Regular Variation of Arbitrary Order

Edward Omey

HUB

Brussels - Belgium

www.edwardomey.com

Topics of research

- Regular variation and extensions in analysis and in probability theory

- Univariate and multivariate regular variation

- Univariate and multivariate subexponential distribution functions

- Renewal Theory and Weighted Renewal Theory

- Subordination (univariate and multivariate)

- $0 - 1$ sequences

- etc

Outline

Introduction

Regular variation of order $1, 2, ..., n$

Regularly varying rate vectors

Regular varation of order n

Some applications

Further research

1 Introduction

1.1 Regular Variation of order 1

RV: a positive and measurable function $f(x)$ is regularly varying if, as $t \to \infty$ we have

$$\frac{f(tx)}{f(t)} \to \lambda(x) < \infty, \ \forall x > 0.$$

Questions

Q1: What is $\lambda(x)$?

Q2: For a given λ, characterize f.

Answers

Q1. To answer Q1 we proceed as follows: for $x, y > 0$, we write

$$\frac{f(txy)}{f(t)} = \frac{f(txy)}{f(ty)} \frac{f(ty)}{f(t)},$$

and we obtain that

$$\lambda(xy) = \lambda(x) \times \lambda(y).$$

Under minimal assumptions, we find $\lambda(x) \equiv 0$, or, for some real number α,

$$\lambda(x) = x^{\alpha}.$$

In this case we use the notation $f \in RV(\alpha)$.

Q2 To answer Q2, we need the following uniform convergence theorem:

Theorem 1 *If $f \in RV(\alpha)$, then $f(tx)/f(t) \to x^{\alpha}$ locally uniform in $x > 0$.*

As a Corollary, we obtain the following Representation Theorem

Theorem 2 *(Representation Theorem) Suppose that f is a positive and measurable function. Then $f \in RV(\alpha)$ if and only if f can be represented as*

$$f(x) = x^{\alpha} c(x) \exp\left(\int_a^x \varepsilon(t) \frac{1}{t} dt\right)$$

where $c(x)$ and $\varepsilon(x)$ are measurable functions such that $c(x) \to c > 0$ and $\varepsilon(t) \to 0$.

1.2 Regular variation of order 2

Taking for simplicity $\alpha = 0$, we have

$$\frac{f(tx)}{f(t)} \to 1,\ \forall x > 0. \tag{1}$$

To analyse the speed of convergence in (1), we study second-order relations of the following type:

$$\frac{f(tx) - f(t)}{g_1(t)} \to h_1(x) < \infty, \forall x > 0. \tag{2}$$

Again some questions arise: characterize h_1, characterize g_1, characterize f.

1.2.1 To characterize h_1 and/or g_1 in (2)

We replace x by xy. On the one hand we have

$$\frac{f(txy)-f(t)}{g_1(t)} \to h_1(xy)$$

On the other hand, we have

$$\begin{aligned} & \frac{f(txy)-f(t)}{g_1(t)} \\ = \; & \frac{f(txy)-f(ty)}{g_1(ty)}\frac{g_1(ty)}{g_1(t)} + \frac{f(ty)-f(t)}{g_1(t)} \\ = \; & h_1(x)(1+o(1))\frac{g_1(ty)}{g_1(t)} + h_1(y) + o(1) \end{aligned}$$

It follows that we should have

$$h_1(xy) = h_1(x)\frac{g_1(ty)}{g_1(t)} + h_1(y) + o(1)\frac{g_1(ty)}{g_1(t)} + o(1)$$

Now assume that $h_1(x) \neq 0$ for at least one value of x.

In this case we find that

$$\frac{g_1(ty)}{g_1(t)} \to \lambda_1(y) := \frac{h_1(xy) - h_1(y)}{h_1(x)}, \forall y > 0.$$

Under minimal assumptions on h_1, we find $\lambda_1(y) = y^\beta$ and then also that $g_1 \in RV(\beta)$.

If so, it follows that

$$h_1(xy) - h_1(y) = y^\beta h_1(x).$$

We can easily solve this functional equation:

If $\beta = 0$, we find that

$$h_1(x) + h_1(y) = h_1(xy)$$

and hence $h_1(x) = c_1 \log(x)$.

If $\beta \neq 0$, by interchanging x and y, we have

$$h_1(xy) = h_1(y) + y^{\beta} h_1(x) = h_1(x) + x^{\beta} h_1(y)$$

and we obtain that $h_1(x) = c_1(1 - x^{\beta})$.

