Aims

The goal of this course is to provide a bridge between school and university mathematics. In particular, the central role of proofs in mathematics will be emphasized throughout the course. Proofs tell us why things are true; results in mathematics are not the result of belief or experiment but of proof. There is some overlap with A-level and Advanced Highers, though I assume you have neither, and there will be elements of revision but the tone of the course is different.

Syllabus

The conceptual aspects of mathematics:

- What is mathematics? Mathematics in history and contemporary mathematics.
- Reasoning and logic. What is an argument? The notion of proof in mathematics with simple first examples such as the irrationality of $\sqrt{2}$, the triangle theorem, Pythagoras’ theorem, and the proof that there are infinitely many primes.
- Abstraction and rules. The meaning of key algebraic terms such as: associativity, commutativity, distributivity, identity, inverse. The rules of high-school algebra. Proof that $-1 \times -1 = 1$.
- Problem-solving.
- The need for checking.

1. Combinatorics:
   - Counting.
   - Manipulate sets and their elements. This includes the Boolean operations and set product.
   - Answer simple counting questions involving permutations and combinations. Connections with probability touched on.
   - Statement and proof of the Binomial theorem (by a counting argument). Applications.

2. Complex numbers and polynomials:
   - How complex numbers were discovered.
   - Add, subtract, multiply and divide complex numbers.
   - Find square roots of complex numbers.
   - Solve quadratics by completing the square.
   - Represent complex numbers in the complex plane.
   - Understand the geometric interpretations of addition and multiplication of complex numbers.
   - Proof that a polynomial of degree $n$ over the complex numbers has at most $n$ roots.
• The fundamental theorem of algebra.
• Proof of the fundamental theorem of algebra for real polynomials.
• Find $n$th roots.
• Use De Moivre’s theorem to find expressions for $\sin^n \theta$ and $\cos^n \theta$.
• Euler’s theorem and its proof.
• Find rational roots of polynomials with integer coefficients.
• Factorize real and complex polynomials appropriately.
• Understand the difference between trigonometric solutions and radical solutions.

3. Matrices:
• Why matrices are important.
• Add, subtract, and multiply two matrices, and multiply a matrix by a scalar; be able to carry out sequences of such operations to obtain a single matrix as a result. The main emphasis will be on ‘small’ matrices often $2 \times 2$ or $3 \times 3$ throughout.
• Proof of associativity for matrix multiplication.
• Solve linear equations using Gaussian elimination.
• Proof of the fundamental theorem of linear equations.
• Compute determinants by first row expansion.
• Compute matrix inverses using the adjugate method.
• Calculate the characteristic polynomial of a matrix.
• Statement of the Cayley-Hamilton theorem and proof in the $2 \times 2$ case.

4. Vectors:
• What is Euclidean geometry?
• Compute with vectors using inner products, vector products, and scalar triple products.
• Find the equation of the unique line determined by two points or a point and a vector in space.
• Find the equation of the unique plane determined by three points or by a point and a normal.
• Calculate intersections of lines or planes.
• Derivation of the volume of a parallelepiped using scalar triple products and connection with determinants.

Teaching and Assessment

Contact Hours: 3 lectures and 1 tutorial per week
Assessment: up to 30% by class tests or other continuous assessment
at least 70% by end of course 2-hour exam
Resit Type: exam
By the end of the course, students should be able to:

- carry out all the procedures described in the syllabus;
- be able to read and construct simple proofs;
- understand how and why mathematics at university is different from mathematics at school.