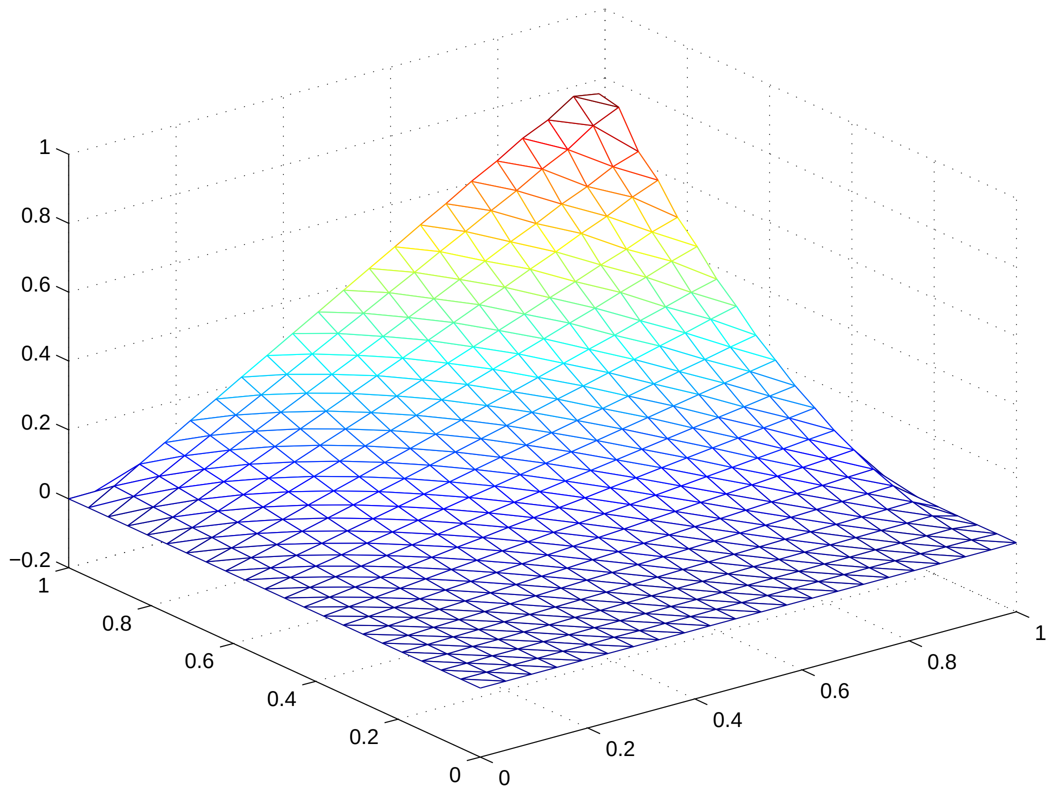


Galerkin approximation u_h on a uniform 21×21 mesh.

Example 2



The new approximation v_h

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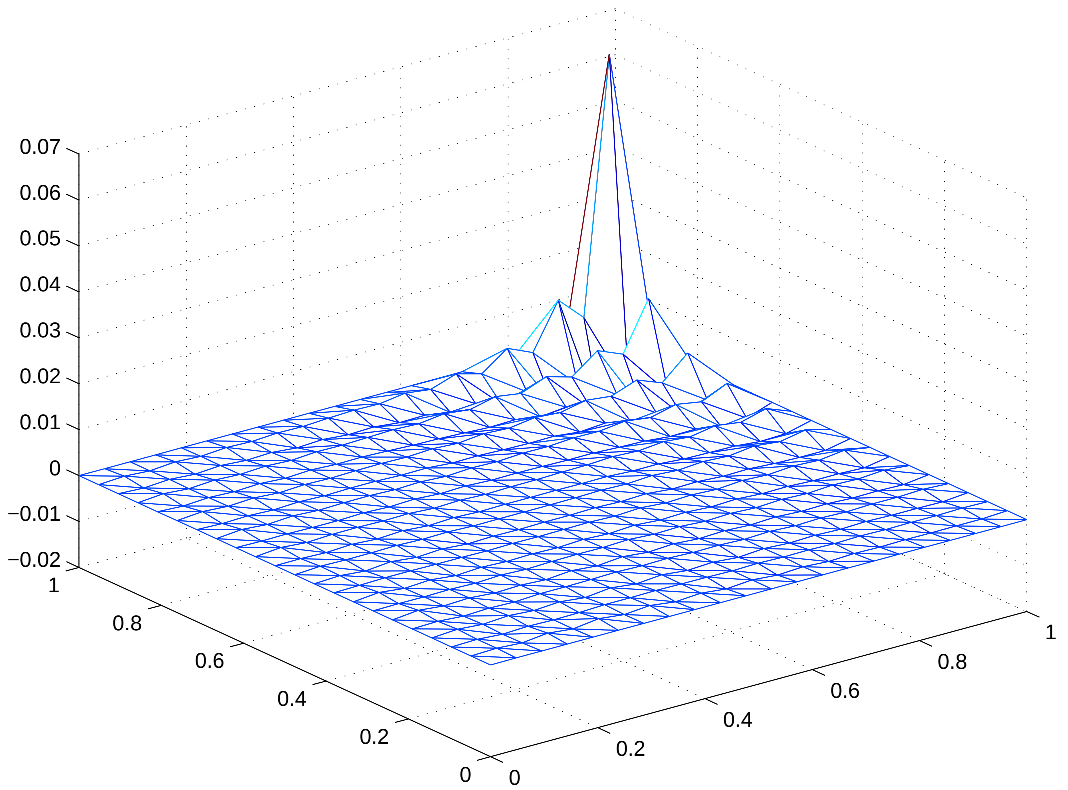

