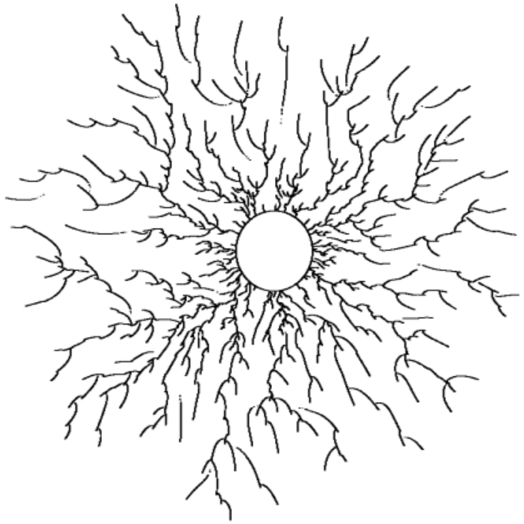
Planar aggregation and the coalescing Brownian flow



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(Joint work with James Norris, Cambridge)

Overview

- Planar random growth models
- Conformal mappings and the Hastings-Levitov model
- Stochastic flows and the Brownian web
- Scaling limit of the aggregation model

Planar random growth

Simplest examples are lattice-based models: Eden (Eden 1961), Diffusion-limited Aggregation (Witten and Sander 1981), Dielectric breakdown (Niemeyer ea 1984).

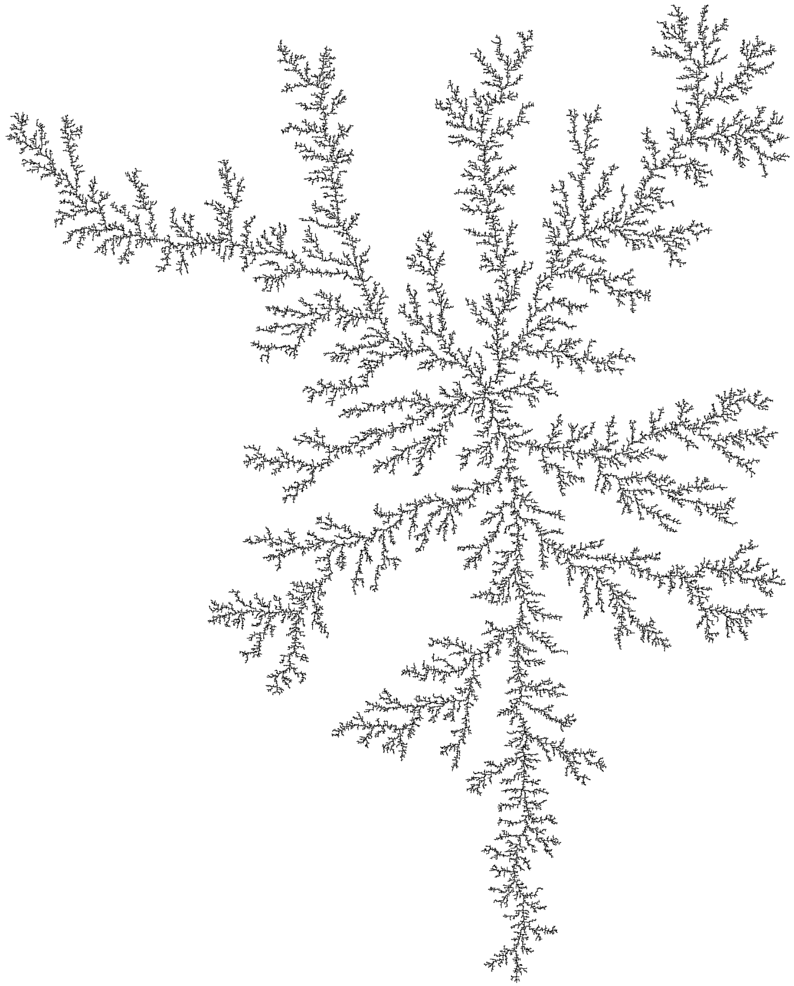
Primary interest is in the asymptotic behaviour of large clusters.

Computational investigations are highly sensitive to model variations.

Few notable mathematical results (Kesten 1987).

