Aims

This is a course on functional analysis for mathematics students. It aims to study normed linear spaces and some of the linear operators between them and give some applications of their use. The normed linear spaces which are complete metric spaces are especially important and the first part of the course is an introduction to the theory of Lebesgue integration with the aim of providing examples of complete spaces normed linear spaces of integrable functions.

Syllabus

\( \sigma \) algebras, measures and measurable functions: \( \sigma \) algebras, measures, measurable functions, characteristic functions, simple measurable functions, approximation of positive measurable functions by simple measurable functions. \( (6 \text{ lectures}) \)

Integration: The integral of simple positive measurable functions, positive measurable functions and measurable functions, the monotone convergence theorem, Fatou's lemma and the dominated convergence theorem, the spaces of functions \( L^1, L^2 \) and \( L^\infty \). \( (6 \text{ lectures}) \)

Normed spaces and inner product spaces: Cauchy Sequences, Complete spaces, Normed spaces, Banach spaces, inner product spaces, Hilbert spaces, Cauchy Schwarz inequality, orthogonality, Pythagoras theorem, orthogonal complement. \( (6 \text{ lectures}) \)

Linear operators: Continuous linear transformations, norm of continuous linear transformations, properties of the space \( B(X, Y) \), the Riesz-Fr\'echet theorem. \( (3 \text{ lectures}) \)

Operators on Hilbert spaces: The adjoint of an operator, normal, self-adjoint and unitary operators. \( (3 \text{ lectures}) \)

Inverses and spectra of operators: Invertible operators, characterisations of invertible operators, the spectrum of an operator, positive operators, polar decomposition of invertible operators. \( (6 \text{ lectures}) \)

Additional reading: An additional reading topic or topics on measure theory and or functional analysis

Revision and problem solving: \( (3 \text{ lectures}) \)

Teaching and Assessment

Contact Hours: 3 lectures and 1 tutorial per week
Assessment: 0% by class tests or other continuous assessment
Resit Type: exam for MSc

Content: September 2009
By the end of the course, students should be able to:

- define a $\sigma$ algebra, a measure, and a measurable function, be able to check whether a given set of subsets, a given function on a sigma algebra or a function on a set is a $\sigma$ algebra, a measure, and a measurable function respectively, and to prove simple results (seen or unseen) about $\sigma$ algebras, measures, or measurable functions.
- integrate (in turn) a simple positive measurable function, a positive measurable function and a general $L^1$ function and be able to derive properties of integrals using this path, evaluate integrals of continuous functions on the real line with respect to Lebesgue measure and evaluate integrals of sequences with respect to counting measure on the positive integers, state prove and use the Monotone convergence theorem, Fatou’s Lemma and the Dominated convergence theorem, define the $L^p$ spaces and determine whether functions are in $L^p$.
- define a Cauchy sequence, determine whether sequences are Cauchy, define a complete metric space and know whether standard metric spaces are complete, define a norm and an inner product and calculate the standard inner products or norms on $\mathbb{R}^n$, $\mathbb{C}^n$, $C[0,1]$, $L^p$ and $\ell^p$, verify metric space properties of normed spaces, determine whether elements of an inner product space are orthogonal, define the orthogonal complement of a set in an inner product space, prove it is a closed linear subspace and find orthogonal complements.
- characterise continuous (that is bounded) linear transformations between normed spaces and to determine whether a linear transformation is bounded and to work out its norm in simple cases, find the product of linear transformations (if defined)
- define the adjoint of a bounded operator between Hilbert spaces and to compute adjoints of operators, define normal, self-adjoint and unitary operators, determine whether operators are normal, self-adjoint or unitary and be able to prove simple results (seen or unseen) about adjoints, normal, self-adjoint and unitary operators.
- define what is meant by an operator being invertible, know alternative ways of showing an operator is invertible and use these to determine if an operator is invertible, define the spectrum of an operator and find spectrum of an operator (in simple cases) for example using the conditions that it is compact, the relationships between the spectrum of $T$ and $T^*$ and the polynomial spectral mapping theorem, use the spectral radius to work out the norms of both self adjoint and non self adjoint operators, define positive operators and determine whether operators are positive, be able to prove simple results (seen or unseen) about positive operators, use the polar decomposition of invertible operators and find polar decompositions for invertible matrices.
- read and critically evaluate further topics in measure theory and or functional analysis.

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