The error of the new approximation v_h

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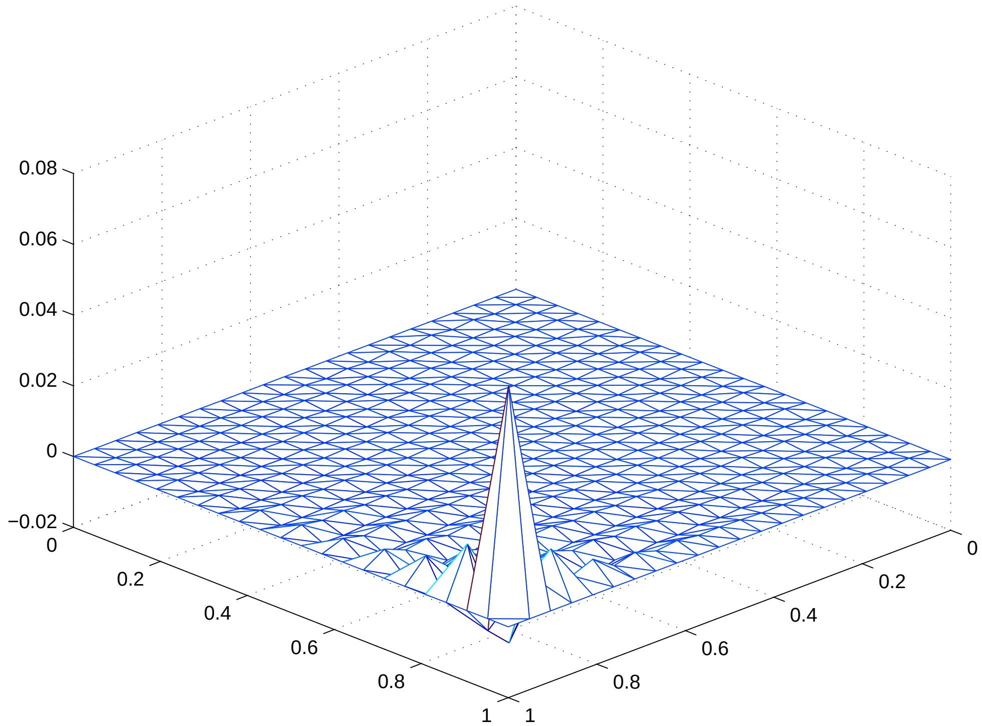
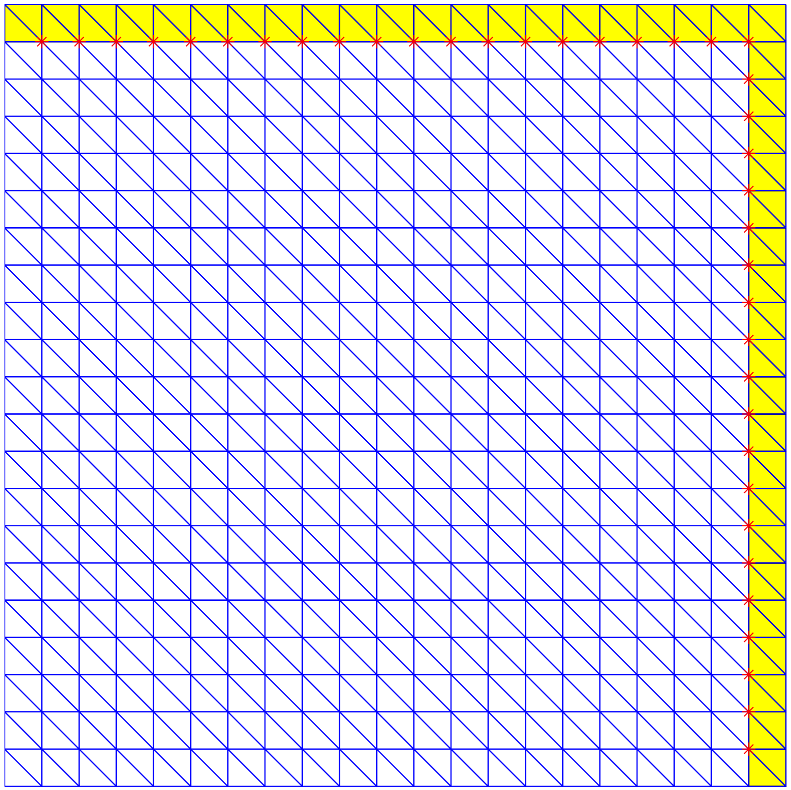


