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A Short Course on Stochastic Geometry

5. Mean value formulae for stationary tessellations

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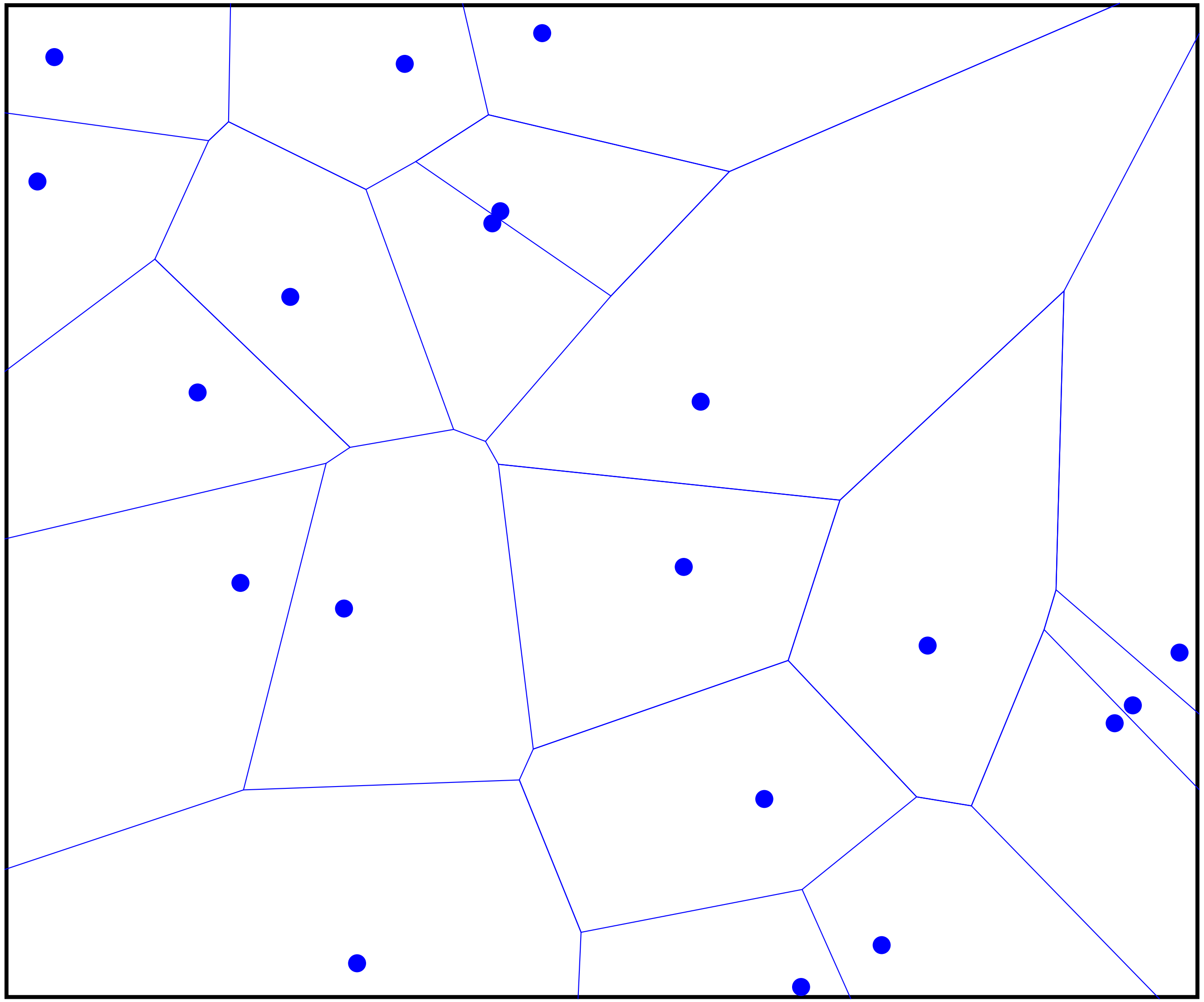
5.1 Voronoi tessellations



Reminder:

(i) The space of all *point configurations* in $\mathbb{R}^d$ is defined as

$$\mathbf{N}_s \equiv \mathbf{N}_s(\mathbb{R}^d) := \{\varphi \subset \mathbb{R}^d : \varphi \text{ is locally finite}\}.$$

(ii) Any $\varphi \in \mathbf{N}_s$ is identified with a counting measure:

$$\varphi(B) := \operatorname{card}\{x \in \varphi : x \in B\}, \quad B \subset \mathbb{R}^d.$$

(iii) The σ-field $\mathcal{N}_s$ is the smallest σ-field of subsets of $\mathbf{N}_s$ making the mappings $\varphi \mapsto \varphi(B)$ for all Borel sets $B \subset \mathbb{R}^d$ measurable.

Definition: The points of $\varphi \in \mathbf{N}_s$ are in *general quadratic position* if the following two conditions are satisfied.

(i) Any $k \in \{2, \dots, d+2\}$ points of φ are in general position.

(ii) No $d+2$ points of φ lie on the boundary of some ball.