Note that in both cases, we find that h_1 is of the form

$$h_1(x) = c_1 \int_1^x u^{\beta-1} du$$

1.2.2 To characterize f

we consider 3 cases.

Case $\beta = 0$ Here $g_1 \in RV(0)$ and this case corresponds to the class $\Pi(g_1)$ of De Haan. We have the following representation theorem.

Theorem 3 *We have $f \in \Pi(g_1)$ if and only if f is of the form*

$$f(x) = c + dg_1(x) + \int_a^x g_1(t)\frac{1}{t}dt.$$

Case $\beta > 0$ or $\beta < 0$ In this case, relation (2) brings nothing new since we have

$$\begin{aligned} \beta &> 0: \; f(x) \sim cg_1(x); \\ \beta &< 0: \; f(x) \to d \text{ and } f(x) - d \sim cg_1(x). \end{aligned}$$

1.3 Regular variation of order 3

To study the rate of convergence in (2), we study third-order relations of the form

$$f(tx) - f(t) = h_1(x)g_1(t) + h_2(x)g_2(t) + o(1)g_2(t) \tag{3}$$

where the new function g_2 is a measurable, ultimately of one sign and satisfies $g_2(t) = o(1)g_1(t)$.

Using vector notation, we set $\overrightarrow{h}(x) = (h_1(x), h_2(x))$ and $\overrightarrow{g}(t) = (g_1(t), g_2(t))^T$ to get that

$$f(tx) - f(t) = \overrightarrow{h}(x)\overrightarrow{g}(t) + o(1)g_2(t).$$

Again, we replace x by xy, and we obtain:

$$f(txy) - f(t) = \overrightarrow{h}(xy)\overrightarrow{g}(t) + o(1)g_2(t),$$

and

$$\begin{aligned} & f(txy) - f(t) \\ = \; & f(txy) - f(tx) + f(tx) - f(t) \\ = \; & \overrightarrow{h}(y)\overrightarrow{g}(tx) + \overrightarrow{h}(x)\overrightarrow{g}(t) \\ & o(1)g_2(t). \end{aligned}$$

It follows that we should have

$$\begin{aligned} \overrightarrow{h}(y)\overrightarrow{g}(tx) &= (\overrightarrow{h}(xy) - \overrightarrow{h}(x))\overrightarrow{g}(t) \\ &\quad + o(1)g_2(tx) + o(1)g_2(t). \end{aligned}$$

Under suitable conditions on $\overrightarrow{h}$, this suggests a relation of the form

$$\overrightarrow{g}(tx) = A(x)\overrightarrow{g}(t) + o(1)g_2(t)$$

1.4 Regular variation of order n

In this report we study regular variation of arbitrary type.

We start from a vector

$$\overrightarrow{g}(t) = (g_1(t), g_2(t), ..., g_n(t))^T$$

and assume for $n \geq 1$ that

$$g_{i+1}(t) = o(1) g_i(t), 1 \leq i \leq n-1.$$

We call f a **regularly varying function of order** n (Notation: $f \in RV(\overrightarrow{h}, \overrightarrow{g})$) if it satisfies:

$$f(tx) = f(t) + \overrightarrow{h}(x) \overrightarrow{g}(t) + o(1) g_n(t), \qquad (4)$$

where $\overrightarrow{h}(x)$ is to be determined later.

For the functions $\overrightarrow{g}$, it turns out that under minimal conditions we have

$$\overrightarrow{g}(tx) = A(x)\overrightarrow{g}(t) + o(1)g_n(t),\ \forall x > 0, \quad (5)$$

where $A(x)$ is a matrix to be determined later.

Vectors satisfying a relation of the form (5) will be called **rate vectors**. A precise definition folows later.

In the next section we discuss some properties of rate vectors. Later we discuss regular variation of arbitrary order.

2 Rate vectors

2.1 Definition and first properties

We first precisely define the type of rate vectors that we are going to study.