The error of the new approximation v_h

from a different point of view.

Example 2



The mesh. In yellow the region excluded from the l. s. problem.
Marked with $*$ the δ points.

Example 2

Delta points $x_1, \dots, x_n$:

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and

$$\min_{\alpha^{(1)},\dots,\alpha^{(n)}} \left\| \nabla \left(u_h - \sum_{j=1}^{n} \alpha^{(j)} q_h^{(j)} \right) \right\|$$

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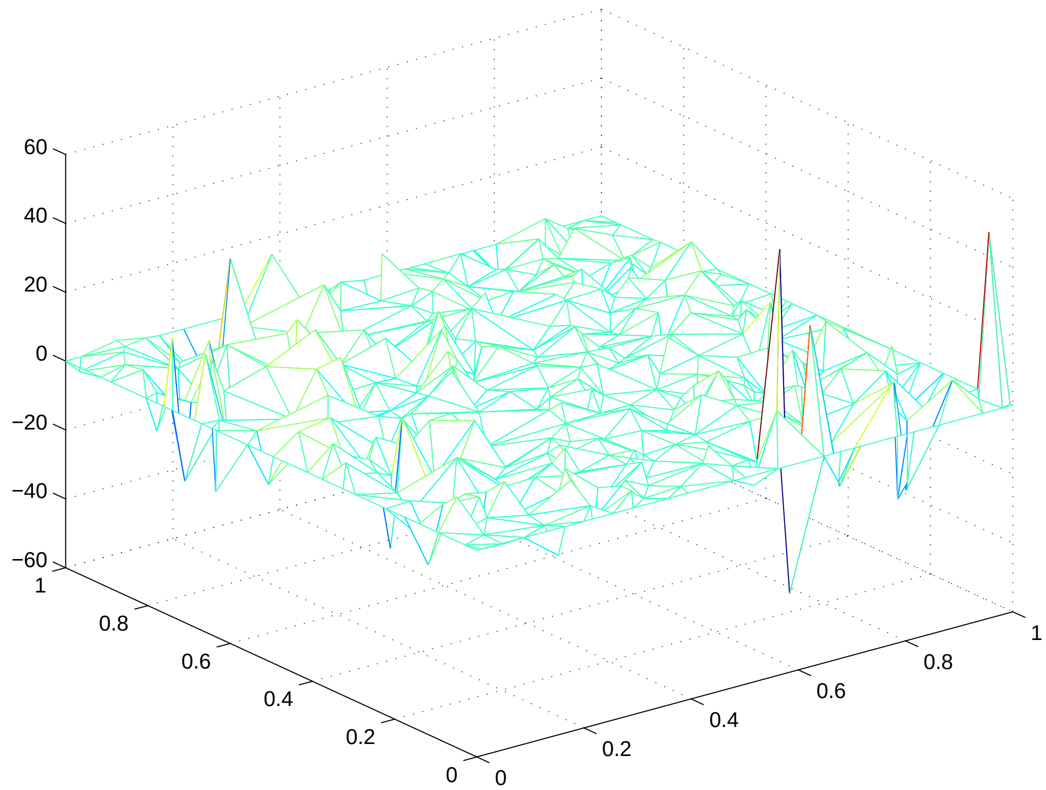

Galerkin approximation u_h on random mesh.

