Günter Last
Institut für Stochastik
Universität Karlsruhe (TH)

A Short Course on Stochastic Geometry

2. Poisson Processes

Günter Last

Lectures presented at the

School of Mathematical and Computer Sciences

Heriot-Watt University, Edinburgh

28.02.–13.03. 2007

2.1 Definition

Definition: Let Λ be a measure on $\mathbb{X}$. A *Poisson process* with intensity measure Λ is a point process N on $\mathbb{X}$ with the following two properties:

(i) The random variables $N(B_1), \ldots, N(B_m)$ are stochastically independent whenever $B_1, \ldots, B_m$ are measurable and pairwise disjoint.

(ii) We have

$$\mathbb{P}(N(B) = k) = \frac{\Lambda(B)^k}{k!} \exp[-\Lambda(B)], \quad k \in \mathbb{N}_0, B \in \mathcal{X},$$

where $\infty^k e^{-\infty} := 0$ for all $k \in \mathbb{N}_0$.

2.2 The characteristic functional

Theorem: *Let Λ be a locally finite measure on $\mathbb{X}$. Then a point process N on $\mathbb{X}$ is a Poisson process with intensity measure Λ if and only if*

$$\mathbb{E}\left[\exp\left[-\int f(x)\, N(dx)\right]\right] = \exp\left[-\int \left(1 - e^{-f(x)}\right) \Lambda(dx)\right]$$

for all measurable $f : \mathbb{X} \to [0, \infty)$.

Corollary: *Assume that N is a mixed sample process with sample distribution F and a sample size distribution that is Poisson with intensity $\gamma \geq 0$. Then N is a Poisson process with intensity measure γF.*

2.3 Existence of Poisson processes

Theorem: *Let Λ be a locally finite measure on $\mathbb{X}$. Then there exists a Poisson process N on $\mathbb{X}$ with intensity measure Λ. (This might require an extension of the original probability space.)*

Corollary: (simple Poisson processes) *A Poisson process N on $\mathbb{X}$ is simple if and only if its intensity measure Λ is* diffuse, *i.e.*

$$\Lambda(\{x\}) = 0, \quad x \in \mathbb{X}.$$

2.4 Mecke's formula

Theorem: *Let Λ be a locally finite measure on $\mathbb{X}$. Then a point process N on $\mathbb{X}$ is a Poisson process with intensity measure Λ if and only if*

$$\mathbb{E}\left[\int f(N,x)\,N(dx)\right] = \mathbb{E}\left[\int f(N+\delta_x,x)\,\Lambda(dx)\right]$$

for all measurable $f : \mathbf{N} \times \mathbb{X} \to [0,\infty)$.

2.5 Stationary Poisson processes

Theorem: *Let N be Poisson process on $\mathbb{R}^d$ with intensity measure Λ. Then N is stationary if and only if*

$$\Lambda(B) = \lambda \int \mathbf{1}_B(x)\,dx, \quad B \in \mathcal{B}(\mathbb{R}^d),$$

for some $\lambda \geq 0$.

Remark: The number λ is the intensity of the point process N. A stationary Poisson process is also called *homogeneous*.

Theorem: *Let $\lambda \geq 0$. Then a stationary point process N on $\mathbb{R}^d$ is a Poisson process with intensity λ if and only if*

$$\mathbb{E}\left[\int f(\theta_x N, x)\, N(dx)\right] = \lambda \mathbb{E}\left[\int f(N + \delta_0, x)\, dx\right]$$

for all measurable $f : \mathbf{N}(\mathbb{R}^d) \times \mathbb{X} \to [0, \infty)$.

Corollary:(Slivnyak's theorem) *A stationary point process N on $\mathbb{R}^d$ with positive and finite intensity is a Poisson process if and only if its Palm distribution $\mathbb{P}_N^0$ satisfies*

$$\mathbb{P}_N^0 = \mathbb{P}(N + \delta_0 \in \cdot).$$

2.6 Rényi's theorem

Theorem: *Consider two simple point processes N and N'. Then $N \stackrel{d}{=} N'$ if and only if*

$$\mathbb{P}(N(B) = 0) = \mathbb{P}(N'(B) = 0), \quad B \in \mathcal{X}_b.$$

Theorem: *Assume that N is a simple point process with diffuse and locally finite intensitiy measure Λ. Then N is a Poisson process if and only if*

$$\mathbb{P}(N(B) = 0) = \exp[-\Lambda(B)], \quad B \in \mathcal{X}_b.$$

2.7 The marking theorem

Theorem: *Let Φ be a marked point process on $\mathbb{X}$ with mark space $\mathbb{Y}$ and ground process*

$$N := \Phi(\cdot \times \mathbb{Y}).$$

Consider a probability measure $\mathbb{Q}$ on $(\mathbb{Y}, \mathcal{Y})$. Then the following two assertions are equivalent:

(i) *N is a Poisson process and Φ is a $\mathbb{Q}$-marking of N.*

(ii) *Φ is a Poisson process on $\mathbb{X} \times \mathbb{Y}$ with intensity measure $\Lambda \otimes \mathbb{Q}$.*