Aims

This course aims to introduce students to the idea of rigorous mathematical arguments and, in particular, to discuss the rigorous foundations of calculus. An important feature of the course is the use of careful, rigorous proofs of the theorems used, and one of the aims of the course is to improve student’s ability to understand such arguments and to develop such proofs for themselves.

A central concept in analysis is the idea of convergence, either of sequences, series or of functions. This course aims to discuss this concept rigorously, and provide the basic results which will be used in later courses.

Of course, we also discuss precise definitions of the derivative and the integral, and prove their basic properties. Everyone will be aware of these, but will probably not have seen rigorous proofs of them. In addition, we describe methods for obtaining inequalities, and tests for convergence of series and power series.

Syllabus

**Sequences:** Briefly recall the idea of a sequence of real numbers. Bounded and convergent sequences, and the definition of the limit of a sequence. General theorems about limits. (6 lectures)

**Suprema and infima:** Sup and inf of sets of real numbers. The completeness axiom for real numbers. (1 lecture)

**Monotone sequences:** Monotone sequences and the monotone convergence theorem. Use of the monotone convergence theorem to prove convergence of sequences without knowing the limit. (1 lecture)

**Subsequences:** Subsequences and the Bolzano-Weierstrass theorem. (1 lecture)

**Continuous functions:** Limits of functions, one-sided limits. General theorems about limits of functions. Continuity, combinations of continuous functions. Boundedness of continuous functions on closed intervals. The intermediate value theorem. Uniform continuity. (6 lectures)

**Differentiability:** The definition of the derivative of a function. Continuity of differentiable functions. Local maxima and minima. Rolle’s theorem. (4 lectures)

**First mean value theorem:** Statement and proof of the first mean value theorem, and applications to inequalities (2 lectures).

**Series and power series:** Convergence of series. The comparison, ratio, zero, absolute convergence and alternating series tests for series. Power series, and the radius of convergence of a power series. (6 lectures).

**Riemann integration and convergence of integrals:** Partitions, upper and lower sums, Riemann integrable functions. (6 lectures).
Teaching and Assessment

Contact Hours: 3 lectures and 1 tutorial per week
Assessment: up to 15% by class tests or other continuous assessment
up to 85% by end of course 2-hour exam
Resit Type: exam
By the end of the course, students should be able to:

- know what a sequence of numbers is.
- be able to prove from the definition that a sequence of numbers is bounded.
- understand the definition of convergence of a sequences of numbers, and be able to prove from the definition that simple sequences of numbers are convergent.
- prove results on combinations of sequences.
- prove that a convergent sequence is bounded.
- prove results on inequalities satisfied by limits, e.g., \( a_n < a \Rightarrow \lim_{n \to \infty} a_n \leq a \).
- understand the definition of the sup and inf of a set of numbers.
- understand the completeness axiom for the set of real numbers.
- be able to prove from the definition that a set of numbers is bounded above or below.
- understand the idea of monotone sequences and the monotone convergence theorem.
- use the monotone convergence theorem to show that sequences are convergent, without knowing the limit. Use this to retrospectively find the limit. In particular, to do this for iteratively defined sequences.
- understand the idea of subsequences and construct examples illustrating the behaviour of subsequences.
- state the Bolzano-Weierstrass theorem, and use it to show that sequences have convergent subsequences.
- limits of functions
- understand the limit and the \( \epsilon-\delta \) definition of continuity of functions.
- prove simple results involving continuous functions and their combinations.
- determine if given functions are continuous or discontinuous.
- prove that a continuous function on a closed interval is bounded and attains its bounds.
- prove the intermediate value theorem.
- use the intermediate value theorem to prove that certain equations have solutions in appropriate intervals.
- define the derivative of a function.
- determine from the definition if given functions are differentiable.
- show that the derivative of a function is zero at a local maximum or minimum of the function.
- prove Rolle's theorem.
- use Rolle's theorem to prove that certain equations have unique solutions in appropriate intervals.
- state and prove the first mean value theorem and use it to obtain inequalities
- know the definition of convergence of a series.
- state and prove the comparison test, the ratio test, the zero test, the absolute convergence test and the alternating series test for convergence of series, and use them to determine whether series are convergent.
- understand what is meant by the radius of convergence of a power series.
- be able to find the radius of convergence of a power series.
- define and compute the upper and lower sums of a bounded function \( f \) with respect to a partition.
- understand the effect of refining a partition.
- determine whether a function is Riemann integrable, either directly from the definition or from the sequential characterisation of integrability, and be aware that continuous and monotone functions are integrable.
- be aware of the basic properties of Riemann integrals.
- state and prove the fundamental theorem of calculus.

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