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A Short Course on Stochastic Geometry

1. Spatial Point Processes

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1.1 Framework

(i) The phase space $\mathbb{X}$ equipped with a σ-field σ-field $\mathcal{X}$. It is assumed that $(\mathbb{X}, \mathcal{X})$ is a *Borel space*, i.e. Borel isomorphic to a Borel subset of $[0,1]$.

(ii) A ring $\mathcal{X}_b$ of *bounded* sets, such that $\mathbb{X}$ is the countable union of sets from $\mathcal{X}_b$.

(iii) The space $\mathbf{N} = \mathbf{N}(\mathbb{X})$ of all *locally finite* (i.e. finite on $\mathcal{X}_b$) counting measures on $\mathbb{X}$.

(iv) The space $\mathbf{N}_s = \mathbf{N}_s(\mathbb{X})$ of all *simple* counting measures on $\mathbb{X}$. This is the space of *point configurations* in $\mathbb{X}$

(v) The σ-field $\mathcal{N} = \mathcal{N}(\mathbb{X})$ is the smallest σ-field of subsets of $\mathbf{N}$ making the mappings $\varphi \mapsto \varphi(B)$ for all measurable sets $B \subset \mathbb{X}$ measurable.

Remarks:

(i) Any *Polish* (i.e. complete and separable metric) space is a Borel space.

(ii) If $\mathbb{X}$ is a Borel subset of a Polish space, then $\mathcal{X}$ is always chosen as the system of all Borel subsets of $\mathbb{X}$. One possible choice of $\mathcal{X}_b$ is then the systems $\mathcal{B}(\mathbb{X})$ of all Borel sets that are bounded with respect to the underlying metric.

(iii) A locally compact, second-countable Hausdorff space is Polish. In this case $\mathcal{X}_b$ is chosen as the system of all relatively compact subsets of $\mathcal{X}$.

Lemma: Any $\varphi \in \mathbf{N}$ can be measurably decomposed as

$$\varphi = \sum_{j=1}^{n} \delta_{x_j}$$

for some $n \in \mathbb{N} \cup \{\infty\}$, and $x_1, x_2, \ldots \in \mathbb{X}$. In particular, $\mathbf{N}_s$ is a measurable subset of $\mathbf{N}$.

1.2 Definition of a point process

Definition: A *point process* on $\mathbb{X}$ is a measurable mapping N from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into $(\mathbf{N}, \mathcal{N})$. In the *canonical case*, $\mathbb{P}$ is a probability measure on $(\mathbf{N}, \mathcal{N})$ and N is the identity on $\mathbf{N}$. A point process N is *simple* if $\mathbb{P}(N \in \mathbf{N}_s) = 1$.

Lemma: Let N be a point process on $\mathbb{X}$. Then there are random elements $X_1, X_2, \dots$ in $\mathbb{X}$ such that

$$N = \sum_{n=1}^{N(\mathbb{X})} \delta_{X_n}.$$

Remark: A simple point process can be identified with the (random) set $\{X_n : n \le N(\mathbb{X})\}$.

1.3 The intensity measure

Definition: The *intensity measure* Λ of a point process N is the measure Λ on $(\mathbf{X}, \mathcal{X})$ defined by

$$\Lambda(B) := \mathbb{E}[N(B)], \qquad B \in \mathcal{X}.$$

Theorem: (Campbell's theorem) *Let N be a point process and consider a measurable function $f : \mathbb{X} \to [0, \infty)$. Then*

$$\int f(x)\, N(dx) \equiv \sum_{x \in N} f(x)$$

is a random variable and

$$\mathbb{E}\left[\int f(x)\, N(dx)\right] = \int f(x)\, \Lambda(dx),$$

where Λ is the intensity measure of N.

1.4 Binomial and mixed binomial point processes

Definition: Let $X_1, X_2, \ldots, X_m$ be m independent random vectors in $\mathbb{R}^d$ with common distribution F. Then

$$N := \sum_{n=1}^{m} \delta_{X_n}$$

is called *binomial process* with sample size m and sample distribution F.

