Günter Last
Institut für Stochastik
Universität Karlsruhe (TH)

A Short Course on Stochastic Geometry

6. Distributional properties of Poisson Voronoi tessellations

Günter Last

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6.1 The neighbours of a typical vertex

General assumption: We consider the Voronoi tessellation $\{C(N,x) : x \in N\}$ based on a stationary Poisson process N of intensity $\lambda > 0$.

Definition: Consider the probability measure $\mathbb{P}^0_{N_0}$.

(i) Almost surely there are exactly $d+1$ different points $X_0, \ldots, X_d \in N$ (lexicographically ordered) such that

$$\{0\} = C(N, X_0) \cap \ldots \cap C(N, X_d).$$

The points $X_0, \ldots, X_d$ are the *neighbours* of the origin.

(ii) Let $R := |X_0| = \cdots = |X_d|$ denote the distance to the neighbours and define the unit vectors

$$U_0 := \frac{X_0}{R}, \ldots, U_d := \frac{X_d}{R}.$$

Theorem: *Under the probability measure $\mathbb{P}^0_{N_0}$ the following holds.*

(i) *The random variables* $(\{x \in N : |x| > R\}, R)$ *and* $(U_0, \ldots, U_d)$ *are independent.*

(ii) R^d *is Gamma distributed with shape parameter d and scale parameter* $\lambda\kappa_d$.

(iii) *The conditional distribution of* $\{x \in N : |x| > R\}$ *given* $R = r$ *is the distribution of a Poisson process restricted to the complement of the ball* $B(0, r)$.

(iv) $\{U_0, \ldots, U_d\}$ *has distribution*

$$c_0^{-1} \int \cdots \int \mathbf{1}\{\{u_0, \ldots, u_d\} \in \cdot\}\Delta_d(u_0, \ldots, u_d)\, \mathbb{S}(du_0) \ldots \mathbb{S}(du_d)$$

where $\Delta_d(u_0, \ldots, u_d)$ *is the volume of the simplex spanned by the vectors* $u_0, \ldots, u_d$, $\mathbb{S}$ *is the uniform distribution on the unit sphere* S^{d-1} *and* c_0 *is an explicitly known constant.*

6.2 The length-biased distribution of the neigbours of a typical face

Definition: Consider the probability measure $\mathbb{P}^0_{M_k}$ for some fixed $k \in \{1, \dots, d-1\}$.

(i) Almost surely there is exactly one k-face $F_k \in \mathcal{S}_k(N)$ such that $0 \in F_k$.

(ii) Almost surely there are exactly $d - k + 1$ different points $X_0, \dots, X_{d-k} \in N$ (lexicographically ordered) such that

$$F_k = C(N, X_0) \cap \ldots \cap C(N, X_{d-k}).$$

The points $X_0, \dots, X_{d-k}$ are the *neighbours* of F_k.

(iii) Let

$$R := |X_0| = \cdots = |X_{d-k}|$$

denote the distance of the origin from the neighbours.

(iv) There is a unique $(d-k)$-dimensional ball in the affine hull of the neighbours containing the neighbours on its boundary. We let Z denote the centre of this ball.

