Aims

In this course we examine classical results about linear operators on Hilbert spaces. We begin by studying the concepts of projection and dimension on separable Hilbert spaces. We then study the notions of weak and strong topology on a Hilbert space. We then study the concept of compact operators which is fundamental in functional analysis. We then study the notions of weak, strong and norm topology in the algebra of bounded operators. We subsequently focus on the spectral theorem for compact selfadjoint operators. Then we study the solution of Fredholm and Volterra integral equations.

Syllabus

**Projections, bases and orthogonality:** Projections in infinite dimension. The notion of basis for separable Hilbert spaces. Orthonormal bases. The general Fourier’s Theorem. *(1 revision + 5 lectures)*

**Compact operators:** Definition. Every compact operator is bounded. The product of bounded and compact is always a compact operator. No compact operator in a separable Hilbert space is invertible. *(3 lectures)*

**Hilbert-Schmidt operators:** Definition. Every Hilbert-Schmidt operator is compact. The product of bounded and Hilbert-Schmidt is always a Hilbert-Schmidt operator. Trace of a Hilbert-Schmidt operator. *(3 lectures)*

**The spectrum of compact operators in separable Hilbert spaces:** Point spectrum and continuous spectrum. Discrete spectrum and essential spectrum. The origin as a point in the spectrum. Non-zero eigenvalues. The spectrum of every compact operator is at most countable. *(3 lectures)*

**The Fredholm alternative:** Homogeneous and inhomogeneous equations for linear operators. Dichotomy between unique solvability and solvability if and only if a finite set of conditions holds. Consequences of the Fredholm alternative. *(3 lectures)*

**Selfadjoint compact operators:** Invariant subspaces. Extrema of the spectrum for self-adjoint compact operators in terms of the operator norm. The Riesz-Schauder Theorem. *(3 lectures)*

**The Spectral Theorem:** Multiplicity. The Hilbert-Schmidt Theorem. Orthonormal bases of eigenvectors. *(3 lectures)*

**Fredholm integral equations:** Integral equations of the first and second kind. Compactness of integral operators. Hermitian integral operators. *(3 lectures)*

**Volterra integral equations:** Continuous and discontinuous kernels. Simple criterion for the solvability of Volterra integral equations. *(2 lectures)*

**The weak topology in a separable Hilbert space:** Weak convergence. Norm convergence. *(extra material lectures)*
Weak, strong and norm topologies: The weak topology in the algebra of bounded operators. The strong topology in the algebra of bounded operators. The norm topology in the algebra of bounded operators. (extra material lectures)

The Calkin algebra: Quotient algebras. Semi-Fredholm operators. Fredholm operators. The Fredholm Alternative in algebraic terms. (extra material lectures)

Teaching and Assessment

Contact Hours: 3 lectures and 1 tutorial per week
Assessment: 0% by class tests or other continuous assessment
100% by end of course 2-hour exam
Resit Type: 100% in the summer diet for the MSc
By the end of the course, students should be able to:

- understand the concept of projection on a separable Hilbert space,
- understand the notion of basis of a separable Hilbert space,
- understand the notion of orthonormal basis of a separable Hilbert space,
- formulate the infinite-dimensional general Fourier's Theorem,
- understand the concept of compact operators,
- Determine whether examples of simple operators are compact or not,
- understand the distinction between point and continuous spectrum,
- understand the distinction between discrete and essential spectrum,
- formulate the spectral decomposition of simple Hermitian compact operators,
- Formulate the Fredholm Alternative,
- solve various Fredholm integral equations,
- solve various Volterra integral equations,
- distinguish the notions of weak and norm convergence for separable Hilbert spaces,
- distinguish the notions of weak, strong and norm convergence for bounded operators,
- define the Calkin algebra,
- characterise compact operators in algebraic terms,
- define Fredholm operators,
- formulate the Fredholm alternative in algebraic terms.