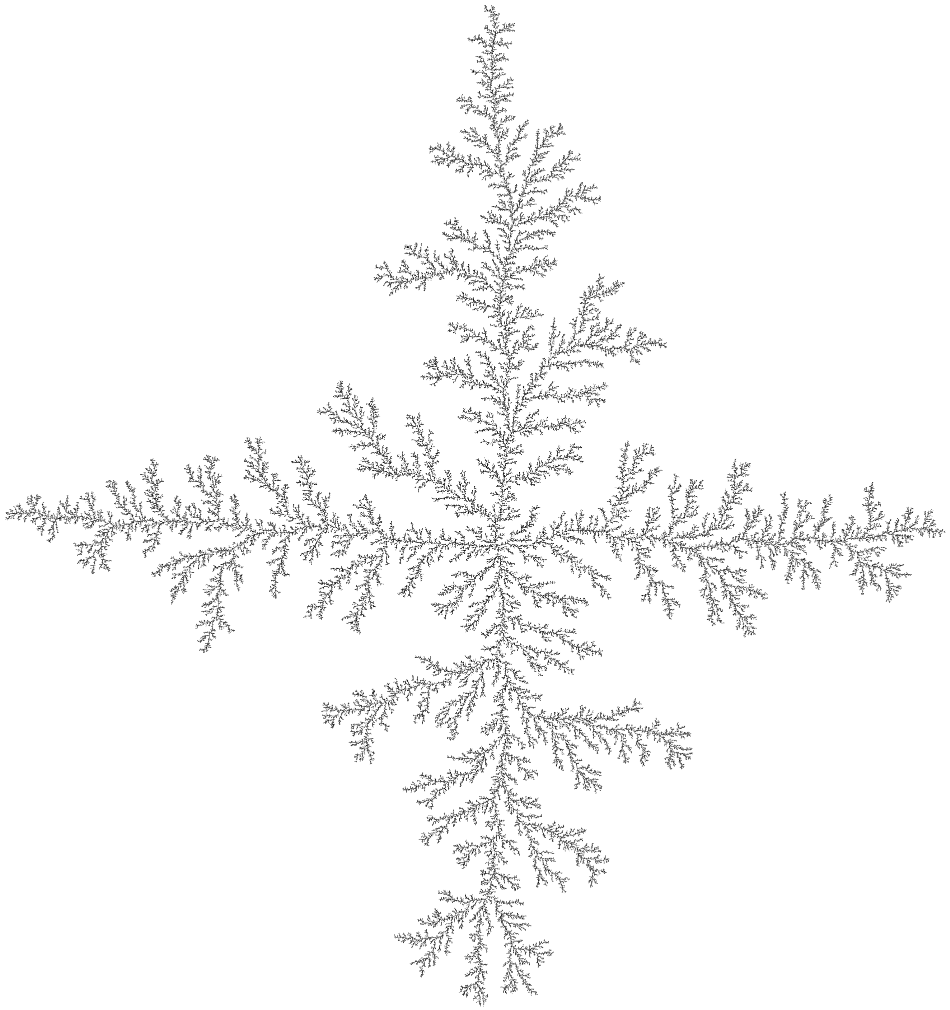
Not good models for continuum phenomena.

A DLA cluster of size 2000



Simulation due to Vincent Beffara http://www.umpa.ens-lyon.fr/~vbeffara.

A DLA cluster of size 4096



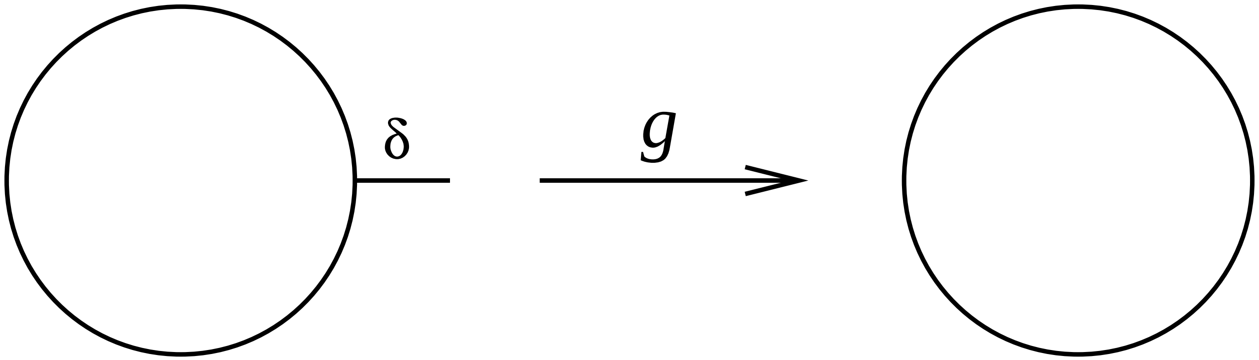
Simulation due to Vincent Beffara http://www.umpa.ens-lyon.fr/~vbeffara.

The Riemann mapping theorem

Theorem: Let K be a connected compact subset of $\mathbb{C}$ larger than a single point such that $\mathbb{C} \setminus K$ is connected.

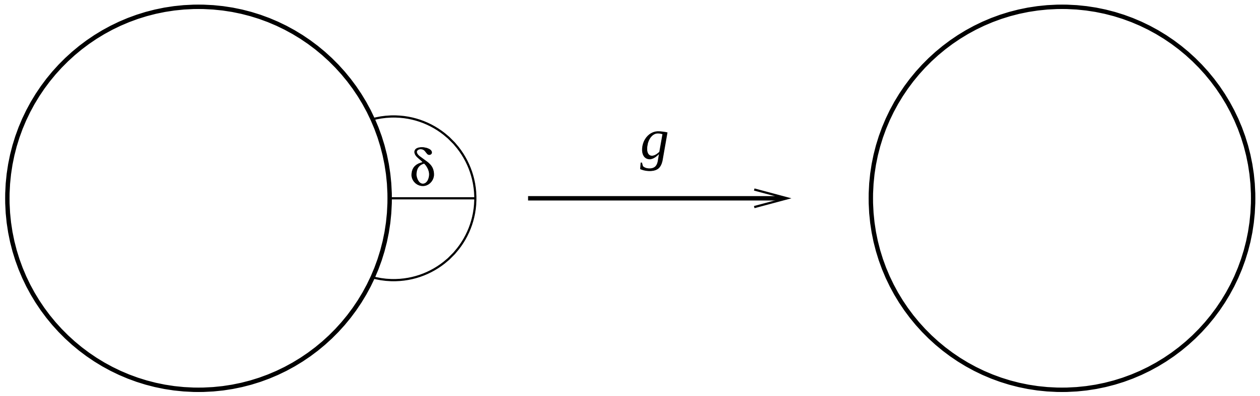
Then there exists a unique conformal transformation $g : \mathbb{C} \setminus K \longrightarrow \mathbb{C} \setminus K_0$ with $g(z) \sim e^{-\kappa} z$ for some $\kappa \in \mathbb{R}$ as $z \longrightarrow \infty$, where $K_0 = \overline{\mathbb{D}}$.