(v) Let

$$R' := |X_0 - Z|, \qquad R'' := |Z|.$$

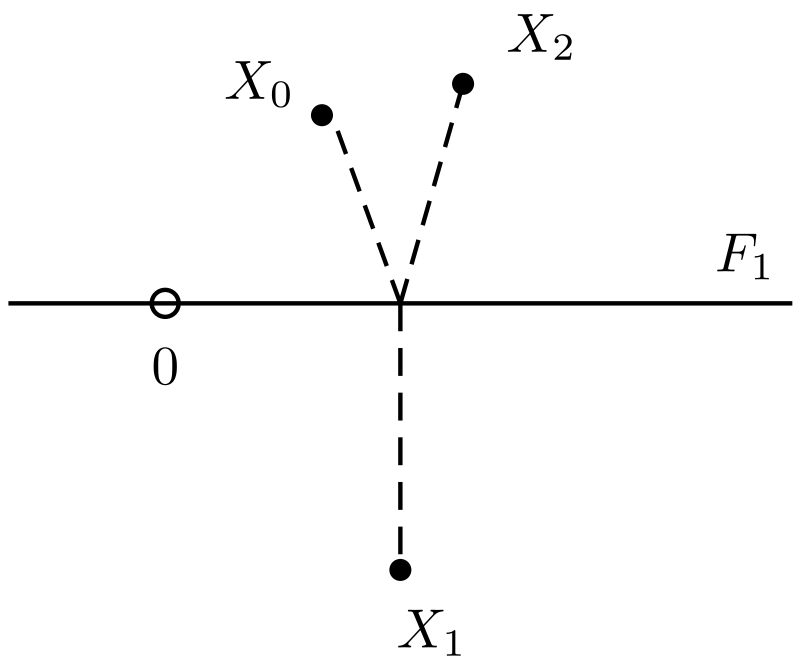
so that

$$R^2 = {R'}^2 + {R''}^2.$$

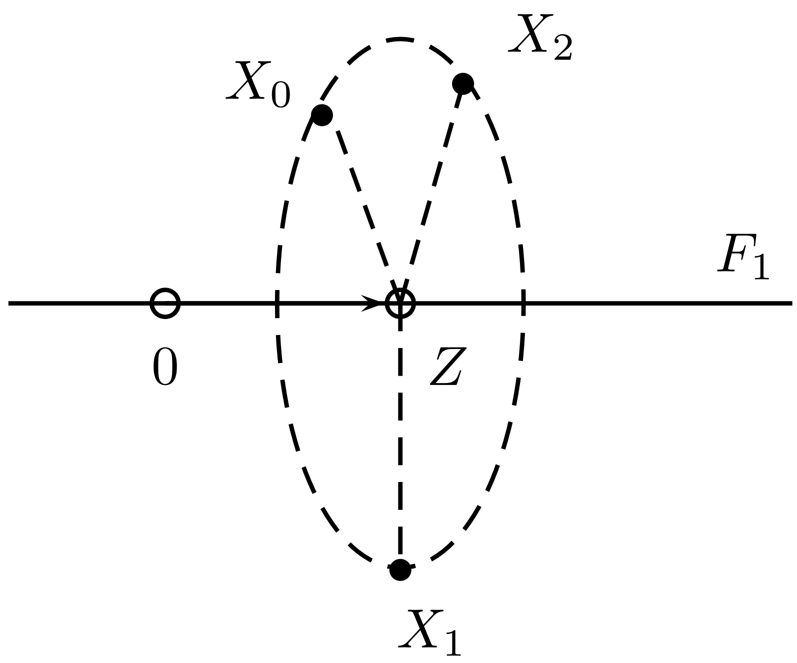
(vi) Define the unit vectors

$$U_0 := \frac{X_0 - Z}{R'}, \ldots, U_{d-k} := \frac{X_{d-k} - Z}{R'}, \quad U := \frac{Z}{|Z|}.$$

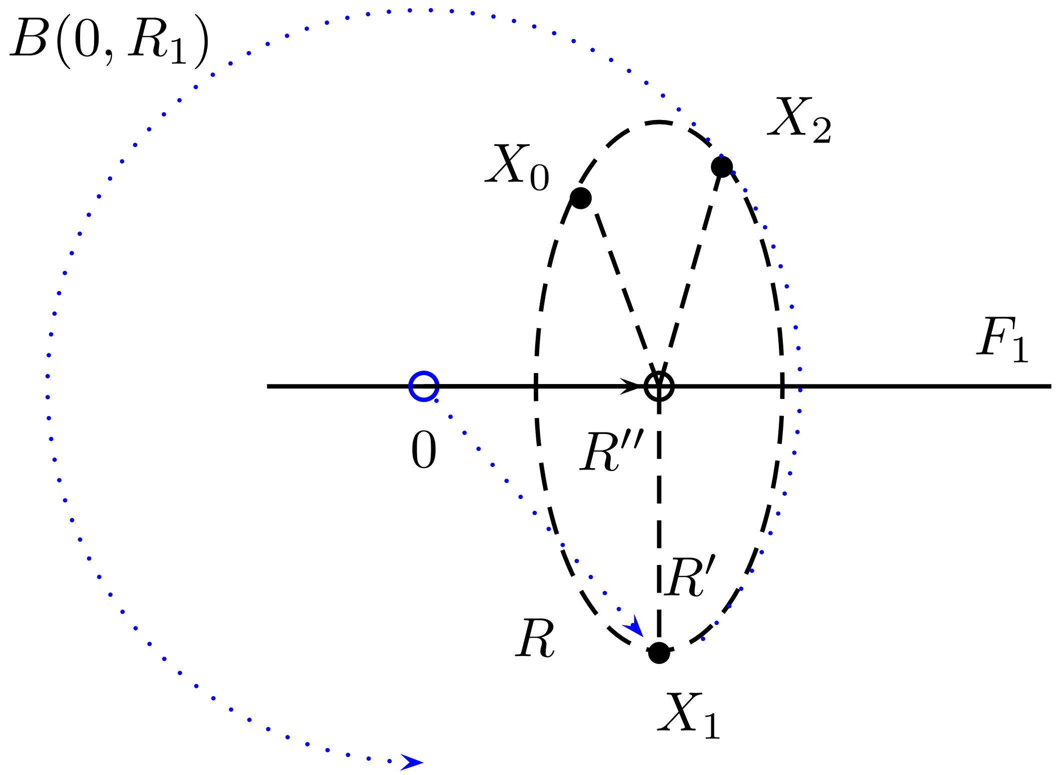
Situation under $\mathbb{P}^0_{M_k}$ for $k = 1$, $d = 3$.



