

The reflected process of a random walk or Lévy process

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Queue length, drawdown process, segmental score, fluid
queue..

These are all guises for the reflected process:

If S a random walk and $\bar{S}_n = \max_{r \leq n} S_r$, this is

$$R = (R_n, n \geq 0) \text{ where } R_n = \bar{S}_n - S_n.$$

If X is a Lévy process and $\bar{X}_t = \sup_{s \leq t} X_s$, this is

$$R = (R_t, t \geq 0) \text{ where } R_t = \bar{X}_t - X_t.$$

R is a Markov process, it moves like $-S$ (or $-X$) with a barrier at 0.

Its distribution is specified by F -the distribution of the steps $Y_1, Y_2 \dots$ in the rw case, or (γ, σ, Π) -the characteristic triple of the Lp.

AIM: to describe some recent results about the reflected process, viewed as a stochastic process in its own right.

1 BASICS

For each **fixed** n ,

$$R_n \stackrel{D}{=} \left(- \min_{1 \leq j \leq n} S_j \right) = -\underline{S}_n.$$

In all cases we have $\limsup R_n \stackrel{a.s.}{=} \infty$, but as for rws, 3 distinct cases:

- $S_n \xrightarrow{a.s.} -\infty$, then $\overline{S}_n \xrightarrow{a.s.} \overline{S}_\infty < \infty$, and $R_n \simeq \overline{S}_\infty - S_n$ for large n .
- $S_n \xrightarrow{a.s.} +\infty$, then $-\underline{S}_n \xrightarrow{a.s.} -\underline{S}_\infty < \infty$, and $R_n \xrightarrow{D} R \in (0, \infty)$.
- S oscillates, then R oscillates, in the sense that $P(R_n = 0 \text{ i.o.}) = 1$.

2 Passage times

In the rw case we put $\tau_\kappa(r) = \min\{n : R_n > rn^\kappa\}$. The first question is: when is this a.s. finite for every $r > 0$, or equivalently, when is

$$\limsup \frac{R_n}{n^\kappa} \stackrel{a.s.}{=} \infty? \quad (1)$$

Since $R_n \geq -S_n$, $\liminf \frac{S_n}{n^\kappa} \stackrel{a.s.}{=} -\infty$ is a sufficient condition, and the NASC for this is known from Kesten and Maller, 1998, Doney and Maller, 2005. The complete answer is:

1. For $\kappa = 0$ (1) always happens, and in fact $\tau_0(r)$ has a finite mgf.
2. For $\kappa > 1$ (1) $\iff E(Y^-)^{1/\kappa} = \infty$.

3. For $0 < \kappa \leq 1$ (1) $\iff E(Y^-)^{1/\kappa} = \infty$ OR $\liminf \frac{S_n}{n^\kappa} \stackrel{a.s.}{=} -\infty$.

Note that

- Result 2 is quite different from the NASC for $\liminf \frac{S_n}{n^\kappa} \stackrel{a.s.}{=} -\infty$, which is an integral test involving both tails of F .
- In 3, $E(Y^-)^{1/\kappa} = \infty$ and $\liminf \frac{S_n}{n^\kappa}$ finite is possible, and $E(Y^-)^{1/\kappa} < \infty$ and $\liminf \frac{S_n}{n^\kappa} \stackrel{a.s.}{=} -\infty$ can also occur.
- The case $\kappa \in (1/2, 1)$, $EY = 0$ is particularly interesting: then the NASC when $E(Y^-)^{1/\kappa} < \infty$ is that $E(Y^+)^{1/\kappa} = \infty$ and for every $\lambda > 0$

$$\int_1^\infty \exp \left\{ -\lambda \left(\frac{y^{\frac{2\kappa-1}{\kappa}}}{W(y)} \right)^{\kappa/(1-\kappa)} \right\} \frac{dy}{y} = \infty ,$$

where

$$W(y) = \int_0^y \int_z^\infty x F(dx) dz.$$

2.1 Lévy processes-large times

We can deduce exactly analogous results for Lps from the rw results. (We should work with $\tilde{\tau}_\kappa(r) = \inf\{t : X_t > r(1+t)^\kappa\}$ to avoid problems at 0. To get the results just replace Y by X_1)

One way is easy:

$$R(n) = \sup_{s \leq n} X_s - X_n \geq \max_{r \leq n} X_r - X_n$$

gives a bound in one direction, and for the other we use the stochastic bounds in Doney (2005). We write $X_t = \tilde{X}_t + Z_t$, where Z is the compound Poisson process

of jumps exceeding 1 in absolute value: let J_1, J_2, \dots denote those jumps, and $\theta_1, \theta_2, \dots$ their occurrence times. Also put $\theta_0 = 0$, and

$$\begin{aligned} M_n &= \sup_{\theta_n \leq t < \theta_{n+1}} X_t, \quad n \geq 0, \\ \nu_n &= \inf\{t \geq \theta_n : X_t \geq M_n\}. \end{aligned}$$

Then

$$\begin{aligned} M_n &= M_{n-1} + J_n + \inf_{\nu_{n-1} \leq t < \theta_n} (X_t - M_{n-1}) \\ &\quad + \sup_{\theta_n < t \leq \nu_n} (X_t - X_{\theta_n}) \\ &= M_{n-1} + Y_n, \quad \text{where } Y_n \stackrel{D}{=} X(\theta_1). \end{aligned}$$

Thus $M_n = M_0 + S_n$, and, for $t \in (\theta_n, \theta_{n+1}]$, $R_t \leq \max_{r \leq n} M_r - X_t = R_n^{(S)} + M_n - X_t$ and

$$\begin{aligned} M_n - X_t &= M_n - \sum_1^n J_r - (X_t - \sum_1^n J_r) \\ &= \sup_{(\theta_n, \theta_{n+1}]} \tilde{X}_s - \tilde{X}_t \leq \Delta_n \\ &: = \sup_{(\theta_n, \theta_{n+1}]} \tilde{X}_s - \inf_{(\theta_n, \theta_{n+1}]} \tilde{X}_s. \end{aligned}$$

Thus $R_t \leq R_n^{(S)} + \Delta_n$, and this is good enough.