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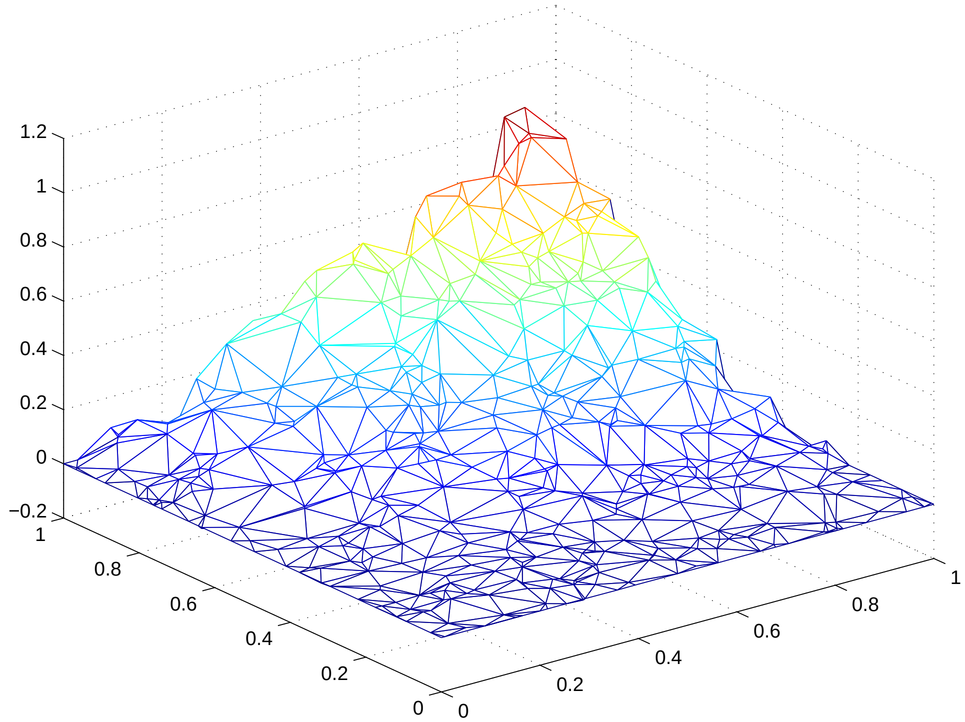

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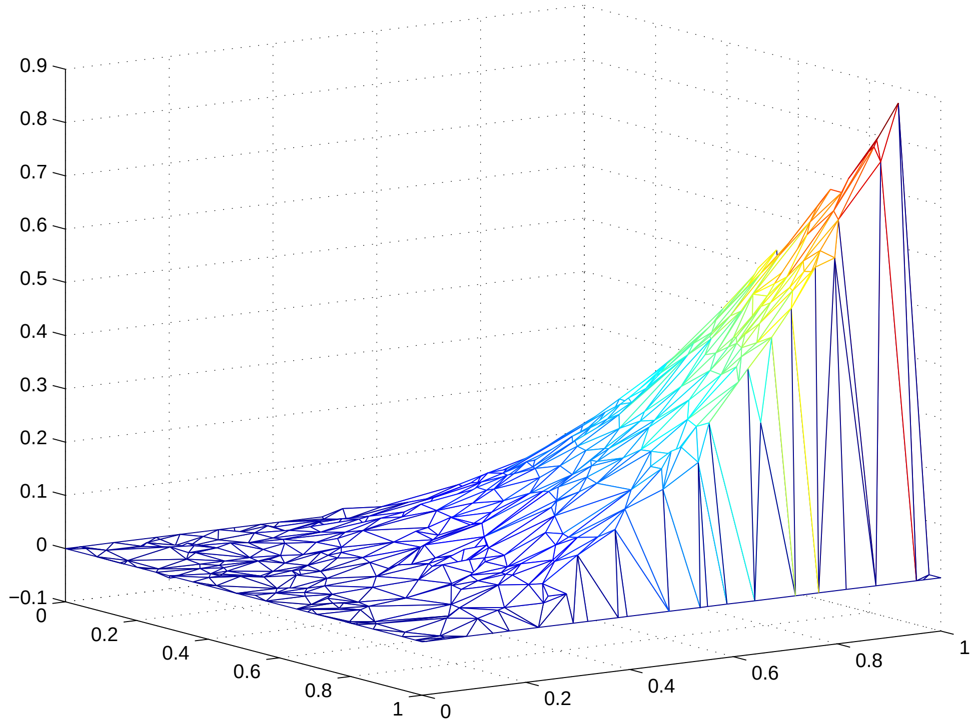

