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A Short Course on Stochastic Geometry

4. The Boolean model

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4.1 Definition of the Boolean model

(i) We consider the space $\mathcal{C}' := \mathcal{C} \setminus \{\emptyset\}$ of compact *particles* in $\mathbb{R}^d$. Being a measurable subset of $\mathcal{F}$ it is equipped with the Borel σ-field

$$\mathcal{B}(\mathcal{C}') := \{\mathcal{H} \cap \mathcal{C}' : \mathcal{H} \in \mathcal{B}(\mathcal{F})\}.$$

(ii) We consider marked point processes Φ on $\mathbb{R}^d$ with marks in $\mathcal{C}'$.

(iii) The ball centred at $x \in \mathbb{R}^d$ and having radius $r \geq 0$ is denoted by $B(x, r)$. The unit ball is denoted by $B^d := B(0, 1)$.

Definition: Let $N = \{X_n : n \in \mathbb{N}\}$ be a homogeneous Poisson process with intensity λ and $Z_1, Z_2, \dots$ a sequence of independent random particles ($\mathcal{C}'$-valued random variables) with common distribution $\mathbb{Q}$ satisfying the integrability assumption

$$\int V_d(C+K)\,\mathbb{Q}(dC) < \infty, \quad K \in \mathcal{C}'.$$

Assume that N and (Z_n) are independent. Then

$$Z := \bigcup_{n\in\mathbb{N}} (Z_n + X_n)$$

is called *Boolean model* with *grain distribution* $\mathbb{Q}$.

Convention: We often consider a *typical grain* of a Boolean model, i.e. a random closed set Z_0 with distribution $\mathbb{Q}$.

Proposition: Let $N = \{X_n : n \in \mathbb{N}\}$ be a homogeneous Poisson process and $R_1, R_2, \ldots$ a sequence of independent non-negative random variables with common distribution G. Define

$$Z = \bigcup_{n\in\mathbb{N}} B(X_n, R_n).$$

Then $\mathbb{P}(Z = \mathbb{R}^d) = 1$ iff

$$\int r^d \, G(dr) = \infty.$$

Remark: Let $\mathbb{Q}$ be the distribution of $B(0, R)$, where R is a non-negative random variable. Then $\mathbb{Q}$ satisfies the integrability condition required in the definition of the Boolean model iff

$$\mathbb{E}R^d < \infty.$$

Proposition: Let Z be a Boolean model as above. Then, almost surely, any bounded set is intersected by only finitely many of the (secondary) grains $Z_n + X_n$. In particular we have $Z \in \mathcal{F}$ outside a measurable set of $\mathbb{P}$-measure 0.

4.2 Basic properties of the Boolean model

Theorem: *A Boolean model satisfies*

$$\mathbb{P}(Z \cap B = \emptyset) = \exp\left\{-\lambda \int_{\mathcal{C}'} V_d(C + B^*)\, \mathbb{Q}(dC)\right\}, \quad B \in \mathcal{C},$$

where $B^* := \{-x : x \in B\}$.

Remark: Since the capacity functional $B \mapsto \mathbb{P}(Z \cap B \neq \emptyset)$ determines the distribution of Z, it easily follows that a Boolean model is stationary. It is isotropic if the grain distribution $\mathbb{Q}$ is isotropic, i.e.

$$\mathbb{Q}(A) = \mathbb{Q}(\vartheta A), \quad \vartheta \in SO_d,\ A \subset \mathcal{C}.$$

Proposition: *The volume fraction* $p := \mathbb{E}[V_d(Z \cap [0,1]^d)]$ *of a Boolean model* Z *is given by*

$$p = 1 - \exp\left[-\lambda \mathbb{E} V_d(Z_0)\right].$$

4.3 Contact distribution functions

Definition: Let $B \subset \mathbb{R}^d$ be a convex set with $0 \in B$ and consider a stationary random closed set Z whose volume fraction is strictly less than 1. The *contact distribution function* of Z with structuring element B is defined by

$$H_B(r) := \mathbb{P}(rB \cap Z \neq \emptyset \,|\, 0 \notin Z), \quad r \geq 0.$$

Examples:

(i) In case B equals the unit ball B^d in $\mathbb{R}^d$, H_{B^d} is called *spherical contact distribution function* of Z.