Example 1:



$$g(z) = \left(1 - \left(\frac{\delta}{\delta+2}\right)^2\right) \frac{z+1}{2z} \left(z + 1 + \sqrt{z^2 + 1 - 2z \frac{1 + \left(\frac{\delta}{\delta+2}\right)^2}{1 - \left(\frac{\delta}{\delta+2}\right)^2}} \right) - 1$$

Example 2:



$$g(z) = \frac{(e^{i\theta}z - 1)^a - (e^{-i\theta}z - 1)^a}{(z - e^{-i\theta})^a - (z - e^{i\theta})^a},$$

where

$$\theta = 2\tan^{-1}\delta\sqrt{1 - \frac{\delta^2}{4}} \quad \text{and} \quad a = \frac{\pi}{\pi - \cos^{-1}(\delta/2)}.$$

Hastings-Levitov planar aggregation

Set $D_0 = (\mathbb{C} \cup \{\infty\}) \setminus K_0$ and let Z_1 be a point on ∂K_0 chosen uniformly at random.

Let P_1 be a connected subset of D_1 of diameter δ_0 containing Z, such that $D_1 = (\mathbb{C} \cup \{\infty\}) \setminus (K_0 \cup P_1)$ is simply connected.

The set P_1 models an incoming particle, which is attached to K_0 at Z_1.

Let F_1 be the unique conformal map $D_0 \rightarrow D_1$.

The map F_1 corresponds to a particle attached at a random point Z_1 on the unit circle, of radius δ_0.

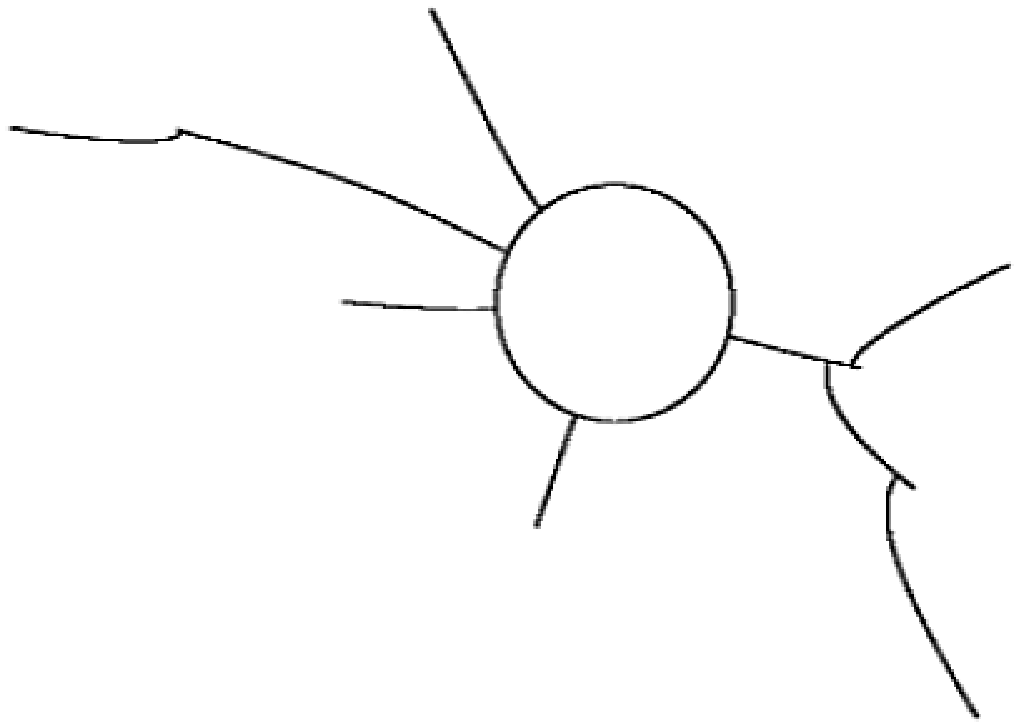
In 1998, Hastings and Levitov formulated a family of continuum growth models in terms of sequences of iterated conformal maps, indexed by a parameter $\alpha \in [0, 2]$ as follows.

Identify the cluster after the arrival of n particles with a conformal map $\Phi_n = F_1 \circ \cdots \circ F_n$ on D_0, where, conditional on Φ_n, the map F_{n+1} corresponds to a particle attached at a random point Z_{n+1} on the unit circle, but now of radius $\delta_{n+1} = \delta_0 |\Phi'_n(Z_{n+1})|^\alpha$.