Definition 1 *A vector* $\overrightarrow{g}(t) = (g_1(t), ..., g_n(t))^T$ *of measurable functions* g_i *is called a rate vector if*

(i) $g_i(t)$ *is ultimately of constant sign and* $|g_i|$, $1/|g_i|$ *are bounded on finite intervals far enough to the right;*

(ii) If $n \geq 2$, $g_{i+1}(t) = o(1)g_i(t)$, $1 \leq i \leq n-1$;

(iii) $\forall x > 0$ *there is a matrix* $A(x) \in \Re^{nxn}$ *such that*

$$\overrightarrow{g}(tx) = A(x)\overrightarrow{g}(t) + o(1)g_n(t) \tag{6}$$

In our first result we formulate some elementary properties of rate vectors.

Proposition 4 *Assume that* $\overrightarrow{g}(t)$ *is a rate vector. Then*

(i) The matrix $A(x)$ *is upper triangular and uniquely defined;*

(ii) If $A_{k,k}(x) \neq 0$ *for* $k \geq 2$*, then* $A_{i,i}(x) \neq 0$ *for all* $i = 1, 2, ..., k-1$.

(iii) $A(x)$ *is invertible if and only if* $A_{n,n}(x) \neq 0$.

(iv) $A(xy) = A(x)A(y)$

As an example, take $\overrightarrow{g}(t) = (t\,\mathsf{log}(t), t, 1)^t$.

We have

$$
\begin{aligned}
g_1(tx) &= tx\,\mathsf{log}(tx) = xg_1(t) + x\,\mathsf{log}(x)g_2(t) \\
g_2(tx) &= tx = xg_2(t) \\
g_3(tx) &= 1 = g_3(t)
\end{aligned}
$$

It follows that $\overrightarrow{g}(tx) = A(x)g(t)$ where

$$
A(x) = \begin{bmatrix} x & x\,\mathsf{log}(x) & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

In general, since $A(x)$ is upper triangular, for $i = 1, ..., n$, we have

$$g_i(tx)/g_i(t) \rightarrow A_{i,i}(x),\ \forall x > 0.$$

A relation of this type is well known in the theory of regular variation. We find that

$$A_{i,i}(x) = x^{\alpha(i)},\ 1 \leq i \leq n, \tag{7}$$

Hence the matrix $A(x)$ is invertible.

Note that $g_{i+1}(t) = o(1)g_i(t)$ implies that

$$\alpha(1) \geq \alpha(2) \geq ... \geq \alpha(n)$$

2.2 Regularly varying rate vectors

To formulate the next result, for $x > 0$ and $B \in \Re^{nxn}$, we put

$$\exp(B) = \sum_{k=0}^{\infty} \frac{1}{k!} B^k \text{ and } x^B = \exp(B \log(x)).$$

We already showed that $A(xy) = A(x)A(y)$. In the next proposition, we solve this functional equaltion.

Proposition 5 *Suppose that (6) and (7) hold.*

(i) Then $A(x)$ is of the form $A(x) = x^B$ for some upper triangular matrix B.

(ii) The diagonal elements of B are nonincreasing.

Definition 2 *(index matrix) The matrix B is called an index matrix, i.e. B is a square upper triangular real matrix with nonincreasing diagonal elements.*

Definition 3 *A rate vector $\overrightarrow{g}$ of dimension n is regularly varying with index matrix $B \in \Re^{nxn}$ if for all $x > 0$, $\overrightarrow{g}$ satisfies $\overrightarrow{g}(tx) = A(x)\overrightarrow{g}(t) + o(1)g_n(t)$, where $A(x) = x^B$. Notation $\overrightarrow{g} \in R_B$.*

Examples

(1) For $n = 1$, consider

$$g(t) = \exp(-\exp(\lfloor \log(t) \rfloor)),\ t \geq 1.$$

Then $g(tx)/g(t) \to 0$, for all $x \geq e$ and $g(xt)/g(t)$ does not converge for $0 < x < e$.

Hence, $g(t)$ is not a regularly varying rate function.

(2) Let $n = 2$ and consider

$$\overrightarrow{g}(t) = ((\log(t))^{\alpha}, 1)^{t}, 0 < \alpha < 1$$

We have $\overrightarrow{g}(tx) = A(x)\overrightarrow{g}(t) + o(1)$ where $A(x) = I$, the identity matrix. It follows that $\overrightarrow{g} \in R_B$ where the index matrix is a 2×2 zero matrix.