(ii) In case $B = [0, u]$ for some unit vector $u \in \mathbb{R}^d$, H_B is the *linear contact distribution function* of Z in direction u.

Proposition: *Assume that Z is a Boolean model and let B be a structuring element as above. Then*

$$H_B(r) = 1 - \exp\left\{-\lambda \int_{\mathcal{C}'} [V_d(C + rB^*) - V_d(C)]\, \mathbb{Q}(dC)\right\}, \quad r \geq 0.$$

Example: Let Z be a Boolean model with typical grain $B(0,R)$, where R is a non-negative random variable. Then the spherical contact distribution of the Boolean model is given by

$$H_{B^d}(r) = 1 - \exp\left\{-\lambda\kappa_d \sum_{j=0}^{d-1} \binom{d}{j} r^{d-j}\mathbb{E}[R^j]\right\},$$

where κ_d is the volume of the unit ball B^d.

Remark: A convex set $K \in \mathcal{C}$ satisfies the *Steiner formula*

$$V_d(K + rB^d) = \sum_{j=0}^{d} r^j \kappa_j V_{d-j}(K), \quad r \geq 0,$$

where $V_0(K), \ldots, V_d(K)$ are the *intrinsic volumes* of K. $V_d(K)$ is the volume of K, $V_{d-1}(K)$ is half the surface area, $V_{d-2}(K)$ is proportional to the integral mean curvature, ..., $V_1(K)$ is proportional to the mean width of K, and $V_0(K)$ is the Euler characteristic (which is 1 if K is non-empty and 0 if K is the empty set).

Proposition: *The spherical contact distribution of the Boolean model is given by*

$$H_{B^d}(t) = 1 - \exp\left[-\lambda \sum_{i=0}^{d-1} \kappa_{d-i} \bar{V}_i t^{d-i}\right], \quad t \geq 0,$$

where

$$\bar{V}_i := \mathbb{E}[V_i(Z_0)], \quad i = 0, \dots, d-1,$$

is the mean of the i-th intrinsic volume of the typical grain.

Remark: A similar result holds for any contact disribution function. One has to use *mixed volumes* instead of intrinsic volumes.

4.4. Geometric densities of the Boolean model

Theorem: *Assume that Z is a Boolean model with an almost surely convex typical grain. Let W be convex body and let $j \in \{0,\dots,d\}$. Then the limit*

$$\lambda_j := \lim_{r\to\infty} \frac{1}{V_d(rW)} \mathbb{E}\left[V_j(Z \cap rW)\right]$$

exists and satisfies

$$\lambda_j = \mathbb{E}[V_j(Z \cap C^d) - V_j(Z \cap \partial^+ C^d)],$$

where $C^d := [0,1]^d$ is the unit cube and $\partial^+ C^d$ the set of all points in C^d with at least one coordinate equal 1. Here the intrinsic volume V_j has to be extended in an appropriate way.

Remark: The number λ_d is just the volume fraction of Z.

Remark: We also have the almost sure convergence

$$\lambda_j = \lim_{r\to\infty} \frac{V_j(Z \cap rW)}{V_d(rW)}.$$

Theorem: *Assume that Z is a Boolean model with non-empty convex grains and define*

$$\bar{\lambda}_i := \lambda \bar{V}_i = \lambda \int V_i(K)\, \mathbb{Q}(dK), \quad i = 0, \dots, d.$$

(Note that $\bar{\lambda}_0 = \lambda$.) Then $\lambda_d = 1 - e^{-\bar{\lambda}_d}$ and

$$\lambda_{d-1} = e^{-\bar{\lambda}_d} \bar{\lambda}_{d-1}.$$

If the grain distribution $\mathbb{Q}$ is isotropic, then the numbers $\lambda_0, \dots, \lambda_{d-2}$ are polynomials in λ_d and $\bar{\lambda}_0, \dots, \bar{\lambda}_{d-1}$. In case $d = 3$, we have in particular

$$\lambda_1 = (1-p)\left(\bar{\lambda}_1 - \frac{\pi^2}{8}\bar{\lambda}_2^2\right),$$

$$\lambda_0 = (1-p)\left(\lambda - \frac{1}{4}\bar{\lambda}_1\bar{\lambda}_2 + \frac{\pi}{48}\bar{\lambda}_2^3\right).$$