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The arithmetic average of geometric (fractional) Brownian motion admits 1-dimensional ~~marginal~~ martingale marginals.

1. Our original motivation: the P. Carr et al [ ] result

(1.1) In [ ], the authors show that, for $(B_s, s \geq 0)$ a 1-dimensional Brownian motion, and any $\lambda \in \mathbb{R}$, the arithmetic average:

$$\mathcal{A}_t := \frac{1}{t}\int_0^t ds \exp\left(\lambda B_s - \frac{\lambda^2 s}{2}\right), \quad t \geq 0$$

of the geometric Brownian motion: $\left(\mathcal{E}_s^{(\lambda)} = \exp\left(\lambda B_s - \frac{\lambda^2 s}{2}\right), s \geq 0\right)$

is increasing in the convex order sense, that is: if $g : \mathbb{R}_+ \to \mathbb{R}$ is a convex function such that for every $t > 0$, $E[|g(\mathcal{A}_t)|] < \infty$,

then the function: $G(t) = E[g(\mathcal{A}_t)]$

is increasing.

(1.2) In the present paper, a very short proof of the above result is provided, as a consequence of the following

Theorem 1: Keeping the above notation, there exists a martingale $(M_t, t \geq 0)$ such that:

$$\text{for any } t \geq 0, \qquad \mathcal{A}_t \overset{(law)}{=} M_t \qquad (1)$$

Precisely, if $(W_{u,t} ; u \geq 0, t \geq 0)$ denotes the standard Brownian sheet, then one may take:

$$M_t = \int_0^1 du \exp\left(\lambda W_{u,t} - \frac{\lambda^2}{2} ut\right), \quad t \geq 0 \qquad (2)$$

which is a martingale with respect to the filtration: $\mathcal{W}_t \equiv \sigma\{W_{h,v} ; h \geq 0, v \leq t\}$.

Assuming the validity of Theorem 1 for one instant, the P. Carr et al [ ] result follows immediately, as a consequence of Jensen's inequality —

We now provide the

Proof of Theorem 1: a) Note that:

$$\mathcal{A}_t = \int_0^1 du \exp\left(\lambda B_{ut} - \frac{\lambda^2}{2} u t\right)$$

for all $t \geq 0$. Now, fix $t > 0$; then: $(B_{ut}, u \geq 0) \overset{(law)}{=} (W_{u,t};\, u \geq 0)$

$$\left(\overset{(law)}{=} (\sqrt{t}\, B_u, u \geq 0) \right)$$

Thus, for $t > 0$, fixed, one has:

$$\mathcal{A}_t \overset{(law)}{=} \int_0^1 du \exp\left(\lambda W_{u,t} - \frac{\lambda^2 u t}{2}\right) \equiv M_t .$$

b) Now, $(M_t, t \geq 0)$ is a $(\mathcal{W}_t)$ martingale, since, for any $u \geq 0$,

$\exp\left(\lambda W_{u,t} - \frac{\lambda^2 u t}{2}\right)$ is a $(\mathcal{W}_t)$ martingale, as a consequence of the independence of $(W_{u,t+h} - W_{u,t})$ from $\mathcal{W}_t$. $\square$

(1.3) The simplicity of the proof of Theorem 1 invites to look for a number of ~~ext~~ extensions, after inspecting carefully the properties of Brownian motion which made the above arguments work.

In this light, it seems reasonable to look for ~~the~~ centered Gaussian processes $(G_s, s \geq 0)$ such that the associated arithmetic average:

$$A_t^G \overset{def}{=} \frac{1}{t} \int_0^t ds \exp\left(\lambda G_s - \frac{\lambda^2}{2} E[G_s^2]\right), \quad t \geq 0,$$

admits ~~one~~ 1-dimensional martingale marginals.

As a particularly interesting example, we obtained the following

Theorem 2: Let $0 < H < 1$, and $(B_s^H, s \geq 0)$ denote the fractional Brownian motion with Hurst index H.

Then,
$$\mathcal{A}_t^{(H)} \overset{def}{=} \frac{1}{t} \int_0^t ds \exp\left(\lambda B_s^H - \frac{\lambda^2}{2} s^{2H}\right), \quad t \geq 0,$$

admits 1-dimensional martingale marginals.

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Section 2 of this paper is devoted to the proof of Theorem 2, while, in Section 3, we derive a number of consequences from Theorems 1 and 2.

2. A proof of Theorem 2.

(2.1) This subsection may be considered as a warm-up, after which the proof of Theorem 2 shall follow without difficulty ---

Rather than focussing directly upon the case study of $(\mathcal{A}_t^{(H)}, t \geq 0)$, we first consider $(\mathcal{A}_t^{G}, t \geq 0)$, for a centered gaussian process $(G_s, s \geq 0)$ such that:

i) for any $c > 0$, $(G_{sc}, s \geq 0) \overset{(law)}{=} (c^H G_s, s \geq 0)$

ii) $G_s = \int_0^\infty h(s,u)\, dB_u$, for a deterministic Borel function h

$h: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that, for each s: $\int_0^\infty h^2(s,u)\, du < \infty$.