Definition: Let $\varphi \in \mathbf{N}_s$.

(i) The *Voronoi cell* $C(\varphi, x)$ of $x \in \varphi$ is the set of all sites $y \in \mathbb{R}^d$ whose distance from x is smaller or equal than the distances to all other points of φ.

(ii) The *Voronoi tessellation* based on φ is the system

$$\mathcal{S}_d(\varphi) := \{C(\varphi, x) : x \in \varphi\}.$$

Remark: If the convex hull of φ coincides with $\mathbb{R}^d$, then all Voronoi cells are bounded and $\mathcal{S}_d(\varphi)$ is a *face-to-face* tessellation. Moreover, if the point of φ are in general quadratic position, then $\mathcal{S}_d(\varphi)$ is also *normal* in the sense that any k-face is contained in exactly $d-k+1$ cells.

Definition: Let C be a convex polytope. Then

$$C = \bigcup_{k\in\{0,\dots,d\}} \bigcup_{C\in\mathcal{S}_k(C)} \operatorname{relint} F,$$

where $\mathcal{S}_k(C)$ is a finite set of k-dimensional polytopes whose affine hulls are pairwise not equal. A polytope $F \in \mathcal{S}_k(C)$ is called a *k-face* of C.

Definition: Let $\varphi \in \mathbf{N}_s$ and $k \in \{0,\dots,d\}$. The system of all k-faces of the Voronoi tessellation $\mathcal{S}_d(\varphi)$ is defined by

$$\mathcal{S}_k(\varphi) := \bigcup_{C\in\mathcal{S}_d(\varphi)} \mathcal{S}_k(C).$$

5.2 Typical faces of stationary Voronoi tessellations

Assumption: N is a stationary simple point process on $\mathbb{R}^d$ such that almost surely N is non-empty and the points of N in general quadratic position. We consider the (random) Voronoi tessellation

$$\mathcal{S}_d(N) = \{C(N,x) : x \in N\}$$

generated by N.

Definition: For a k-dimensional convex set C let $\pi_k(C)$ denote the centre of gravity of C. Define the stationary point process of centres of k-faces by

$$N_k := \{\pi_k(F) : F \in \mathcal{S}_k(N)\}.$$

Assumption: For any $k \in \{0, \dots, d\}$ the intensity

$$\gamma_k := \mathbb{E}[N_k([0,1]^d)]$$

is assumed to be finite.

Remark: We have $\gamma_d = \lambda$.

Definition: Let $k \in \{0, \dots, d\}$. Under the Palm probability measure $\mathbb{P}^0_{N_k}$ we denote by $C_k \in \mathcal{S}_k(N)$ the k-face satisfying $\pi_k(C_k) = 0$. The distribution

$$\mathbb{P}^0_{N_k}(C_k \in \cdot)$$

is the distribution of the *typical* k-face of the Voronoi tessellation based on N.

Definition: For any $k \in \{0, \dots, d\}$ we define the stationary random measure M_k on $\mathbb{R}^d$ by

$$M_k(B) := \sum_{F \in \mathcal{S}_k(N)} |F \cap B|_k,$$

where $|C|_k$ denotes the k-dimensional area (Hausdorff measure) of a measurable set $C \subset \mathbb{R}^d$.

Assumption: For any $k \in \{0, \dots, d\}$ the intensity

$$\mu_k := \mathbb{E}[M_k([0,1]^d)]$$

is assumed to be finite.

Remark: We have $M_0 = N_0$ and hence $\gamma_0 = \mu_0$.

Definition: Let $k \in \{0,\dots,d\}$. Under the Palm probability measure $\mathbb{P}^0_{M_k}$ we denote by $F_k \in \mathcal{S}_k(N)$ the k-face satisfying $0 \in F_k$. The distribution

$$\mathbb{P}^0_{M_k}(F_k \in \cdot)$$

can be interpreted as an *area-biased* version of the distribution of the *typical* k-face.

Proposition: *Consider $k \in \{0,\dots,d\}$ and a measurable and shift-invariant function $g : \mathbf{N}_s \to [0,\infty)$. Then*

$$\mu_k \mathbb{E}^0_{M_k}[g] = \gamma_k \mathbb{E}^0_{N_k}\left[|C_k|_k \cdot g\right],$$

$$\gamma_k \mathbb{E}^0_{N_k}[g] = \mu_k \mathbb{E}^0_{M_k}\left[|F_k|_k^{-1} \cdot g\right].$$