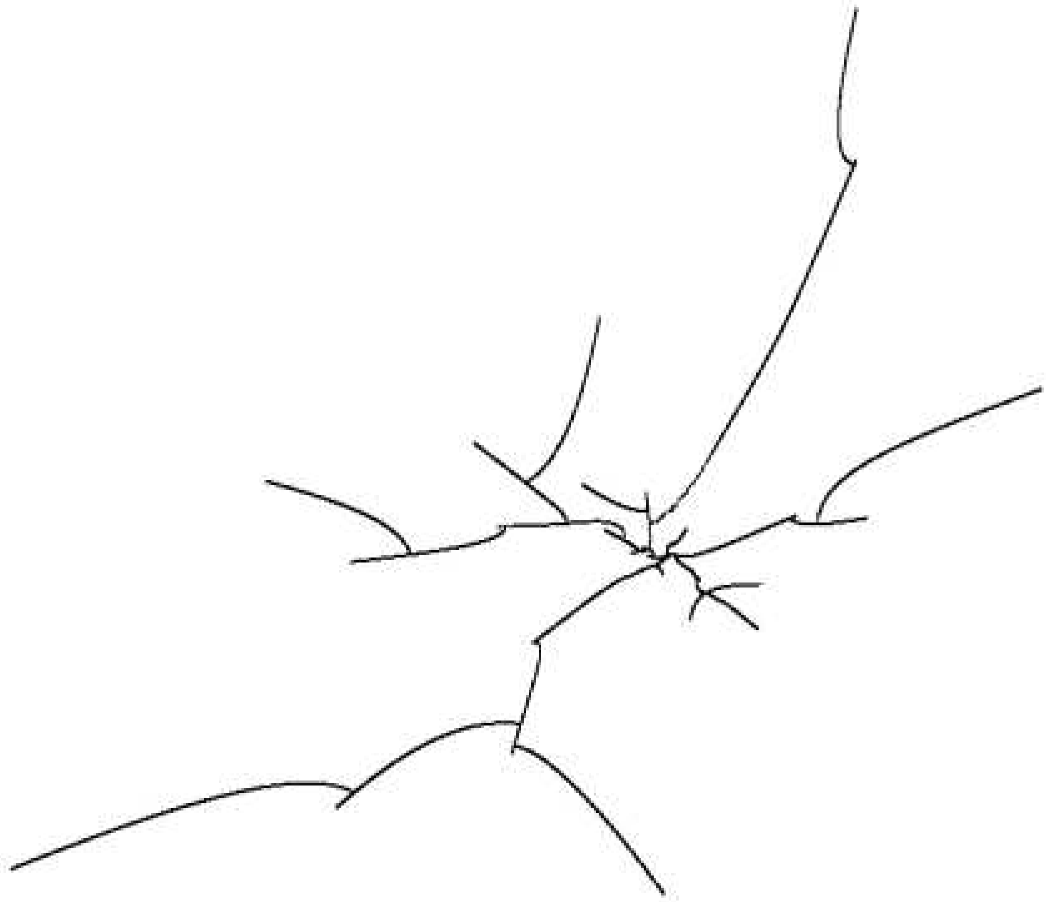
$\alpha \in [1, 2]$ correspond to dielectric-breakdown models, with $\alpha = 1$ giving the Eden model and $\alpha = 2$ giving DLA.

We shall establish a scaling limit for $HL(0)$, the $\alpha = 0$ case, where the maps F_n are iid.