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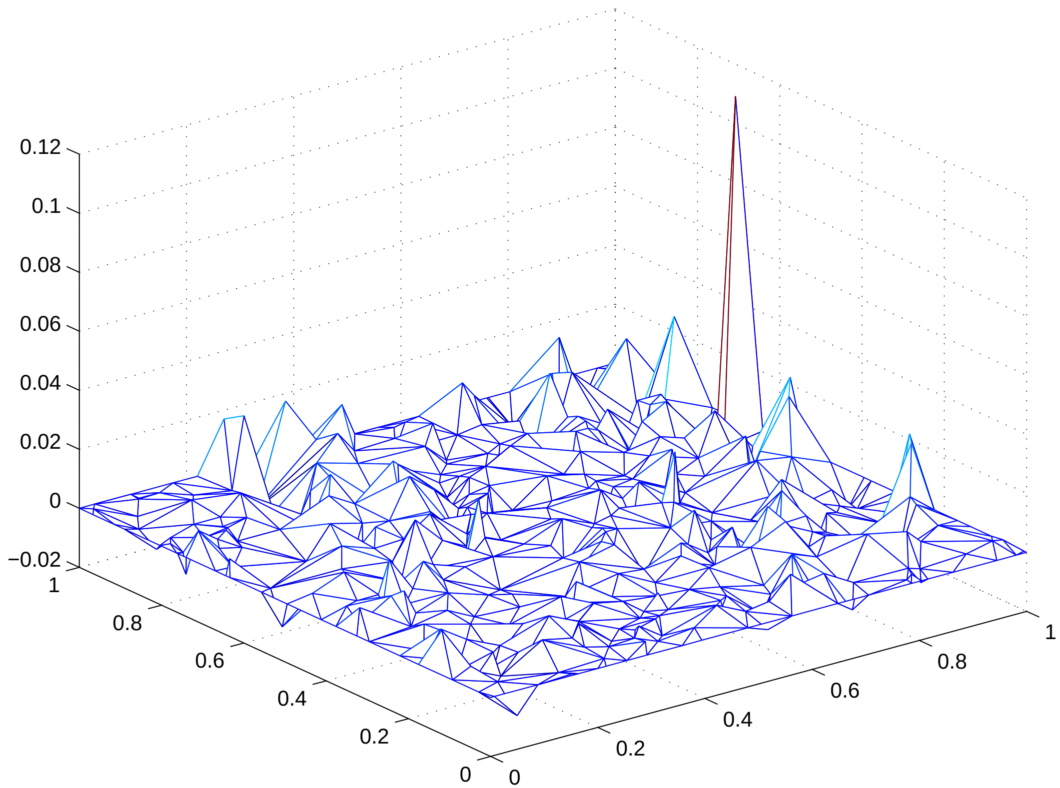

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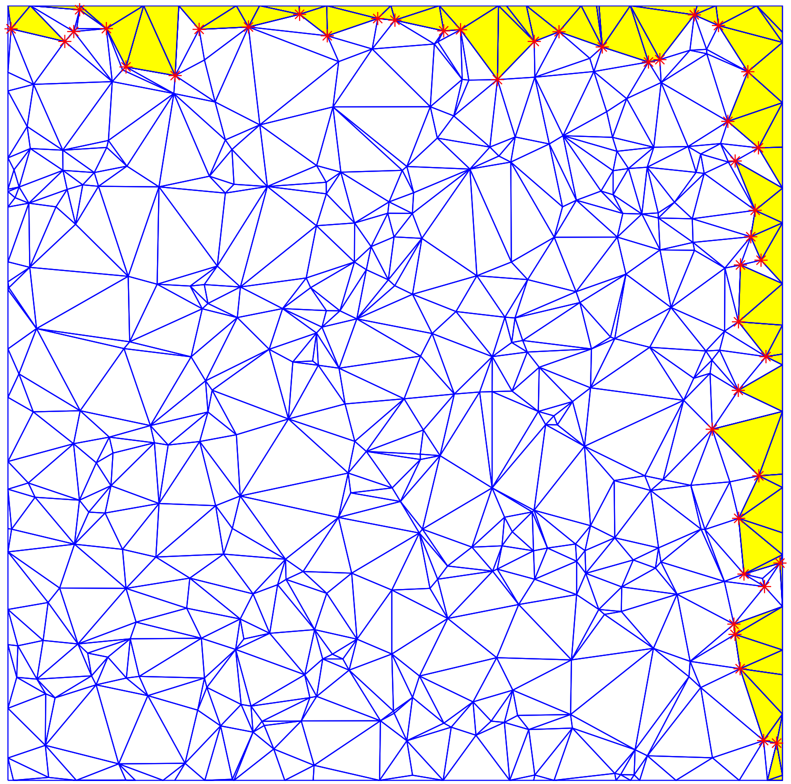

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IN PLAIN WORDS: TOO MUCH WORK

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Enjoy your retirement, David!

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But keep on joining us from time to time!