Lemma: Assuming ii), then i) is satisfied as soon as:

$$(3) \quad h(sc, vc) = c^\beta h(s,v), \text{ where } \beta = H - 1/2.$$

Proof: By Brownian scaling, we get:

$$(G_{sc}, s \geq 0) \equiv \left(\int_0^\infty h(sc, vc)\, dB_{vc} \right.$$

$$\overset{(law)}{=} \sqrt{c} \int_0^\infty h(sc, vc)\, dB_v$$

$$\overset{(law)}{=} c^{\frac{1}{2}+\beta} \int_0^\infty h(s,v)\, dB_v$$

~~(To be continued).~~

Thus, taking $c = 1/s$ in (3), we obtain:

$$h\left(1, \frac{v}{s}\right) = \frac{1}{s^\beta} h(s,v),$$

so that the function h may be written as: $h(s,v) = s^\beta k\left(\frac{v}{s}\right)$,

with: $k(v) \equiv h(1,v)$.

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Consequently, the Gaussian process under consideration in this subsection (2.1) may be defined by:

$$G_s = s^{(H-\frac{1}{2})} \int_0^\infty k\left(\frac{v}{s}\right) dB_v ,$$

for a function $k \in L^2(dv)$.

We shall now show the analogue of Theorem 2 for the process:

$$\mathcal{A}_t^{(k)} \overset{\text{def}}{=} \frac{1}{t}\int_0^t ds \exp\left(\lambda s^\beta \int_0^\infty k\left(\frac{v}{s}\right) dB_v - \frac{\lambda^2}{2} s^{2\beta} \int_0^\infty k^2\left(\frac{v}{s}\right) dv\right)$$

$$(4) \qquad \equiv \frac{1}{t}\int_0^t ds \exp\left(\lambda s^{H-\frac{1}{2}} \int_0^\infty k\left(\frac{v}{s}\right) dB_v - \frac{\lambda^2}{2} s^{2H} K\right)$$

where: $K = \int_0^\infty k^2(w)\,dw$.

We may now prove the following

Theorem 3: Consider for $\lambda \in \mathbb{R}$, $H > 0$, and $k \in L^2(dw)$ the process $\mathcal{A}_t^{(k)}$, defined by (4). Then, there exists a martingale $(M_t, t \geq 0)$ such that, for any $t \geq 0$, one has:

$$\mathcal{A}_t^{(k)} \overset{(law)}{=} M_t \qquad (5).$$

Proof: From (4), we make the change of variables: $s = tu$

Thus:

$$\mathcal{A}_t^{(k)} = \int_0^1 du \exp\left(\lambda (tu)^{H-\frac{1}{2}} \int_0^\infty k\left(\frac{v}{tu}\right) dB_v - \frac{\lambda^2}{2}(tu)^{2H} K\right)$$

$$= \int_0^1 du \exp\left(\lambda (tu)^{H-1/2} \int_0^\infty k\left(\frac{w}{u}\right) d_w(B_{wt}) - \frac{\lambda^2}{2}(tu)^{2H} K\right).$$

Thus, for fixed t, we obtain:

$$\mathcal{A}_t^{(k)} \overset{(law)}{=} \int_0^1 du \, \exp\left(\lambda \, t^H u^\beta \int_0^\infty k\left(\frac{w}{u}\right) d_w(B_w) - \frac{\lambda^2}{2}(ut)^{2H} K\right)$$

Introducing now, similarly as to the argument in the proof of Theorem 1, the Wiener sheet, we get (again, for fixed t):

$$\mathcal{A}_t^{(k)} = \int_0^1 du \, \exp\left(\lambda u^\beta \int_0^\infty k\left(\frac{w}{u}\right) d_w(W_{w,\, t^{2H}}) - \frac{\lambda^2}{2}(ut)^{2H} K\right).$$

For simplicity, let us define: $T = t^{2H}$, and denote:

$$M_T^{(k)} = \int_0^1 du \, \exp\left(\lambda u^\beta \int_0^\infty k\left(\frac{w}{u}\right) d_w(W_{w,T}) - \frac{\lambda^2}{2}(u^{2H} K)\, T\right)$$

Then, the same argument as in the proof of Theorem 1 shows that $(M_T^{(k)}, T \geq 0)$ is a $(\mathcal{W}_T, T \geq 0)$ martingale.

Finally, we have obtained (5), with $M_t = M^{(k)}_{t^{2H}}$, $t \geq 0$. $\square$

(2.2) The proof of Theorem 2 is now easy —

Indeed, from the Mandelbrot-Van Ness & [ ] representation of the fractional Brownian motion $(B_s^H, s \geq 0)$, one has:

$$(B_s^H, s \geq 0) \overset{(law)}{=} \int_{-\infty}^{\infty} h(s,u)\, dB_u$$

for a certain function $h : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$

Precisely:

$$h(s,u) = \qquad (6)$$

Now, the arguments used in subsection 2.1 extend to the case where (G_s) is represented as: $\int_{-\infty}^{\infty} h(s,u)\, dB_u$, with $(B_u, u \in \mathbb{R})$ a "2-sided" Brownian motion. We then need only consider

$$\left(W_{(u,t)} \; ; \; u \in \mathbb{R}, \; t \geq 0 \right)$$

a two-sided Wiener sheet to extend the validity of Theorem 3 to this situation.

Theorem 2 has been proven.

3. Comments and Corollaries. (To be continued).