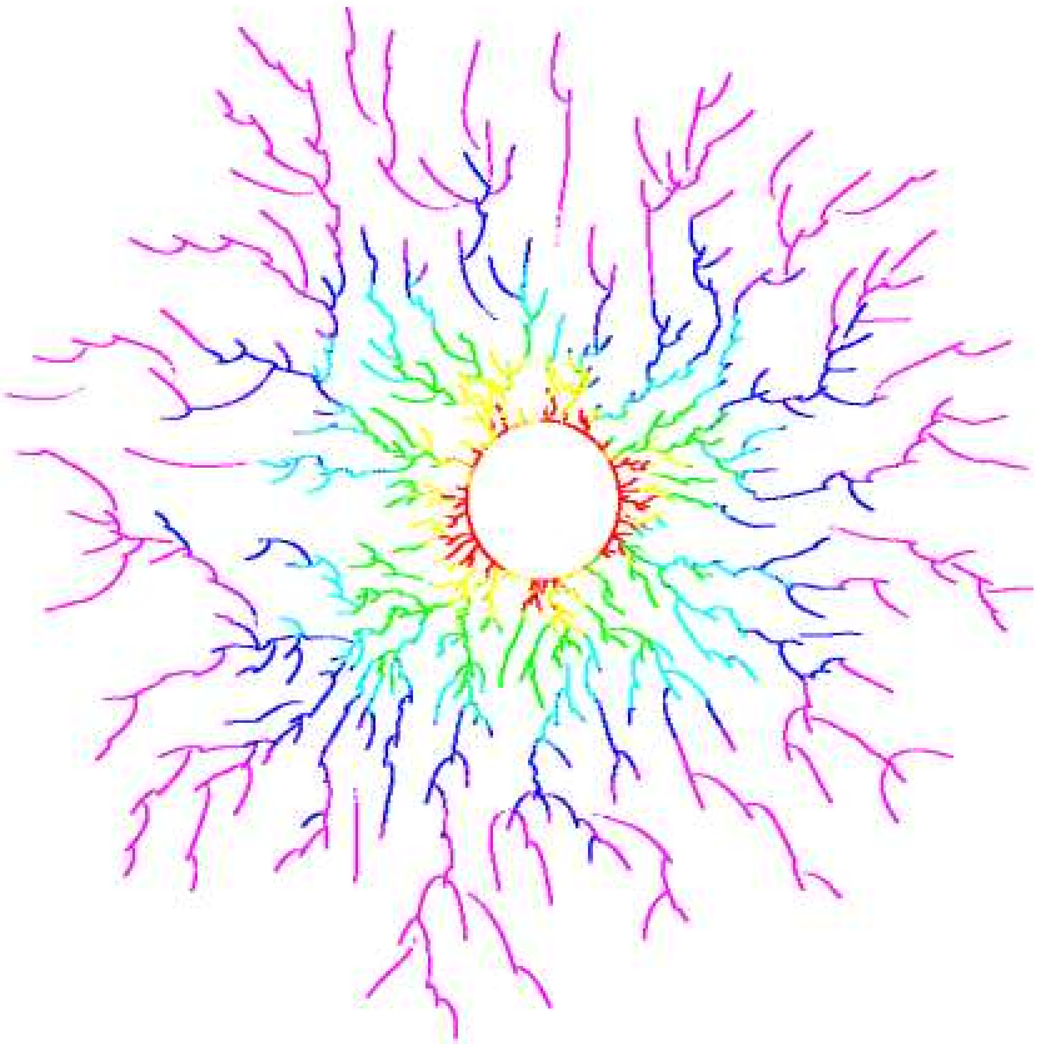
The slit model after a few arrivals with $\delta = 1$



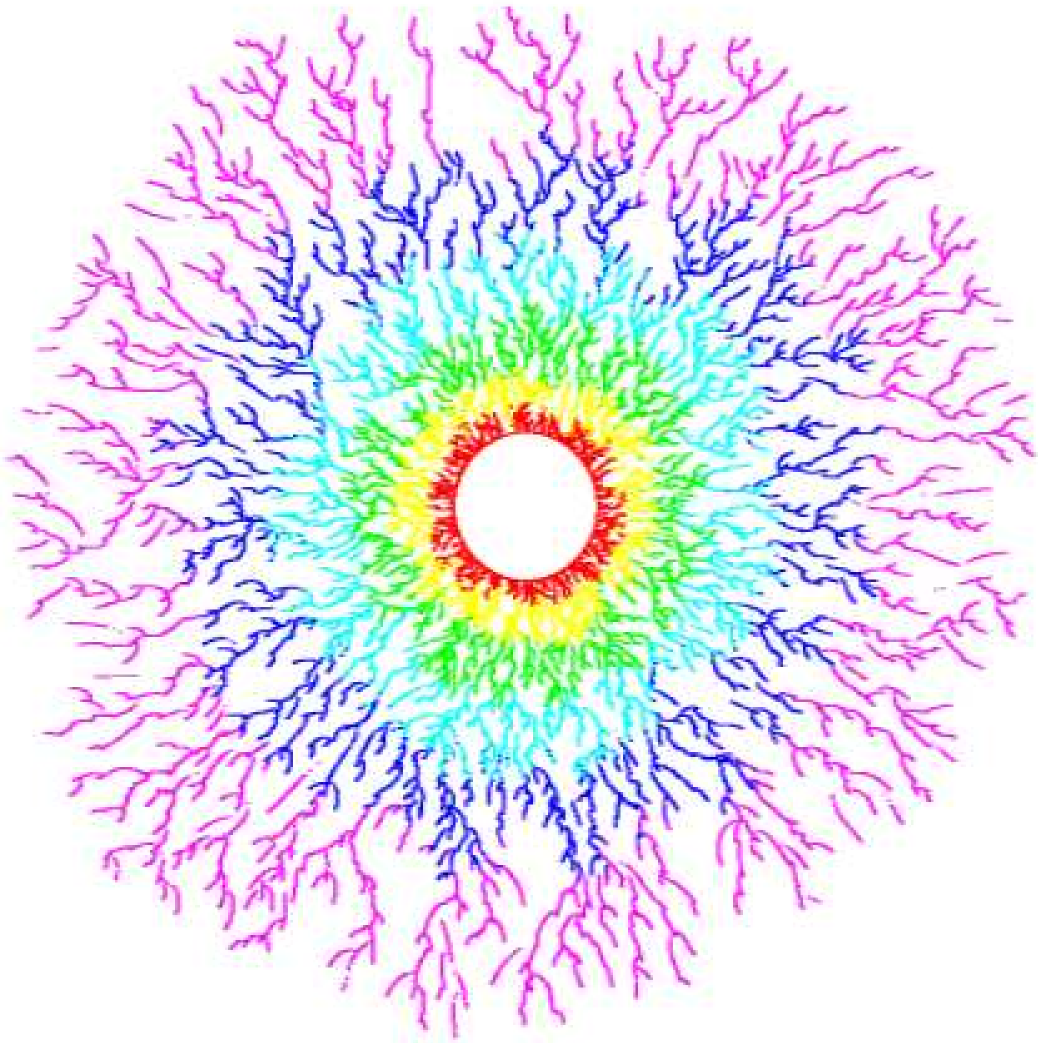
The slit model after a 100 arrivals with $\delta = 1$