2.2 Small-time behaviour of R .

Here we look for the NASC for

$$\limsup_{t \rightarrow 0} \frac{R_t}{t^\kappa} \stackrel{a.s.}{=} \infty; \quad (2)$$

clearly there is no rw analogue, but in fact there is a formal similarity between the results at 0 and ∞ .

The results here are in Savov (2007), and rely on results from Bertoin, Doney and Maller (2007), where the same question is answered for X .

We put $\bar{\Pi}^-(x) = \Pi((-\infty, -x))$ for $x > 0$, noting that

$$\int_0^1 \bar{\Pi}^-(x^\kappa) dx = \kappa^{-1} \int_{-1}^0 |y|^{1/\kappa} \Pi(dy).$$

Assume first that X has bounded variation, and drift d . Then (2) happens iff $\kappa > 1$ and either $d < 0$ or $\int_0^1 \bar{\Pi}^-(x^\kappa) dx = \infty$.

If X has infinite variation then

(i) (2) always holds if $\kappa \geq 1$, and never holds if $\kappa < 1/2$.

(ii) It holds for $\kappa \in [1/2, 1)$ iff $\int_0^1 \bar{\Pi}^-(x^\kappa) dx = \infty$ OR $\liminf \frac{X_t}{t^\kappa} \stackrel{a.s.}{=} -\infty$.

This last can occur in several ways:

- If $1/2 < \kappa < 1$ it requires $\sigma > 0$ or $\sigma = 0$, $\int_0^1 \bar{\Pi}^-(x^\kappa) dx < \infty$, $\int_0^1 \bar{\Pi}^+(x^\kappa) dx = \infty$, and for every $\lambda > 0$,

$$\int_0^1 \exp \left\{ -\lambda \left(\frac{y^{\frac{2\kappa-1}{\kappa}}}{\omega(y)} \right)^{\kappa/(1-\kappa)} \right\} \frac{dy}{y} = \infty$$

where

$$\omega(y) = \int_0^y \int_z^1 x \Pi^-(dx).$$

- The case $\kappa = 1/2$ is quite different; here the NASC when $\sigma = 0$ is that

$$\int_0^1 \exp \left\{ -\frac{\lambda^2}{2V(x)} \right\} \frac{dx}{x} = \infty \text{ for every } \lambda > 0,$$

where

$$V(x) = \int_{-x}^x y^2 \Pi(dy).$$

3 Expected Passage times

The general question: when is $E\tau_\kappa(r)$ finite? is extremely difficult, so we focus on 2 apparently straightforward cases.

- Case1: $EY = 0, \text{Var}(Y) = s^2 < \infty$.
- Case2: $E|Y| < \infty, EY = \mu < 0$.

In Case1, we ask is $E\tau_{1/2}(r)$ finite?

Here the correct rw comparison is with $T_{1/2}(r) = \min\{n : |S_n| > r\sqrt{n}\}$.

In 1967 Gundy and Siegmund proved that $ET_{1/2}(r) < \infty \iff r < s$, and in Doney and Maller, 2007, we show the same holds for R .

Their proof relies on the fact that $S_n^2 - ns^2$ is a martingale.

Ours use that, with $\phi(x) = 2 \int_x^\infty y\bar{F}(y)dy$,

$$R_n^2 - ns^2 + \sum_1^n \phi(R_{i-1})$$

is a martingale.

For Case 2, we ask whether $E\tau_1(r)$ is finite, and the obvious comparison is with the passage time of $-S$ over rn ; in particular $E\tau_1(-\mu)$ should be infinite. We can prove this, if we assume that $E(Y^+)^2 < \infty$, but we CANNOT prove it in general!

Similar results hold in the L_p situation.

4 Renewal Theorems and the Overshoot

Write $\tau(r)$ for $\tau_0(r)$: the following mimics known results for $T_r = \min\{n : |S_n| > r\}$:

In Case 1

$$\lim_{r \rightarrow \infty} \frac{E\tau(r)}{r^2} = \frac{1}{s^2},$$

In Case 2

$$\lim_{r \rightarrow \infty} \frac{E\tau(r)}{r} = \frac{1}{-\mu}.$$

When R crosses level r is the overshoot $O_r = R_{\tau(r)} - r$ a.s. asymptotically small compared to r ?

Griffin and Maller (1998) showed the answer to the similar problem for $|S|$ is that it occurs in cases 1 and 2, and also when $\mu \in (0, \infty)$, but in **no other cases**.

In Doney, Maller and Savov (2007) we show that in these cases $r^{-1}O_r \xrightarrow{a.s.} 0$, but the converse is definitely false: it can also happen with $\mu = 0$ and $s^2 = \infty$.