(3) For $\overrightarrow{g}(t) = (t\log(t), t, 1)^t$

we have $\overrightarrow{g}(tx) = A(x)g(t)$ where

$$A(x) = \begin{bmatrix} x & x\log(x) & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(4) For

$$\overrightarrow{g}(t) = (\log(t), (\log(t))^{1/2}, 1)^t$$

, we have $\overrightarrow{g} \in R_B$ with

$$A(x) = \begin{bmatrix} 1 & 0 & \log(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Remarks

a) Suppose that D is an invertible upper diagonal matrix, and let $\widetilde{g}(t) = D\overrightarrow{g}(t)$, where $\overrightarrow{g} \in R_B$.

In this case, we have $\widetilde{g} \in R_{\widetilde{B}}$, where $\widetilde{B} = DBD^{-1}$.

In some cases we can find D so that $\widetilde{B}$ is a diagonal matrix or a matrix in Jordan form.

b) It makes sense to study asymptotic expansions of the form

$$F(t) = \overrightarrow{a}\ \overrightarrow{g}(t) + o(1)g_n(t),$$

where $\overrightarrow{g} \in R_B$. In this case we have

$$F(xt) = \overrightarrow{a}x^B\overrightarrow{g}(t) + o(1)g_n(t)$$

2.3 The matrix A(x)

We found that $A(x) = x^B$ where $A(x)$ and B are upper diagonal matrices and where $A_{i,i}(x) = x^{\alpha(i)}$, $i = 1, 2, ..., n$.

In this section we determine the **precise** form of the other entries of $A(x)$.

Recall that $A(xy) = A(x)A(y)$ and that $A(1) = I$. We obtain that

$$\frac{A(xy) - A(x)}{y - 1} = \frac{A(y) - I}{y - 1} A(x)$$

Letting $y \to 1$, we obtain that

$$xA'(x) = BA(x)$$

This system of differential equations can be solved explicitly.

We find that

$$\begin{aligned} & (A_{i,i+1}(x), ..., A_{i,n}(x)) \\ = \ & x^{\alpha(i)} \overrightarrow{c(i)} \int_1^x \widetilde{A}_{i+1,n}(y) y^{-\alpha(i)-1} dy, \end{aligned}$$

where $\overrightarrow{c(i)} = (B_{i,i+1}, ..., B_{i,n})$ and $\widetilde{A}_{k,l} = (A_{i,j})_{i,j=k}^{l}$

Starting with $A_{i,i}(x) = x^{\alpha(i)}$, we can find recursively all entries of $A(x)$.

Remark.

Suppose that the matrix D is given by

$$D = \left(\begin{bmatrix} 0 & \overrightarrow{c} \\ \mathbf{0} & C \end{bmatrix}\right)$$

where C is a matrix.

In this case x^D is given by

$$x^D = \left(\begin{bmatrix} \mathbf{1} & \overrightarrow{u}(x) \\ \mathbf{0} & x^C \end{bmatrix}\right)$$

where by the prevous result, $\overrightarrow{u}(x)$ is given by

$$\overrightarrow{u}(x) = \overrightarrow{c} \int_1^x y^C y^{-1} dy.$$

2.4 Representation theorem & Potter bounds

In our next result we show that in (6), the convergence holds locally uniformly in $x > 0$.

Theorem 6 *(Uniform convergence theorem) If* $\overrightarrow{g} \in R_B$*, then* $\overrightarrow{g}(tx) = x^B g(t) + o(1) g_n(t)$ *holds locally uniformly in* $x > 0$*.*

This result leads to a representation theorem for $\overrightarrow{g} \in R_B$.

Theorem 7 *(Representation Theorem) We have* $\overrightarrow{g} \in R_B$ *if and only if there exist* $a > 0$*,* $\overrightarrow{v} \in \Re^n$ *and measurable functions* $\overrightarrow{\eta}$*,* $\overrightarrow{\varphi} : [t_0, \infty) \to \Re^{n\times 1}$ *such that* $|\eta_i(t)| + |\phi_i(t)| = o(1)\, |g_n(t)|$ *and such that*

$$\overrightarrow{g}(t) = t^B \overrightarrow{v} + \overrightarrow{\eta}(t) + t^B \int_a^t u^{-B} \overrightarrow{\phi}(u) \frac{1}{u} du, \text{ for } t \geq a. \tag{8}$$

Remark

Given B and $g_n(t)$, we can use the previous representation to construct examples $\overrightarrow{g} \in R_B$.