Remark: If N is a Binomial process as above, then $N(B)$ has a $\text{Bin}(m, F(B))$-distribution for all Borel sets B. In particular N has intensity measure

$$\mathbb{E}[N(B)] = mF(B), \qquad B \in \mathcal{B}(\mathbb{R}^d).$$

Definition: Let $X_1, X_2, \ldots$ be an infinite sequence of independent random vectors in $\mathbb{R}^d$ with common (diffuse) distribution F and let τ be an integer-valued random variable independent of the sequence (X_n). Then

$$N := \sum_{n=1}^{\tau} \delta_{X_n}$$

is called *mixed binomial process* with random sample size τ and sample distribution F.

Proposition: Assume that N is a mixed Binomial process, where τ has a Poisson distribution with parameter $\lambda > 0$. Then $N(B)$ has a Poisson-distribution with parameter $\lambda F(B)$. Moreover, $N(B)$ and $N(B')$ are independent, whenever B and B' are disjoint.

1.5 The distribution of a point process

Definition: The *distribution* of a point process N is the probability measure $\mathbb{P} \circ N^{-1}$ on $(\mathbf{N}, \mathcal{N})$.

Theorem: Let N and N' be point processes on $\mathbb{X}$. Then the following assertions are equivalent:

(i) $N \overset{d}{=} N'$.

(ii) $\int f\, dN \overset{d}{=} \int f\, dN'$ for all measurable $f : \mathbb{X} \to [0, \infty)$.

(iii) For all measurable $f : \mathbb{X} \to [0, \infty)$:

$$\mathbb{E}\left[\exp\left[-\int f(x)\, N(dx)\right]\right] = \mathbb{E}\left[\exp\left[-\int f(x)\, N'(dx)\right]\right].$$

1.6 Marked point processes

Definition: Consider a Borel space $(\mathbb{Y}, \mathcal{Y})$ and consider the product space $\mathbb{X} \times \mathbb{Y}$ equipped with the product σ-field $\mathcal{X} \otimes \mathcal{Y}$. Define

$$(\mathcal{X} \otimes \mathcal{Y})_b := \{B \times C : B \in \mathcal{X}_b, C \in \mathcal{Y}\}.$$

A *marked point process* on $\mathbb{X}$ with *mark space* $\mathbb{Y}$ is then a point process on $\mathbb{X} \times \mathbb{Y}$.

Remark: If Φ is a marked point process on $\mathbb{X}$ with mark space $\mathbb{Y}$ then

$$\Phi(\cdot \times \mathbb{Y})$$

is a point process on $\mathbb{X}$, the *ground process* of Ψ.

Definition: Let $\mathbb{Q}$ be a probability measure on $(\mathbb{Y}, \mathcal{Y})$ and let

$$N = \sum_{n=1}^{N(\mathbb{X})} \delta_{X_n}$$

be a point process on $\mathbb{X}$. A marked point process Φ on $\mathbb{X}$ with mark space $\mathbb{Y}$ is called (independent) $\mathbb{Q}$-marking of N if there is a sequence $Y_1, Y_2, \dots$ of independent random elements in $\mathbb{Y}$ with distribution $\mathbb{Q}$ such that (Y_n) is independent of N and

$$\Phi \overset{d}{=} \sum_{n=1}^{N(\mathbb{X})} \delta_{(X_n, Y_n)}.$$

Lemma: Let N be a point process on $\mathbb{X}$ with intensity measure Λ and let Φ be a $\mathbb{Q}$-marking of N. Then Φ has intensity measure $\Lambda \otimes \mathbb{Q}$.

Example: Let $p \in [0,1]$ and consider a sequence $Y_1, Y_2, \ldots$ of independent $\{0,1\}$-valued random variables with $\mathbb{P}(Y_i = 1) = p$ for $i = 1, 2, \ldots$. Then the point process

$$N_p := \sum_{n=1}^{N(\mathbb{X})} \mathbf{1}\{Y_n = 1\} \delta_{X_n}.$$

is called *p-thinning* of N.

