Günter Last
Institut für Stochastik
Universität Karlsruhe (TH)

A Short Course on Stochastic Geometry

3. Random Measures and Random Closed Sets

Günter Last

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3.1 Random measures

Definition: Let $(\mathbb{X}, \mathcal{X})$ and $\mathcal{X}_b$ be as in Lecture 1.

(i) $\mathbf{M} = \mathbf{M}(\mathbb{X})$ is the set of all *locally finite* (i.e. finite on $\mathcal{X}_b$) measures ν on $\mathbb{X}$.

(ii) The σ-field $\mathcal{M}$ is the smallest σ-field of subsets of $\mathbf{M}$ making the mappings $\nu \mapsto \nu(B)$ for all measurable sets $B \subset \mathbb{X}$ measurable.

Definition: A *random measure* on $\mathbb{X}$ is a measurable mapping M from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into $(\mathbf{M}, \mathcal{M})$. In the *canonical case*, $\mathbb{P}$ is a probability measure on $(\mathbf{M}, \mathcal{M})$ and M is the identity on $\mathbf{M}$.

Definition: The *intensity measure* Λ of a random measure M is the measure Λ on $(\mathbb{X}, \mathcal{X})$ defined by

$$\Lambda(B) := \mathbb{E}[M(B)], \quad B \in \mathcal{X}.$$

Theorem: (Campbell's theorem) *Let M be a random measure and consider a measurable function $f : \mathbb{X} \to [0, \infty)$. Then $\int f(x)\, M(dx)$ is a random variable and*

$$\mathbb{E}\left[\int f(x)\, M(dx)\right] = \int f(x)\, \Lambda(dx),$$

where Λ is the intensity measure of M.

Definition: The *distribution* of a random measure M is the probability measure $\mathbb{P} \circ M^{-1}$ on $(\mathbf{M}, \mathcal{M})$.

Theorem: Let M and M' be random measures on $\mathbb{X}$. Then the following assertions are equivalent:

(i) $M \stackrel{d}{=} M'$.

(ii) $\int f\, dM \stackrel{d}{=} \int f\, dM'$ for all measurable $f : \mathbb{X} \to [0, \infty)$.

(iii) For all measurable $f : \mathbb{X} \to [0, \infty)$:

$$\mathbb{E}\left[\exp\left[-\int f(x)\, M(dx)\right]\right] = \mathbb{E}\left[\exp\left[-\int f(x)\, M'(dx)\right]\right].$$

3.2 Stationary random measures

Definition: Consider a Borel space $(\mathbb{Y}, \mathcal{Y})$ and the product $\mathbb{R}^d \times \mathbb{Y}$ equipped with the product σ-field $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{Y}$. Let

$$\mathbf{M} := \mathbf{M}(\mathbb{R}^d \times \mathbb{Y}),$$

where $A \subset \mathbb{R}^d \times \mathbb{Y}$ is in $(\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{Y})_b$ iff $A \subset B \times \mathbb{Y}$ for some bounded $B \subset \mathbb{R}^d$. Let $\nu \in \mathbf{M}$ and $x \in \mathbb{R}^d$. Then the *shift* $\theta_x : \mathbf{M} \to \mathbf{M}$ is defined by

$$\theta_x \nu(A) := \int \mathbf{1}_A(z - x, y)\, \nu(d(z, y)), \quad A \in \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{Y}.$$

In particular

$$\theta_x \nu(B \times C) = \nu((B + x) \times C), \quad B \in \mathcal{B}(\mathbb{R}^d), C \in \mathcal{Y}.$$

Lemma: The mapping $(x, \nu) \mapsto \theta_x \nu$ from $\mathbb{R}^d \times \mathbf{M}$ to $\mathbf{M}$ is measurable.

Definition: A random measure M on $\mathbb{R}^d \times \mathbb{Y}$ is *stationary* if

$$\theta_x M \overset{d}{=} M, \quad x \in \mathbb{R}^d.$$

Remark: Let M be a stationary random measure on $\mathbb{R}^d \times \mathbb{Y}$ on $\mathbb{R}^d$ and $C \in \mathcal{Y}$. Then

$$M(\cdot \times C)$$

is a stationary random measure on $\mathbb{R}^d$.