5.3 Mean values for typical faces

Corollary: *For any $k \in \{0, \dots, d\}$ we have*

$$\mu_k = \gamma_k \mathbb{E}^0_{N_k} [|C_k|_k] ,$$

$$\gamma_k = \mu_k \mathbb{E}^0_{M_k} \left[|F_k|_k^{-1}\right] .$$

In particular

$$\mathbb{E}^0_{N_d}[V_d(C_d)] = \lambda^{-1},$$

$$\mathbb{E}[V_d(F_d)^{-1}] = \lambda.$$

Proposition: *We have*

$$\sum_{j=0}^{d} (-1)^j \gamma_j = 0.$$

Definition: Let $\mathcal{P}^d$ denote the system of all convex polytopes in $\mathbb{R}^d$. For $k \in \{0, \dots, d\}$ we define $\nu_k : \mathcal{P}^d \to \mathbb{N}$ by

$$\nu_k(F) := \operatorname{card} \mathcal{S}_k(F).$$

Proposition: *Consider the planar case* $d = 2$. *Then* $\gamma_0 = 2\lambda$ *and* $\gamma_1 = 3\lambda$. *Moreover,*

$$\mathbb{E}^0_{N_2}[|C_2|_2] = \frac{1}{\lambda},$$

$$\mathbb{E}^0_{N_2}[|\partial C_2|_1] = \frac{2\mu_1}{\lambda},$$

$$\mathbb{E}^0_{N_2}[\nu_0(C_2)] = 6,$$

$$\mathbb{E}^0_{N_1}[|C_1|_1] = \frac{\mu_1}{3\lambda}.$$

Theorem: *If N is a stationary Poisson process of intensity λ then the intensities μ_k are explicitly known. In case $d = 2$ we have*

$$\mu_0 = 2\lambda, \quad \mu_1 = 2\sqrt{\lambda}$$

and in case $d = 3$ we have

$$\mu_0 = \frac{24\pi^2}{35}\lambda, \quad \mu_1 = \frac{48\pi^2}{35}\lambda, \quad \mu_2 = \Big(\frac{24\pi^2}{35} + 1\Big)\lambda.$$

Problem: Assume that N is a stationary Poisson process. Determine the distributions

$$\mathbb{P}^0_{N_k}(C_k \in \cdot), \quad k = 0, \dots, d,$$

and

$$\mathbb{P}^0_{M_k}(F_k \in \cdot), \quad k = 0, \dots, d.$$

5.4 General tessellations

Definition: A *mosaic* (*tessellation*) in $\mathbb{R}^d$ is a countable system $\mathcal{M}$ of compact subsets of $\mathbb{R}^d$ (*cells*) with the following properties.

(i) Any bounded set is intersected by only a finite number of the cells.

(ii) All cells are convex and have a non-empy interior.

(iii) The union of the cells is all of $\mathbb{R}^d$.

(iv) The interiors of the cells are mutually disjoint.

Remark: The cells of a mosaic are convex polytopes.

Definition: A random tessellation in $\mathbb{R}^d$ is a simple point process X on the space $\mathcal{C}'$ of non-empty compact subsets of $\mathbb{R}^d$ such that $\mathbb{P}(X \in \mathcal{M}) = 1$. The elements of X are called *cells* of X.

Definition: A random tessellation X in $\mathbb{R}^d$ is called *face-to-face* if the faces of different cells do not overlap. In this case the system of all k-faces of X is defined by

$$\mathcal{S}_k(X) := \bigcup_{C \in X} \mathcal{S}_k(C).$$

Definition: A random tessellation X in $\mathbb{R}^d$ is called *normal* if every k-face of X is contained in precisely $d - k + 1$ cells of X.

Definition: A random tessellation X in $\mathbb{R}^d$ is called *stationary* if

$$X \overset{d}{=} X + y, \quad y \in \mathbb{R}^d.$$

Definition: Let X be a stationary random tessellation in $\mathbb{R}^d$ and $k \in \{0, \ldots, d\}$. Then

$$M_k(B) := \sum_{F \in \mathcal{S}_k(X)} |F \cap B|_k,$$

defines a stationary random measure M_k on $\mathbb{R}^d$ and

$$N_k := \{\pi_k(F) : F \in \mathcal{S}_k(X)\}$$

defines a stationary point process N_k (of centres of k-faces) on $\mathbb{R}^d$. The intensities of M_k and N_k are denoted by μ_k and γ_k, respectively.

Proposition: *Let X be a stationary random face-to-face tessellation in $\mathbb{R}^d$. Then the relationships between the Palm probability measures $\mathbb{P}^0_{M_k}$ and $\mathbb{P}^0_{N_k}$ given above for stationary Voronoi tessellations remain valid. In particular*

$$\mathbb{E}^0_{N_d}[V_d(C_d)] = \lambda^{-1},$$
$$\mathbb{E}[V_d(F_d)^{-1}] = \lambda$$

and

$$\sum_{j=0}^{d} (-1)^j \gamma_j = 0.$$

Proposition: *Let X be a stationary random face-to-face and normal tessellation in $\mathbb{R}^2$. Then $\gamma_0 = 2\lambda$ and $\gamma_1 = 3\lambda$. Moreover,*

$$\mathbb{E}^0_{N_2}[|C_2|_2] = \frac{1}{\lambda},$$

$$\mathbb{E}^0_{N_2}[|\partial C_2|_1] = \frac{2\mu_1}{\lambda},$$

$$\mathbb{E}^0_{N_2}[\nu_0(C_2)] = 6,$$

$$\mathbb{E}^0_{N_1}[|C_1|_1] = \frac{\mu_1}{3\lambda}.$$