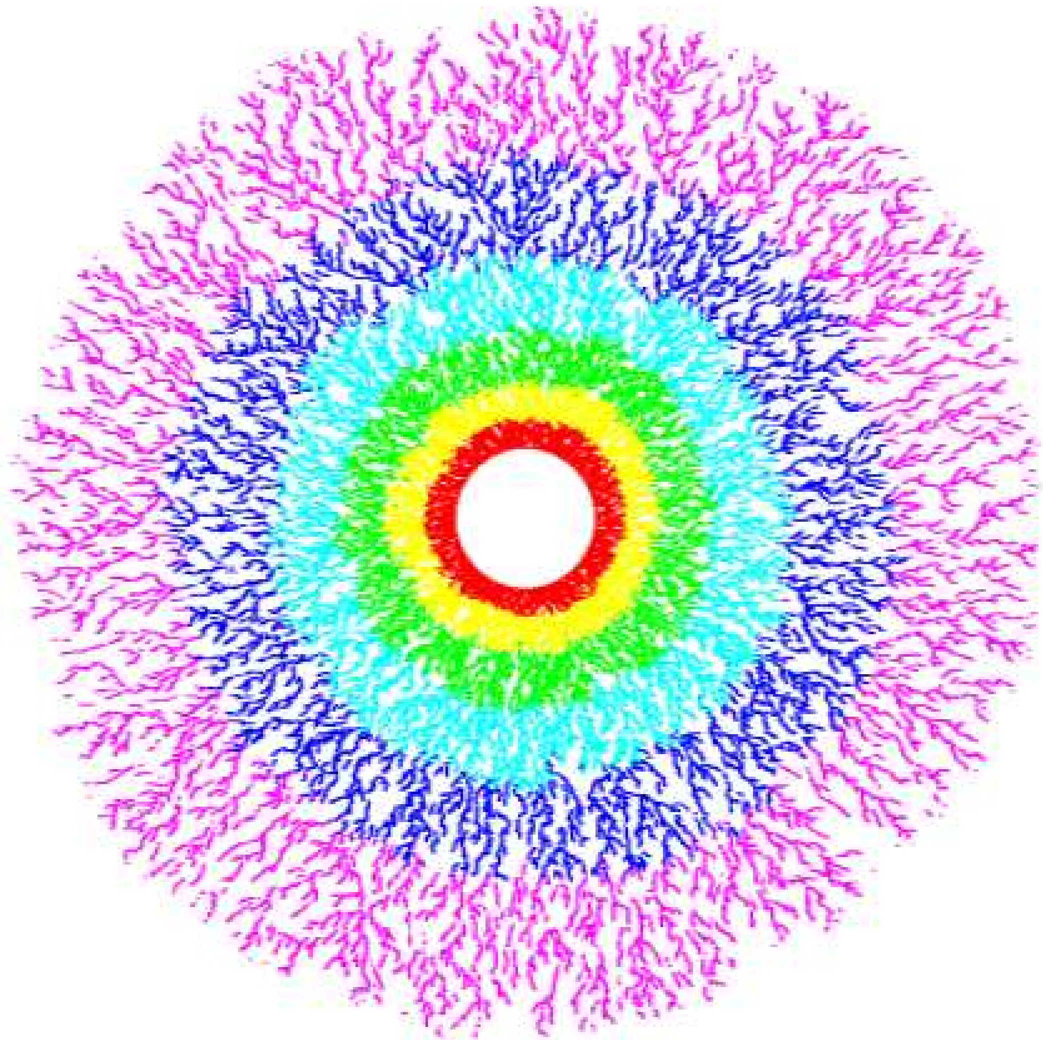
The slit model after 800 arrivals with $\delta = 1/10$