Now we provide a Potter type of upperbound for

$$\overrightarrow{g}(tx) - x^B \overrightarrow{g}(t)$$

Recall that for any square matrix C we have

$$\lim_{m\to\infty} \|C^m\|^{1/m} = \max\{|\lambda|, \lambda \text{ is an eigenvalue of } C\}.$$

Now take $C = \exp(B)$ and observe that

$$0 < e^{\alpha(n)} \le e^{\alpha(n-1)} \le ... \le e^{\alpha(1)}$$

We find that

$$\lim_{m\to\infty} \|\exp(mB)\|^{1/m} = \exp(\alpha(1))$$

Taking $C = \exp(-B)$, we find that

$$\lim_{m\to\infty} \|\exp(-mB)\|^{1/m} = \exp(-\alpha(n))$$

Theorem 8 *(Potter's Theorem for R_B). Let $g \in R_B$. For all $\delta > 0$ and $\epsilon > 0$ there exists $t_0 > 0$ such that*

$$
\begin{aligned}
& \left\| \overrightarrow{g}(tx) - x^B \overrightarrow{g}(t) \right\| \\
\leq \; & \delta x^{\alpha(1)+\epsilon} \left| g_n(t) \right|, \; \forall x \geq 1, t \geq t_0, \\
\leq \; & \delta x^{\alpha(n)-\varepsilon} \left| g_n(t) \right|, \; \forall t \geq t_0, t_0 \leq x \leq 1.
\end{aligned}
$$

These bounds can be used (will be used) to obtain new results in the contextof integral transforms and Abelian theorems for regularly varying rate vectors.

3 GRV of order n

Let $\overrightarrow{g} \in R_B$. Then f is regularly varying of order n if we have

$$f(tx) = f(t) + \overrightarrow{h}(x)\overrightarrow{g}(t) + o(1)g_n(t), \; \forall x > 0.$$

Now we find that $\overrightarrow{h}(x)$ is given by

$$\overrightarrow{h}(x) = \overrightarrow{c} \int_1^x A(u)u^{-1}du$$

We use the notation $f \in GRV(\overrightarrow{g})$, with g-index $\overrightarrow{c}$.

Remark

Taking derivatives w.r.t. x, we formally obtain that

$$tDf(tx) = D\overrightarrow{h}(x)\overrightarrow{g}(t) + o(1)g_n(t).$$

Taking $x = 1$,we find a relation of the form

$$tu(t) = \overrightarrow{a}\,\overrightarrow{g}(t) + o(1)g_n(t).$$

We can again formulate a representation theorem and Potters' bounds.

Theorem 9 *Let $\overrightarrow{g} \in R_B$. We have $f \in GRV(\overrightarrow{g})$ with g-index $\overrightarrow{c}$ if and only if there exists a constants $a > 0, v \in \Re$, and functions η, ϕ such that*

$$|\eta(t)| + |\phi(t)| = o(1)\,|g_n(t)|$$

and such that

$$f(t) = v + \eta(t) + \int_a^t \left\{\overrightarrow{c}\ \overrightarrow{g}(u) + \phi(u)\right\} u^{-1} du,\ t \geq a.$$

Theorem 10 *(Potter) Let $\overrightarrow{g} \in R_B$ and $f \in GRV(\overrightarrow{g})$. Then for all $\delta > 0$ and $\varepsilon > 0$ there exists $t_0 > 0$ such that*

$$\begin{aligned} & \left| f(tx) - f(t) - \overrightarrow{h}(x)\overrightarrow{g}(t) \right| \\ \leq\ & \delta x^{\max(\alpha(1)+\epsilon,0)}\,|g_n(t)|\,,\ \textit{for all } x \geq 1, t \geq t_0, \\ \leq\ & \delta x^{\min(\alpha(n)-\varepsilon,0)}\,|g_n(t)|\,,\ \textit{for all } t \geq t_0, t_0 \leq x \leq 1. \end{aligned}$$

These bounds can be used to study integral transforms of regularly varying functions of order n.

3.1 Examples

- The function $f(x) = \log\lfloor x \rfloor$ is in the class Π, but is not RV of any higher order.

- The function $f(x) = \log(x)$ satisfies $f(tx) - f(t) = \log(x)$

- The function $f(x) = \log(\log(x))$ satisfies

$$
\begin{aligned}
& f(tx) - f(t) \\
= \; & \sum_{0}^{\infty} \frac{(-1)^n}{n+1} \times \frac{(\log(x))^{n+1}}{(\log(t))^{n+1}}
\end{aligned}
$$

and $f \in GRV$ with $g_i(t) = (\log(t))^{-i-1}$.

