On the last time decomposition of a path of a random walk:

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1 Iterates of a stopping time

This note is concerned with the behaviour of

$$S_n = \sum_{i=1}^{n} \xi_i, \quad n = 0, 1, 2, \ldots,$$

a random walk in the real line before and after a stopping time α, up to an independent memoryless time. More precisely, α is a random time (values in $\mathbb{Z}_+ \cup \{\infty\}$) such that $\{\alpha \le n\}$ is determined by $\xi_1, \ldots, \xi_n$ for all $n \in \mathbb{N}$. A memoryless time T is such that

$$P(T \ge n) = q^n, \quad , \quad n = 0, 1, 2, \ldots,$$

for some $q \in (0, 1)$. Then $P(T = n) = q^n p$, $p = 1 - q$. To state the result we are after, we need to consider the *iterates* of α. We can always assume that the ξ_n are defined on some probability space Ω, and $\theta : \Omega \to \Omega$ such that $\xi_n(\theta\omega) = \xi_{n+1}(\omega)$. Let θ^n be the n-the iterate of θ and let θ^α be defined by $\theta^\alpha(\omega) = \theta^{\alpha(\omega)}(\omega)$. Then define

$$\alpha(1) := \alpha + \alpha \circ \theta^\alpha, \quad \alpha(i+1) = \alpha(i) + \alpha(i) \circ \theta^{\alpha(i)}, \quad i \ge 1.$$

In this way, α generates a *point process* which is the random set

$$\alpha(1), \quad \alpha(2), \quad \alpha(3), \quad \ldots$$

We are interested in the last such time before T. To express this formally, let

$$N_t = \sum_{i \ge 0} \mathbf{1}(\alpha(i) \le t), \quad t \ge 0$$

be the counting process associated to the iterates of α. Then, for all $t \ge 0$,

$$\alpha(N_t) \le t < \alpha(N_t + 1).$$

Thus, $\alpha(N_T)$ is the time we are interested in. The result we are after is

Theorem 1. *Assume that T is memoryless and independent of the random walk. Then $(\alpha(N_T), S_{\alpha(N_T)})$ is independent of $(T - \alpha(N_T), S_T - S_{\alpha(N_T)})$.*

The proof will be presented in several steps.

2 Independence up to a stopping time

To start with, we give a definition:

Definition 1 (independence up to a stopping time)**.** *If $\zeta_1, \zeta_2, \dots$ are random elements of a fairly arbitrary space and σ a stopping time, we say that $(\zeta_1, \dots, \zeta_\sigma)$ are independent if there exist probability measures $Q_1, Q_2, \dots$, such that*

$$P(\zeta_1 \in B_1, \dots, \zeta_\sigma \in B_n \mid \sigma = n) = Q_1(B_1) \dots Q_n(B_n),$$

for all Borel sets $B_1, B_2, \dots$, and all $n \in \mathbb{N}$.

A consequence of the definition is

Lemma 1. *If $(\zeta_1, \dots, \zeta_\sigma)$ are independent then $f(\zeta_1, \dots, \zeta_{\sigma-1})$ is independent of $g(\zeta_\sigma)$, for all measurable f, g.*

We also have:

Lemma 2. *If $\zeta_1, \zeta_2, \dots$ are i.i.d. with common distribution μ and if $\sigma = \inf\{n \in \mathbb{N} : \zeta_n \in A\}$, for some set A, then $(\zeta_1, \dots, \zeta_\sigma)$ are independent.*

Proof. Assume $0 < \mu(A) < 1$. For $n \geq 1$, we have

$$\begin{aligned} &P(\zeta_1 \in B_1, \dots, \zeta_n \in B_n, \sigma = n) \\ &\qquad = P(\zeta_1 \in B_1 \cap A^c, \dots, \zeta_{n-1} \in B_{n-1} \cap A^c, \zeta_n \in B_n \cap A, \sigma = n) \\ &\qquad = P(\zeta_1 \in B_1 \cap A^c) \cdots P(\zeta_{n-1} \in B_{n-1} \cap A^c) P(\zeta_n \in B_n \cap A) \\ &\qquad = \frac{\mu(B_1 \cap A^c)}{\mu(A^c)} \cdots \frac{\mu(B_{n-1} \cap A^c)}{\mu(A^c)} \frac{\mu(B_n \cap A)}{\mu(A)} P(\sigma = n), \end{aligned}$$

and so

$$P(\zeta_1 \in B_1, \dots, \zeta_n \in B_n \mid \sigma = n) = Q_1(B_1) \cdots Q_{n-1}(B_{n-1}) Q_n(B_n)$$

where $Q_1 = \cdots = Q_{n-1}$ are equal to the restriction of μ on A^c and Q_n is the restriction of μ on A. □

3 Breaking the path into cycles

We realise the memoryless random variable T as follows. Consider i.i.d. $\{0, 1\}$-valued r.v.'s

$$\delta_1, \delta_2, \dots,$$

independent of the random walk, with

$$P(\delta_1 = 0) = q,$$

and let

$$T = \inf\{n \in \mathbb{N} : \delta_n = 1\} - 1.$$

We define the cycles

$$\mathcal{C}(1) := \Big((\xi_n, \delta_n) : n \leq \alpha \Big), \quad \mathcal{C}(i) := \Big((\xi_n, \delta_n) : \alpha(i-1) < n \leq \alpha(i) \Big), \quad i \geq 2.$$

Notice that

Lemma 3. *The cycles $\mathcal{C}(1), \mathcal{C}(2), \dots$ are i.i.d.*

Proof. By the strong Markov property. □

4 The proof of Theorem 1

We now consider the sums of the ξ's and the δ's over each cycle:

$$X(1) := \sum_{n=1}^{\alpha} \xi_n, \quad \Delta(1) := \sum_{n=1}^{\alpha} \delta_n$$

$$X(i) := \sum_{n=\alpha(i-1)+1}^{\alpha(i)} \xi_n, \quad \Delta(i) := \sum_{n=\alpha(i-1)+1}^{\alpha(i)} \delta_n, \quad i \geq 2.$$

Consider the random index

$$I := \inf\{i \geq 1 : \ \Delta(i) \neq 0\}.$$

Since I is a stopping time relative to the cycles, we have, by Lemma 2,

Lemma 4. $(\mathcal{C}(1), \ldots, \mathcal{C}(I))$ *are independent.*

From the definition of T, we immediately have

$$\alpha(I-1) < T+1 \leq \alpha(I).$$

Thus

$$\alpha(N_T) = \alpha(I-1).$$

Therefore,

$$\big(\alpha(N_T), \ S_{\alpha(N_T)}\big) = \big(\Delta(1) + \cdots + \Delta(I-1), \ X(1) + \cdots + X(I-1)\big)$$

is a function of $\big(\mathcal{C}(1), \ldots, \mathcal{C}(I-1)\big)$, while

$$\big(T - \alpha(N_T), \ S_T - S_{\alpha(N_T)}\big) = \left(\sum_{n=\alpha(I-1)+1}^{T} 1, \quad \sum_{n=\alpha(I-1)+1}^{T} \xi_n \right)$$

is a function of $\mathcal{C}(I)$. By Lemma 1, $\big(\alpha(N_T), \ S_{\alpha(N_T)}\big)$ is independent of $\big(T - \alpha(N_T), \ S_T - S_{\alpha(N_T)}\big)$. □