The slit model after 5000 arrivals with $\delta = 1/25$



The slit model after 20 000 arrivals with $\delta = 1/50$



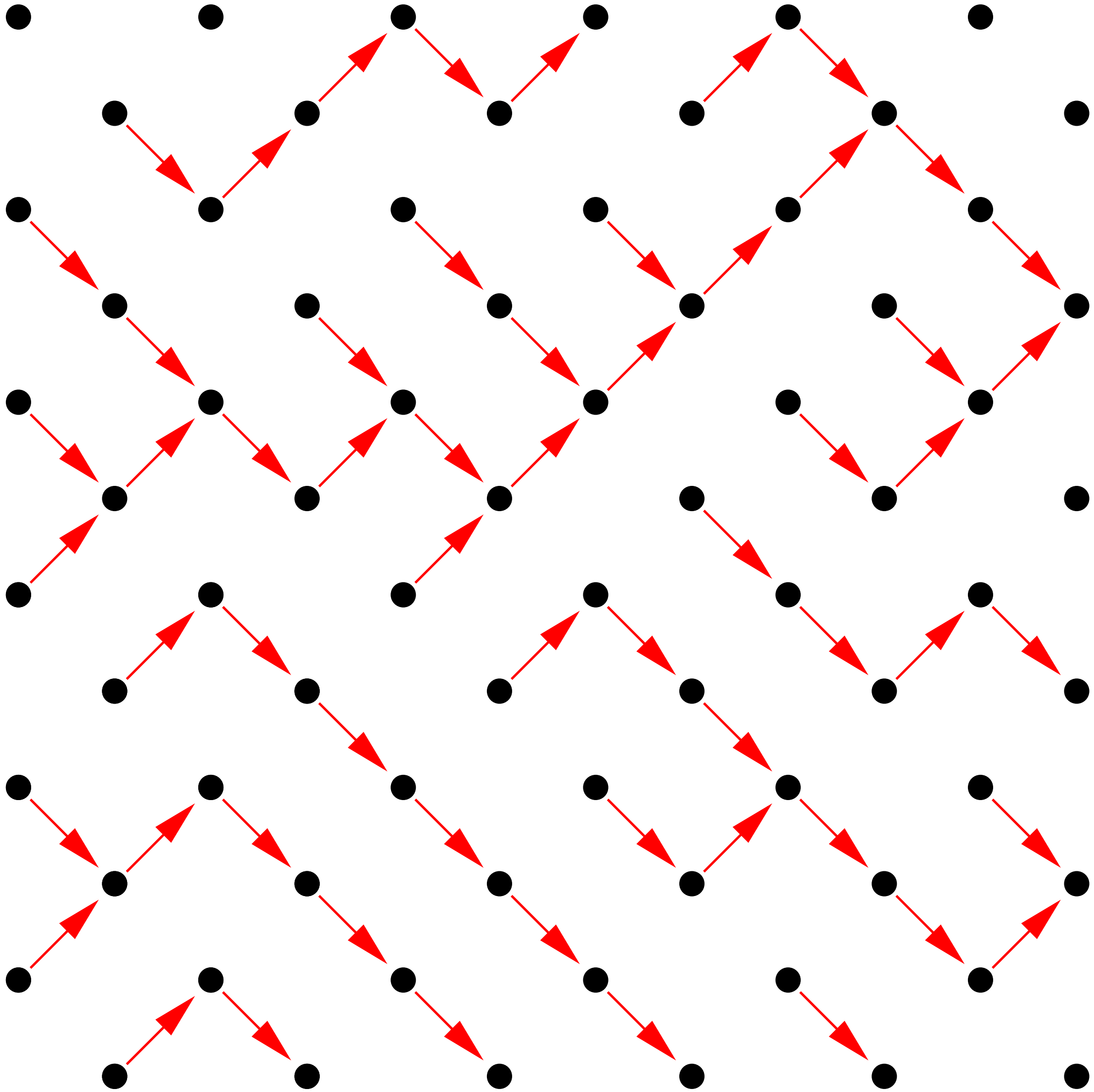
The Brownian web

The Brownian web can be loosely defined as a family of coalescing one-dimensional Brownian motions (with unit diffusion constant and zero drift) starting from all possible points in continuous space-time.

This object was originally formulated by Arratia in 1979 in his PhD thesis.

More recently been studied by Tóth and Werner (1998) and Fontes, Isopi, Newman and Ravishankar (2004).

Coalescing random walks



Arratia's result

For any sequence $E = ((s_k, x_k) : k \in \mathbb{N})$ in $\mathbb{R}^2$, define $C_E = \prod_{k=1}^{\infty} C_{x_k}([s_k, \infty), \mathbb{R})$.

Denote by $(Z_t^k)_{t \geq s_k}$ the kth coordinate process on C_E.

There exists a unique probability measure μ_E on C_E under which the processes $(Z_t^k)_{t \geq s_k}$ are coalescing Brownian motions.

A space for the Brownian web

Fontes et al characterise the Brownian web as a random element of the space of compact collections of $\mathbb{R}$-valued paths with specified starting points.

Disadvantage is there is more than one natural random variable in this space that satisfies the following two conditions.

(i) From any deterministic point (x, t) in space-time, there is almost surely a unique path $W_{x,t}$ starting from (x, t).

(ii) For any deterministic n and $(x_1, t_1), \ldots, (x_n, t_n)$, the joint distribution of $W_{x_1,t_1}, \ldots, W_{x_n,t_n}$ is that of coalescing Brownian motions (with unit diffusion constant).

We define an alternative space which has the advantage of there being a unique Brownian web.

Circle maps

A non-decreasing, right-continuous function $f : \mathbb{R} \to \mathbb{R}$ has the degree 1 property if

$$f^+(x+n) = f^+(x) + n, \quad x \in \mathbb{R}, \quad n \in \mathbb{Z}.$$

Denote the set of such functions by $\mathcal{R}$ and write $\mathcal{L}$ for the analogous set of left-continuous functions.

Each $f^+ \in \mathcal{R}$ has a left-continuous modification $f^- \in \mathcal{L}$, given by $f^-(x) = f(x-)$.

Write $\mathcal{D}$ for the set of all pairs $f = \{f^-, f^+\}$.

A canonical space for the coalescing Brownian flow

Consider $\phi = (\phi_{ts} : s, t \in \mathbb{R}, s < t)$, with $\phi_{ts} \in \mathcal{D}$ for all s, t.