1.7 Stationary point processes

Definition: A simple point process N on $\mathbb{R}^d$ is *stationary* if

$$\sum_{n=1}^{N(\mathbb{R}^d)} \delta_{X_n+x} \overset{d}{=} N = \sum_{n=1}^{N(\mathbb{R}^d)} \delta_{X_n}, \quad x \in \mathbb{R}^d.$$

Remark: A point process N on $\mathbb{R}^d$ is *stationary* if

$$\theta_x N \overset{d}{=} N, \quad x \in \mathbb{R}^d,$$

where $\theta_x N$ is the point process defined by

$$\theta_x N(B) := N(B+x), \quad B \in \mathcal{B}(\mathbb{R}^d).$$

Proposition: Let N be a stationary point process on $\mathbb{R}^d$. Then

$$\mathbb{P}(N(\mathbb{R}^d) = 0) + \mathbb{P}(N(\mathbb{R}^d) = \infty) = 1.$$

Convention: Let N be a stationary point process. If not stated otherwise it will always be assumed that

$$\mathbb{P}(N(\mathbb{R}^d) = \infty) = 1.$$

Definition: Let N be a stationary point process. The *intensity* $\lambda \equiv \lambda_N$ of N is defined by

$$\lambda := \mathbb{E}[N([0,1]^d)].$$

Proposition: *Let N be a stationary point process on $\mathbb{R}^d$ with a finite intensity λ. Then the intensity measure Λ of N is a multiple of Lebesgue measure, i.e.*

$$\Lambda(B) = \lambda \int \mathbf{1}_B(x)\, dx, \quad B \in \mathcal{B}(\mathbb{R}^d).$$

Definition: A marked point process Ψ on $\mathbb{R}^d$ with mark space $\mathbb{Y}$ is *stationary* if

$$\theta_x \Psi \overset{d}{=} \Psi, \quad x \in \mathbb{R}^d,$$

where $\theta_x \Psi$ is the marked point process defined by

$$\theta_x \Psi(B \times C) := \Psi((B + x) \times C), \quad B \in \mathcal{B}(\mathbb{R}^d), C \in \mathcal{Y}.$$

Definition: Let Ψ be a stationary marked point process. The *intensity* $\lambda \equiv \lambda_\Psi$ of Ψ is defined by

$$\lambda := \mathbb{E}[\Psi([0,1]^d \times \mathbb{Y})].$$

Remark: The intensity of Ψ is the intensity of the ground process of Ψ.

1.8 The Palm distribution

Proposition: *Let Ψ be a stationary marked point process with a finite intensity λ. Then there is a probability measure $\mathbb{Q}$ on $\mathbb{Y}$ such that the intensity measure Λ of Ψ is given by*

$$\Lambda(B \times C) = \lambda \mathbb{Q}(C) \int \mathbf{1}_B(x)\, dx, \quad B \in \mathcal{B}(\mathbb{R}^d), C \in \mathcal{Y}.$$

Definition: The measure $\mathbb{Q}$ is called *Palm mark distribution* of the stationary marked point process Ψ.

Example: Assume that Ψ is an independent $\mathbb{Q}$-marking of a stationary ground process N. Then Ψ is stationary. Moreover, $\mathbb{Q}$ is the Palm mark distribution of Ψ.

Lemma: Let N be a stationary point process on $\mathbb{R}^d$. Then

$$\Psi(B \times C) := \int \mathbf{1}_B(x)\mathbf{1}_C(\theta_x N)\, N(dx), \quad B \in \mathcal{B}(\mathbb{R}^d), C \in \mathcal{N}(\mathbb{R}^d),$$

defines a stationary marked point process on $\mathbb{R}^d$ with mark space $\mathbf{N}(\mathbb{R}^d)$.

Definition: Let N be a stationary point process on $\mathbb{R}^d$ with finite intensity. The Palm mark distribution of the stationary marked point process Ψ defined in the above lemma is called *Palm distribution* of N. It is denoted by $\mathbb{P}_N^0$.

Theorem: (refined Campbell's theorem) *Let Ψ be a stationary marked point process on $\mathbb{R}^d$ with finite intensity λ and Palm mark distribution $\mathbb{Q}$. Then*

$$\mathbb{E}\left[\int f(x,y)\,\Psi(d(x,y))\right] = \lambda \iint f(x,y)\,dx\,\mathbb{Q}(dy),$$

for all measurable function $f : \mathbb{R}^d \times \mathbb{Y} \to [0,\infty)$.

Theorem: (refined Campbell's theorem) *Let N be a stationary point process on $\mathbb{R}^d$ with finite intensity λ and Palm distribution $\mathbb{P}_N^0$. Then*

$$\mathbb{E}\left[\int f(x,\theta_x N)\,N(dx)\right] = \lambda \iint f(x,\varphi)\,dx\,\mathbb{P}_N^0(d\varphi),$$

for all measurable function $f : \mathbb{R}^d \times \mathbf{N} \to [0,\infty)$.