Definition: Let M be a stationary random measure on $\mathbb{R}^d \times \mathbb{Y}$. The *intensity* $\lambda \equiv \lambda_M$ of M is defined by

$$\lambda := \mathbb{E}[M([0,1]^d \times \mathbb{Y})].$$

Proposition: *Let M be a stationary random measure $\mathbb{R}^d \times \mathbb{Y}$ with a finite intensity λ. Then there is a probability measure $\mathbb{Q}$ on $\mathbb{Y}$ such that the intensity measure Λ of M is given by*

$$\Lambda(B \times C) = \lambda \mathbb{Q}(C) \int \mathbf{1}_B(x)dx, \quad B \in \mathcal{B}(\mathbb{R}^d), C \in \mathcal{Y}.$$

Definition: The measure $\mathbb{Q}$ is called *Palm mark distribution* of the stationary random measure M.

3.3 Random closed sets

(i) Consider the space $\mathcal{F}$ of all closed subsets of $\mathbb{R}^d$ including the empty set.

(ii) Denote by $\mathcal{O}$ and $\mathcal{C}$ the systems of all open resp. compact subsets of $\mathbb{R}^d$.

(ii) Define for $A \subset \mathbb{R}^d$:

$$\mathcal{F}^A := \{F \in \mathcal{F} : F \cap A = \emptyset\},$$
$$\mathcal{F}_A := \{F \in \mathcal{F} : F \cap A \neq \emptyset\}.$$

(iii) The *Fell topology* is generated by the sets

$$\{\mathcal{F}^C : C \in \mathcal{C}\}, \quad \{\mathcal{F}_G : G \in \mathcal{O}\}.$$

(iv) The Borel σ-field $\mathcal{B}(\mathcal{F})$ is generated by the systems in (iii).

Proposition: *The space $\mathcal{F}$ is a compact second countable Hausdorff space.*

Definition: The *distance* between a point $x \in \mathbb{R}^d$ and a set $A \subset \mathbb{R}^d$ is given by

$$d(A, x) := \inf\{\|x - y\| : y \in A\}.$$

In particular, $d(\emptyset, x) = \infty$.

Proposition: *Let D be a dense subset of $\mathbb{R}^d$ and $F, F_1, F_2, \ldots$ be closed sets. Then* $\lim F_n = F$ *iff*

$$\lim_{n\to\infty} d(F_n, x) = d(F, x), \quad x \in D.$$

Proposition:

(i) *The mapping* $(F, F') \mapsto F \cup F'$ *from* $\mathcal{F}^2$ *to* $\mathcal{F}$ *is continuous.*

(ii) *The mapping* $F \mapsto -F$ *from* $\mathcal{F}$ *to* $\mathcal{F}$ *is continuous.*

(iii) *The mapping* $(\alpha, F) \mapsto \alpha F$ *from* $\mathbb{R} \times \mathcal{F}$ *to* $\mathcal{F}$ *is continuous.*

Definition: A *random closed* set Z (in $\mathbb{R}^d$) is a measurable mapping Z from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$.

Example: A simple point process on $\mathbb{R}^d$ can be considered as a random closed set. More generally, the support of a point process on $\mathbb{R}^d$ is a random closed set.

Example: Let $\{\xi_x : x \in \mathbb{R}^d\}$ be a random field with continuous sample paths and let $t \in \mathbb{R}$. Then the *level set*

$$Z_t := \{x \in \mathbb{R}^d : \xi_x = t\}$$

is a random closed set.

