Aims

This course aims to provide an understanding of the basic facts of complex analysis, in particular the nice properties enjoyed by the derivatives and integrals of functions of a complex variable, and to show how complex analysis can be used to evaluate real integrals. The course also aims to provide an understanding of the basic concepts in analysis in the context of metric spaces showing how these ideas are generalisations of the ideas used in Real Analysis and to improve the students abilities in mathematical reasoning and in expressing themselves accurately in writing by producing correct mathematical proofs.

Syllabus

Revision of Complex numbers: revision of algebraic operations on complex numbers, the Argand diagram and de Moivre’s theorem, (1 lecture)

Metric spaces, convergence and continuity: definition and examples of metric spaces, convergence of sequences, subsequences, continuity;  \( \varepsilon \)-\( \delta \) and sequential definitions (5 lectures)

Paths and Path integrals: paths in the complex plane, path integrals, the length of a path. (4 lectures)

Differentiation of complex functions: the exponential, trig, hyperbolic and log functions, definition and elementary properties of derivatives, the Cauchy Riemann equations, the index of a closed path at a point. (6 lectures)

Cauchy’s theorem and applications: primitives, Cauchy’s theorem for a convex set, Cauchy’s integral formula and Cauchy’s formula for derivatives, Liouville’s theorem, the fundamental theorem of algebra. (4 lectures)

Complex series: convergence of series and power series, radius of convergence of power series, Taylor series, zeros and poles, Laurent series, residues. (4 lectures)

Cauchy Residue Theorem and applications: Cauchy’s residue theorem, evaluation of real integrals, summation of real series. (4 lectures)

Open and closed sets, compact metric spaces: open spheres, open and closed sets in metric spaces, sequential characterisation of closed sets, compact metric spaces and their properties (5 lectures)

Teaching and Assessment

Contact Hours: 3 lectures and 1 tutorial per week
Assessment: up to 15% by class tests or other continuous assessment at least 85% by end of course 2-hour exam
Resit Type: exam
By the end of the course, students should be able to:

- perform standard algebraic operations on complex numbers
- represent complex numbers in an Argand diagram map
- understand the concepts of metric and metric space and be able to determine if a given function is a metric
- know and use the standard metrics on \( \mathbb{R}, \mathbb{R}^N, \mathbb{C}, \mathbb{N}, C([a, b]), \) and discrete metric space
- determine whether a sequence in a metric space is convergent and in particular understand convergence in \( C([a, b]) \) and know the properties of convergent sequences
- solve elementary transcendental complex equations
- find parameterizations for paths in \( \mathbb{C} \)
- compute path integrals in elementary cases
- compute the length of a path
- find the index of a closed path at a point
- state and prove the Estimation Lemma, i.e., \( \left| \int_C f(z) \, dz \right| \leq L(C) \max |f(z)| \)
- determine whether a function from one metric space to another is continuous (either using \( \varepsilon-\delta \) and sequential definitions) and know the properties of continuous functions
- know the definition and derive properties of exponential, logarithmic, trigonometric and hyperbolic functions of complex variable
- state and prove the Cauchy-Riemann equations
- use the Cauchy -Riemann equations to determine where a function is differentiable
- understand the basic results about primitives for complex functions
- appreciate the importance of Cauchy's Theorem
- state and prove the Cauchy Integral Formulae and Cauchy Integral Formulae for derivatives
- use the Cauchy Integral Formulae to find path integrals of functions with algebraic singularities inside the contour of integration
- find an analytic function given its real part
- state and prove Liouville’s Theorem
- use Liouville's theorem to prove that functions are constant
- state and prove the Fundamental Theorem of Algebra;
- find the radii of convergence of complex power series
- use basic facts about termwise differentiation and integration of power series
- find Taylor and Laurent series of simple functions
- classify zeros and singularities of complex functions and calculate zero and pole orders
- calculate residues
- state and prove the Cauchy Residue Theorem
- use the Cauchy Residue Theorem to evaluate path integrals
- use Cauchy Residue Theorem to evaluate real integrals and to find sums to infinity
- determine whether a set in a metric space is open or closed and know the properties of open and closed sets
- determine if a metric space is compact, prove that a compact set is closed and bounded and understand and use the results on continuous images of compact sets

Content: 11 Apr 2008