- $f(x) = \log\Gamma(x)$ is in the class $GRV(\overrightarrow{g})$ with

$$
\begin{aligned}
g_1(x) &= x\log(x) \\
g_2(x) &= x \\
g_3(x) &= 1 \\
g_{3+m}(x) &= x^{1-2m},\; m \geq 1
\end{aligned}
$$

- The tail of many distribution functions can be expanded as a series

$$f(x) = \sum_{0}^{\infty} a(k) x^{-k\alpha}$$

where $\alpha > 0$. For $x > 0$ we have

$$f(xt) - f(t) = \sum_{k=1}^{n} a(k)(x^{-k\alpha} - 1)t^{-k\alpha} + o(1)t^{-n\alpha}.$$

We find

$$\overrightarrow{g}(t) = (t^{-\alpha}, t^{-\alpha 2}, ..., t^{-\alpha n})^{t}$$

and

$$B = diag(-\alpha, -2\alpha, ..., -n\alpha)$$

4 Special cases

We can consider several special cases

4.1 All indices $\alpha(i)$ different from zero

In this case B is invertible and we obtain a very simple representation for f:

$$f(x) = c + dB^{-1}\overrightarrow{g}(x) + o(1)g_n(x).$$

4.2 All indices $\alpha(i)$ different

In this case, we find that f has the following representation:

$$\begin{aligned} f(x) &= a_0 + \sum_{i=1}^{n-1} a_i \int_a^x u^{\alpha(i)-1} du \\ &\quad + \int_a^x (a_n + o(1)) g_n(u) u^{-1} du + o(1) g_n(x). \end{aligned}$$

Another special case is the class Π of order n.

5 The class Π of order n

5.1 Definition

If $\overrightarrow{g} \in R_B$ and all diagonal elements of B are 0, we write $\overrightarrow{g} \in \Pi_B$ and $f \in \Pi(\overrightarrow{g})$.

If $n = 1$, this is the class of de Haan (1970)

If $n = 2$, this class has been studied by Omey and Willekens (1988).

In the next result we show an alternative representation for functions in the class $\Pi(\overrightarrow{g})$. Let $C(t)$ be defined as follows:

$$C(t) = f(t) - \frac{1}{t}\int_a^t f(u)du.$$

This relation can be inverted, and we find

$$f(x) = C(x) + \int_a^x C(u)u^{-1}du.$$

Theorem 11 *Let* $\overrightarrow{g} \in \Pi$. *Then* $f \in \Pi(\overrightarrow{g})$ *with* g-*index* $\overrightarrow{c}$ *if and only if*

$$C(t) = \overrightarrow{d}\ \overrightarrow{g}(t) + o(1)g_n(t),$$

where $\overrightarrow{d} = \overrightarrow{c}(I+B)^{-1}$. *Moreover,* $C \in \Pi(g_2, ..., g_n)$ *of a lower order.*

This theorem gives us a way to construct examples in the class Π.

We define $C_0, C_1, ..., C_n$ as follows: take $g_n \in RV(0)$ and set

$$\begin{aligned} C_n(x) &= g_n(x) + \int_a^x g_n(u)u^{-1}du \\ C_{i-1}(x) &= C_i(x) + \int_a^x C_i(u)u^{-1}du. \end{aligned}$$

In this case, each C_i is in the class $\Pi((C_{i+1}, ...C_n)^t)$.

5.2 Another construction

Another useful way to construct 'nice' functions in the class Π is as follows.

Let D denote the differential operator. We define functions $L_0, L_1, ..., L_n$ as follows:

$$\begin{aligned} L_0(x) &= f(x) \\ L_{i+1}(x) &= xDL_i(x),\ i = 0, ..., n-1. \end{aligned}$$

We have the following result

Theorem 12 *Suppose that L_n is eventually of constant sign and $|L_n| \in RV(0)$. Then we have*

$$f(tx) - f(t) = \sum_{k=1}^{n} \frac{(\log(x))^k}{k!} L_k(t) + o(1)L_n(t)$$

and $f \in \Pi(\overrightarrow{L})$ with L-index $\overrightarrow{c} = (1, 0, ..., 0)$.