3.4 The capacity functional

Definition: The *capacity functional* of a random closed set Z is the mapping $T_Z : \mathcal{C} \to \mathbb{R}$ defined by

$$T_Z(C) := \mathbb{P}(Z \cap C \neq \emptyset), \quad C \in \mathcal{C}.$$

Example: Let ξ be a random variable in $\mathbb{R}^d$. Then

$$Z := \{\xi\}$$

is a random closed set with capacity functional

$$T_Z(C) = \mathbb{P}(\xi \in C), \quad C \in \mathcal{C}.$$

Example: Let $\{\xi_x : x \in \mathbb{R}^d\}$ be a random field with continuous sample paths and $t \in \mathbb{R}$. Then

$$Z := \{x \in \mathbb{R}^d : \xi_x \geq t\}$$

is random closed set with capacity functional

$$T_Z(C) = \mathbb{P}\Big(\sup_{x \in C} \xi_x \geq t\Big), \quad C \in \mathcal{C}.$$

Proposition: *The capacity functional T of a random closed set Z has the following properties.*

(i) $0 \le T(K) \le 1$, $T(\emptyset) = 0$.

(ii) $C_i \downarrow C$ *implies that* $T(C_i) \to T(C)$.

(iii) $S_k(C_0; C_1, \ldots, C_k) \ge 0$, *where* $S_0(C) := 1 - T(C)$ *and, recursively,*

$$S_k(C_0; C_1, \ldots, C_k) \\ := S_{k-1}(C_0; C_1, \ldots, C_{k-1}) - S_{k-1}(C_0 \cup C_k; C_1, \ldots, C_{k-1}).$$

Proposition: *Let Z, Z' be random closed sets. Then $Z \stackrel{d}{=} Z'$ if and only if $T_Z = T_{Z'}$.*

Definition: A *capacity functional* is a function $T : \mathcal{C} \to \mathbb{R}$ having the above properties (i), (ii) and (iii).

Theorem: (Choquet theorem) *Let $T : \mathcal{C} \to \mathbb{R}$ be a capacity functional. Then there exists a random closed set Z such that*

$$T = T_Z.$$

3.5 Stationary and isotropic random closed sets

Definition: Let Z be a random closed set.

(i) Z is called *stationary* if

$$Z \stackrel{d}{=} Z + x, \quad x \in \mathbb{R}^d.$$

(ii) Z is called *isotropic* if

$$Z \stackrel{d}{=} \vartheta Z, \quad \vartheta \in SO_d,$$

where SO_d is the group of rotations on $\mathbb{R}^d$.

Example: If N is a stationary point process, then the support of N is a stationary closed set.

Proposition: *Let Z be random closed set.*

(i) *Z is stationary iff*

$$T_{Z+x} = T_Z, \quad x \in \mathbb{R}^d.$$

(ii) *Z is isotropic iff*

$$T_{\vartheta Z} = T_Z, \quad \vartheta \in SO_d.$$

3.6 The volume fraction

Definition: Let Z be a stationary random closed set. The *volume fraction* of Z is the number

$$p := \mathbb{E}[V_d(Z \cap [0,1]^d)].$$

Proposition: *Let Z be stationary random closed set with volume fraction p.*

(i) *We have*

$$\mathbb{E}[V_d(Z \cap B)] = pV_d(B), \quad B \in \mathcal{B}^d.$$

(ii) *We have*

$$p = \mathbb{P}(x \in Z), \quad x \in \mathbb{R}^d.$$

3.7 The covariance

Definition: Let Z be a random closed set. The *covariance* of Z is the function $C : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$C(x, y) := \mathbb{P}(\{x, y\} \subset Z) = \mathbb{E}[\mathbf{1}_Z(x)\mathbf{1}_Z(y)], \quad x, y \in \mathbb{R}^d.$$

Proposition: *Let Z be stationary random closed set. Then $C(x, y)$ depends only on the difference $x - y$. If, moreover, Z is isotropic, then $C(x, y)$ depends only on the Euclidean norm $\|x - y\|$ of $x - y$.*

Remark: Assume that Z is stationary and ergodic with volume fraction p. Under suitable ergodicity or mixing conditions we have

$$\lim_{r \to \infty} C(r) = p^2.$$

Remark: Assume that Z is stationary and ergodic with volume fraction p. A popular approximation of the covariance function is

$$C(r) \approx p(1-p)e^{-ar} + p^2$$

for some suitable $a > 0$.