Situation under $\mathbb{P}^0_{M_k}$ for $k = 1$, $d = 3$.



Situation under $\mathbb{P}^0_{M_k}$ for $k = 1$, $d = 3$.



Theorem: *Under the probability measure $\mathbb{P}^0_{M_k}$ the following holds.*

(i) *The random variables* $(\{x \in N : |x| > R\}, R)$, R'^2/R^2, *and* $(U_0, \dots, U_{d-k}, U)$ *are independent.*

(ii) R^d *is Gamma distributed with shape parameter* $d-k+k/d$ *and scale parameter* $\lambda\kappa_d$.

(iii) *The conditional distribution of* $\{x \in N : |x| > R\}$ *given* $R = r$ *is the distribution of a Poisson process restricted to the complement of the ball* $B(0,r)$.

(iv) R'^2/R^2 *has a Beta distribution with parameters* $d(d-k)/2$ *and* $k/2$.

(v) *Fix a $d-k$-dimensional linear subspace $L \subset \mathbb{R}^d$. The random pair $(\{U_{k,0},\dots,U_{k,d-k}\},U_k)$ has distribution*

$$c_k^{-1}\int\cdots\int \mathbf{1}\{(\{\vartheta u_0,\dots,\vartheta u_{d-k}\},\vartheta u))\in\cdot\}$$
$$\Delta_{d-k}(u_0,\dots,u_{d-k})^{k+1}\mathbb{S}_L(du_0)\dots\mathbb{S}_L(du_{d-k})\,\mathbb{S}_{L^\perp}(du)\,\nu(d\vartheta),$$

where $\Delta_{d-k}(u_0,\dots,u_{d-k})$ is the $(d-k)$-dimensional volume of the simplex spanned by the vectors $u_0,\dots,u_{d-k}$, ν is the uniform distribution on the rotation group SO_d, c_k is an explicitly known constant, and $\mathbb{S}_L$ and $\mathbb{S}_{L^\perp}$ are the uniform distributions (normalized Haar measures) on the unit spheres in L and the orthogonal complement $L^\perp$ of L, respectively.

6.3 The typical edge and its neighbours

Definition: Assume that N is a stationary Poisson process of intensity $\lambda > 0$ and consider the Palm probability measure $\mathbb{P}^0_{N_1}$.

(i) Let L denote the length of the typical edge C_1.

(ii) Let Z denote the centre of the $(d-1)$-dimensional ball determined by the d neighbours of C_1. Let Φ_1 denote the set of the d unit vectors pointing from Z to the neighbours of C_1.

(ii) Let α' and α'' denote the angles in $[0, \pi]$ spanned by the edge C_1 and the vectors pointing from the endpoints of C_1 to one of the neighbours of C_1.

(iii) Let ξ denote the volume of the union of the two balls centered at the endpoints of the edge C_1 whose radii are given by the respective distances from the endpoints to one of the neighbours of C_1.

Theorem: *Under $\mathbb{P}^0_{N_1}$ we have the following assertions:*

(i) *The random variables (α', α''), ξ and Φ_1 are independent.*

(ii) *The random variable ξ has a Gamma distribution with shape parameter $d+1$ and scale parameter 1.*

(iii) *The distribution of $(\cos \alpha', \cos \alpha'')$ has an explicitly known and integral-free density.*

(iv) *The distribution of Φ_1 is the same as the distribution of the corresponding unit vectors under $\mathbb{P}_{M_1}$. It has been given above.*

Remark: Under $\mathbb{P}^0_{N_1}$ the random variables α', α'', ξ, and Φ determine the edge C_1 and the positions of its neighbours up to a translation.

References for Lectures 5 and 6:

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