Aims

This course aims to provide an understanding of the basic facts of complex analysis, in particular the nice properties enjoyed by the derivatives and integrals of functions of a complex variable, and to show how complex analysis can be used to evaluate real integrals. The course also aims to provide an understanding of the basic concepts in analysis in the context of metric spaces showing how these ideas are generalisations of the ideas used in Real Analysis and to improve the students abilities in mathematical reasoning and in expressing themselves accurately in writing by producing correct mathematical proofs.

Syllabus

Revision of Complex numbers: revision of algebraic operations on complex numbers, the Argand diagram and de Moivre’s theorem, (1 lecture)

Definition of a Metric Spaces, Limits of Sequences and Continuous Functions: definition and examples of metric spaces, sequences and limits, subsequences and bounded sequences, continuous functions; $\epsilon$-$\delta$ and sequential definitions (6 lectures)

Complex Functions Paths and Path integrals: complex functions, paths in the complex plane, path integrals, the exponential, trig, hyperbolic and log functions, the index of a closed path at a point. (6 lectures)

Cauchy’s theorem and its consequences: Differentiation of complex functions, the Cauchy Riemann equations, Cauchy’s theorem for a triangle, Cauchy’s theorem for a convex set, Cauchy’s integral formula. Cauchy’s formula for Derivatives. (6 lectures)

Complex series: Convergence of series and power series, radius of convergence of power series, Taylor series, Liouville’s theorem, the fundamental theorem of algebra. (3 lectures)

Cauchy Residue Theorem: Zeros and Poles, Laurent series, Residues, Cauchy’s residue theorem, evaluation of real integrals, summation of real series. (6 lectures)

Open, closed sets and compact subsets of a metric spaces: open spheres, open sets. closed sets and closure points in metric spaces, sequential characterisation of closed sets, compact metric spaces and their properties (5 lectures)

Teaching and Assessment

Contact Hours: 3 lectures and 1 tutorial per week
Assessment: up to 15% by class tests or other continuous assessment at least 85% by end of course 2-hour exam
Resit Type: exam

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By the end of the course, students should be able to:

- Perform standard algebraic operations on complex numbers. Find the polar form of a complex number and represent complex numbers on an Argand diagram.
- Define a metric for $X$ if $X$ is a non-empty set and show that a function $d$ is or is not a metric on a set.
- Know what is meant by the usual metrics on $\mathbb{R}$, $\mathbb{R}^N$, $\mathbb{C}$ and $B(Y)$.
- Define the limit of a sequence in a metric space and show that a sequence has a limit using an $\varepsilon - N$ argument.
- Know how to find the pointwise limit of a sequence of functions and determine if the convergence is uniform.
- Know how to show that a function between metric spaces is continuous using the $\varepsilon - \delta$ definition of continuity.
- Know how to show that a function between metric spaces is or is not continuous using sequential continuity.
- Given a path $\gamma$, state where it starts and finishes and state whether it is closed.
- Know how to find the length of a path.
- Given a path $\gamma$ sketch $\gamma^*$. Know how to parameterise the paths $L(z, w)$ and $C(w, r)$.
- Given a path $\gamma$ know how to evaluate $\int f(z) \, dz = \int f(\gamma(t)) \, \gamma'(t) \, dt$.
- Define $e^z$, $\sin(z)$, $\cos(z)$, $\sinh(z)$, $\cosh(z)$, $\log(z)$, and $z^\alpha$ and solve equations of the form $e^z = w$, $\sin(z) = w$, or $\cos(z) = w$, where $w$ is a given complex number.
- State the real and imaginary parts of $\cos(z)$, $\sin(z)$ and state and prove what $|\sin(z)|^2$ and $|\cos(z)|^2$ are equal to.
- Define $\text{Ind}(\gamma, w)$ where $\gamma$ be a closed path in $\mathbb{C}$ and $w \in \mathbb{C} \setminus \gamma^*$ and be able to calculate this for a given closed path $\gamma$.
- Check that a function is differentiable from first principles and use the sum, product, quotient and chain rules for differentiation to find derivatives of standard functions.
- State the Cauchy Riemann equations and use them to determine whether a function is differentiable.
- State Cauchy’s theorem for a convex set and know how to use Cauchy’s theorem for a convex set to evaluate line integrals (possibly followed by a parameterisation to evaluate a real integral).
- State and prove Cauchy’s integral formula and know how to use Cauchy’s integral formula to evaluate line integrals (possibly followed by a parameterisation to evaluate a real integral).
- State Cauchy’s formula for derivatives and know how to use Cauchy’s formula for derivatives to evaluate line integrals (possibly followed by a parameterisation to evaluate a real integral).
- Know how to use Cauchy’s estimates for derivatives.
- Determine whether a function $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic and if $u$ is harmonic find $v$ such that $u + iv$ is differentiable.
- Find Taylor series of differentiable functions.
- State and prove Liouville’s theorem.
- Know how to use Liouville’s theorem to show that differentiable functions are constant.
- Find where meromorphic functions have zeros or poles.
- Find the orders of zeros and poles of meromorphic functions.
- Know the formula for $\text{Res}(h, w)$ if $h$ is a meromorphic function with a pole at $w$ and know how to find $\text{Res}(h, w)$ if $h$ is a meromorphic function with a pole at $w$.
- State Cauchy’s residue theorem.
- Know how to use Cauchy’s residue theorem to evaluate line integrals.
- Know how to use properly P5 or Jordan’s lemma to estimate integrals.
- Know how to evaluate real integrals and sum series using the results coming from Cauchy’s residue theorem.
- Define an open set in a metric space, a closed set in a metric space, a closure point of a set in a metric space and a compact subset of a metric space.
- Know how to determine whether a set in a metric space is open or closed and know the properties of open and closed sets.
- State the sequential characterisation of closed sets.
- Know how to show that a set in a metric space is compact directly from the definition.
- Use the definition of a compact subset of a metric space to derive properties of the set for example be able to prove that the continuous image of a compact set is compact.

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