5.3 Inverse of Γ-varying functions

Recall that a function $A(x)$ is in the gamma class $\Gamma(a)$ if we have

$$\frac{A(x + ya(x))}{A(x)} \to \exp(y),\ \forall y \in \Re.$$

In many cases we can choose $a(x)$ as

$$q(x) \equiv q_1(x) = \frac{A(x)}{DA(x)}.$$

If $A(x)$ is an increasing function, we denote by $f(x)$ the inverse function of $A(x)$.

It is well known that $A \in \Gamma(a)$ holds if and only if $f \in \Pi$.

In the next result we provide assumptions on $q_1(x)$ such that f is in the class Π of order n.

Theorem 13 *Suppose $A(x) > 0$ and increasing with inverse function $f(x)$. Let $q_1(x) = A(x)/DA(x)$ and $q_i(x) = q(x)Dq_{i-1}(x)$, $i \geq 2$. Assume that $Dq(x) = o(1)$ and that $\left|D^{n-1}q\right| \in RV(\alpha - n + 1)$ for some $\alpha \leq 1$ where α is not of the form $(m-1)/m$. Then $f \in \Pi$ of order n.*

6 Applications

6.1 Some Quantile functions

In many situations one is interested in quantile functions.

As an example, we take the normal distribution, and more generally, we consider a generalized complementary error function: for $b \in \Re$ and $p > 0$, $p \neq 1$, define $B(x)$ as

$$B(x) = \int_x^\infty y^{b+p-1} \exp(-y^p/p)dy.$$

For $p = 2$ and $b = -1$ this corresponds to the tail of a normal distribution. Let $f(t)$ be defined by the equation

$$B(f(t)) = \frac{1}{t}.$$

Then $f(t)$ corresponds to the quantile function. It is not hard to see that $f(t) \sim (p \log(t))^{1/p} \in \Pi$.

Using the result of the previous section, one can prove that $f \in \Pi$ of any order n.

More precisely, we have

$$\begin{aligned} q(t) &\sim t^{1-p} \\ D^n q(t) &\sim C(p,n) t^{1-p-n}. \end{aligned}$$

Related to the gamma distribution, we define $B(x)$ and $f(x)$ as

$$B(x) = \int_x^\infty y^b \, \mathsf{exp}(-y) dy$$

and

$$B(f(t)) = \frac{1}{t}.$$

The result of the previous section can again be used to show that $f \in \Pi$ of any order n

6.2 Edgeworth expansions in Extreme Value Theory

Let $X_1, X_2, ...$ denote i.i.d. r.v. with d.f. $F(x)$ and let

$$M_n = \max(X_1, X_2, ..., X_n)$$

denote the sequence of partial maxima.

Now suppose that there are constants $a(n) > 0$ and $b(n) \in \Re$ such that

$$\frac{M_n - b(n)}{a(n)} \stackrel{d}{\Longrightarrow} Y.$$

It is well known that the d.f. of Y is of the following type $P(Y(\gamma) \leq x) = G_\gamma(x)$, where

$$G_\gamma(x) = \exp(-(1+\gamma x)^{-1/\gamma},\ \gamma \in \Re,\ 1 + \gamma x \geq 0.$$

Moreover we have

$$P(\frac{M_n - b(n)}{a(n)} \leq x) \to P(Y(\gamma) \leq x)$$

if and only if there exists a function $a(x)$ such that

$$\frac{V(tx) - V(t)}{a(t)} \to \frac{x^\gamma - 1}{\gamma},$$

where

$$V(x) = \sup \left\{ s : \frac{1}{-\log(F(s))} \leq x \right\}.$$

Several authors considered rates of convergence and discussed the difference

$$P(\frac{M_n - b(n)}{a(n)} \leq xt) - P(Y(\gamma) \leq x)$$

To this end, the basic starting assumption is that V is regularly varying of order 2, this is

$$V(tx) - V(t) = h_1(x)g_1(t) + h_2(x)g_2(t) + o(1)g_2(t).$$

where $(g_1(t), g_2(t))^t \in R_B$.

Some other authors use a Von Mises type of condition:

$$tDV(xt) = Dh_1(x)g_1(t) + Dh_2(x)g_2(t) + o(1)g_2(t)$$

Using one of these assumptions, the second order condition implies that

$$\begin{aligned} & \frac{1}{A(n)}\left(P(\frac{M_n - b(n)}{a(n)} \le xt) - P(Y(\gamma) \le x)\right) \\ \to \ & K(x) \end{aligned}$$

locally uniformly in x.

Moreover, de Haan and Ferreira (2006) showed that the second order condition is best possilbe.

In Wang and Cheng (2006), the authors use regular variation of order 3 to obtain estimates for the difference

$$\frac{1}{A(n)}\left(P(\frac{M_n - b(n)}{a(n)} \le xt) - P(Y(\gamma) \le x)\right) - K(x)$$

Using our results, the proofs of Wang and Cheng (2006) can be simplified considerably.

Other references here are

* Lin, Peng and Nadarajah, Bull. Korean Math. Soc. 45 (2008), N$^\circ$ 1, 75-93.

* de Haan and Resnick, Ann.Prob. 24 (1996), 97 - 124.

* de Haan and Ferreira, Extreme value Theory, an introduction, Springer 2006.

* Wang and Cheng, Acta Mathematica Sinica, English Series, Vol. 22 (1), 2006, 27 - 40.

6.3 The difference between the product and convolution product

Suppose that F is a distribution function on $\Re^+$. We say that F is subexponential (notation $F \in S$) if F satisfies

$$\frac{1 - F^{*2}(x)}{1 - F(x)} \to 2.$$

Looking at the rate of convergence here, we arrive at

$$\begin{aligned} R(x) &= 1 - F^{*2}(x) - 2(1 - F(x) \\ &= F^2(x) - F^{*2}(x) - (1 - F(x))^2 \end{aligned}$$

and we are interested in the first term.

More generally, we study the following difference:

$$R(x) = F(x)G(x) - F * G(x)$$

where $F(x) = P(X \leq x)$ and $G(x) = P(Y \leq x)$.

It is easy to show that $R(x) = I + II + III$, where

$$
\begin{aligned}
I(x) &= \int_0^{x/2} (F(x) - F(x-y))dG(y); \\
II(x) &= \int_0^{x/2} (G(x) - G(x-y))dF(y); \\
III(x) &= (G(x) - G(x/2))(F(x) - F(x/2)).
\end{aligned}
$$

In Omey (long ago) we used the following class of functions.:

$$
D(m, \alpha) = \{g : (g(x+y) - g(x))/m(x) \to \alpha y\}.
$$

This is the class Π written under additive form...

Under suitable conditions, we find that

$$
\begin{aligned}
\frac{I(x)}{m_F(x)} &= \int_0^{x/2} \frac{F(x)-F(x-y)}{m_F(x)} dG(y) \\
&\to \int_0^{\infty} \alpha_F y dG(y) \\
&= \alpha_F E(Y),
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
II(x)/m_G(x) &\to \alpha_G E(X), \\
\text{and } III(x) &= o(1) m_F(x)
\end{aligned}
$$

We want to improve this result and introduce the class $D(m, n, \alpha, \beta)$: g satisfies

$$\frac{g(x+y) - g(x) - \alpha m(x)}{n(x)} \to \beta y^2$$

This is Π of higher order written in additive form:

In our problem, we rewrite $I(x)$ as follows

$$\begin{aligned} & \frac{I(x) - \alpha_F m_F(x) E(Y)}{n_F(x)} \\ = \; & \int_0^{x/2} \frac{F(x) - F(x-y) - \alpha_F m_F(x) y}{n_F(x)} dG(y) \\ & -\alpha_F \frac{m_F(x)}{n_F(x)} \int_{x/2}^{\infty} y dG(y) \end{aligned}$$

and under suitable conditions, we get that

$$\frac{I(x) - \alpha_F m_F(x) E(Y)}{n_F(x)} \to \beta_F E(Y^2)$$

and similarly for $II(x)$.

7 The future

- Other special cases?? (for example: part of the $\alpha(i)$ are 0 and the others are $\neq 0$)

- the class Γ of order n (under construction)

- Abelian - Tauberian - Mercerian theorems for rate vectors and for GRV of order n

- If a distribution function F is in the class $F \in GRV(\overrightarrow{g}, \overrightarrow{c})$, how to estimate the indices $\alpha(i)$ and the other entries of B?

- ???

8 The end

8.1 Tyou Hank

8.2 Hyou K'ant

8.3 Thank you

8.4 Questions?