Say that ϕ is a weak flow if

$$\phi^-_{ut} \circ \phi^-_{ts} \le \phi^-_{us} \le \phi^+_{us} \le \phi^+_{ut} \circ \phi^+_{ts}, \quad s < t < u.$$

Say that ϕ is continuous if, for all $t \in \mathbb{R}$, as $s \uparrow t$ and $u \downarrow t$,

$$\phi_{ts} \rightarrow id, \quad \text{and} \quad \phi_{ut} \rightarrow id.$$

Write $C^\circ(\mathbb{R}, \mathcal{D})$ for the set of all continuous weak flows.

Define D_E and $D^\circ(\mathbb{R}, \mathcal{D})$ similarly to C_E and $C^\circ(\mathbb{R}, \mathcal{D})$ but allowing for cadlag discontinuities in time.

Characterisation of the Brownian web

Define $Z^{E,+} : C^{\circ}(\mathbb{R}, \mathcal{D}) \longrightarrow C_E$ by

$$Z^{E,+}(\phi)^k = (\phi^{+}_{ts_k}(x_k))_{t \geq s_k}.$$

Theorem: There exists a unique Borel probability measure μ_W on $C^{\circ}(\mathbb{R}, \mathcal{D})$ such that, for any finite set $E \subset \mathbb{R}^2$, we have

$$\mu_W \circ (Z^{E,+})^{-1} = \mu_E.$$

A class of Lévy flows on the circle

For $f \in \mathcal{D}^*$, let $\tilde{f}(x) = f(x) - x$ and define constants $\rho = \rho(f) > 0$ and $\beta = \beta(f) \in \mathbb{R}$ by

$$\rho \int_0^1 \tilde{f}(x)^2 dx = 1, \quad \beta = \rho \int_0^1 \tilde{f}(x) dx.$$

Let $T_1, T_2, \ldots$ be times in a Poisson process with intensity ρ and $Z_1, Z_2, \ldots$ independent random variables, uniform on $(0, 1)$.

When $\beta = 0$ define $F_{T_i} = \{F^-_{T_i}, F^+_{T_i}\} \in \mathcal{D}^*$ by

$$F^{\pm}_{T_i}(x) = Z_i + f^{\pm}(x - Z_i).$$

For each interval $I \subset \mathbb{R}$, define $X \in D^\circ(\mathbb{R}, \mathcal{D})$ by setting

$$X^{\pm}_I = F^{\pm}_{T_n} \circ \ldots \circ F^{\pm}_{T_m},$$

where $T_m < \ldots < T_n$ are the times of the Poisson process that lie in I.

We take $X^{\pm}_I = id$ if there are no such times.

The case $\beta \neq 0$ has similar construction but compensated for drift.

Convergence to the Brownian web

Let $\lambda = \lambda(f)$ be the smallest constant in $(0,1]$ such that

$$\rho \int_0^1 |\tilde{f}(x+a)\tilde{f}(x)|dx \leq \lambda, \quad a \in [\lambda, 1-\lambda]$$

Measure of how well-localized a perturbation f is of the identity.

Let μ^f denote the distribution of X on $D^\circ(\mathbb{R}, \mathcal{D})$.

Theorem: The law $\mu^f \rightarrow \mu_W$ weakly on $D^\circ(\mathbb{R}, \mathcal{D})$, uniformly in $f \in \mathcal{D}^*$, as $\rho(f) \rightarrow \infty$ and $\lambda(f) \rightarrow 0$.

Scaling limit of the aggregation model

Returning to $HL(0)$, let K_0 denote the closed unit ball in $\mathbb{C}$ with centre at 0.

Set $D_0 = (\mathbb{C} \cup \{\infty\}) \setminus K_0$.

Let P be a closed, connected, simply connected subset of D_0 of diameter $\delta \in (0, 1]$ such that $P \cap K_0 = \{1\}$.

Set $D = (\mathbb{C} \cup \{\infty\}) \setminus (K_0 \cup P)$.

The set P models an incoming particle, attached to K_0 at 1.

Write G for the unique conformal isomorphism $D \rightarrow D_0$.

There exists a unique $g \in \mathcal{D}$ such that g restricts to a continuous map from the interval $(0, 1)$ to itself, and such that

$$G(e^{2\pi i x}) = e^{2\pi i g(x)}, \quad x \in (0, 1).$$

Define $\rho(P) = \rho(g)$ and $\beta(P) = \beta(g)$.

Note that, if P is symmetric in the real axis, then g is an odd function, so $\beta(P) = 0$.

Proposition: There is a universal constant $C < \infty$ such that $\delta^3/C \leq \rho(P) \leq C\delta^{-3}$ and $\lambda(P) < C\delta^{1/4}$.

Construct the flow X from map $g \in \mathcal{D}$ as above and set $\mu_P = \mu_g$.

Theorem: $\mu_P \to \mu_W$ weakly on $D^\circ(\mathbb{R}, \mathcal{D})$ as $\delta(P) \to 0$.

Evolution of harmonic measure on the boundary

Corollary: Let $x_1, \ldots, x_n$ be a positively oriented set of points in $\mathbb{R} \setminus \mathbb{Z}$ and set $x_0 = x_n$.

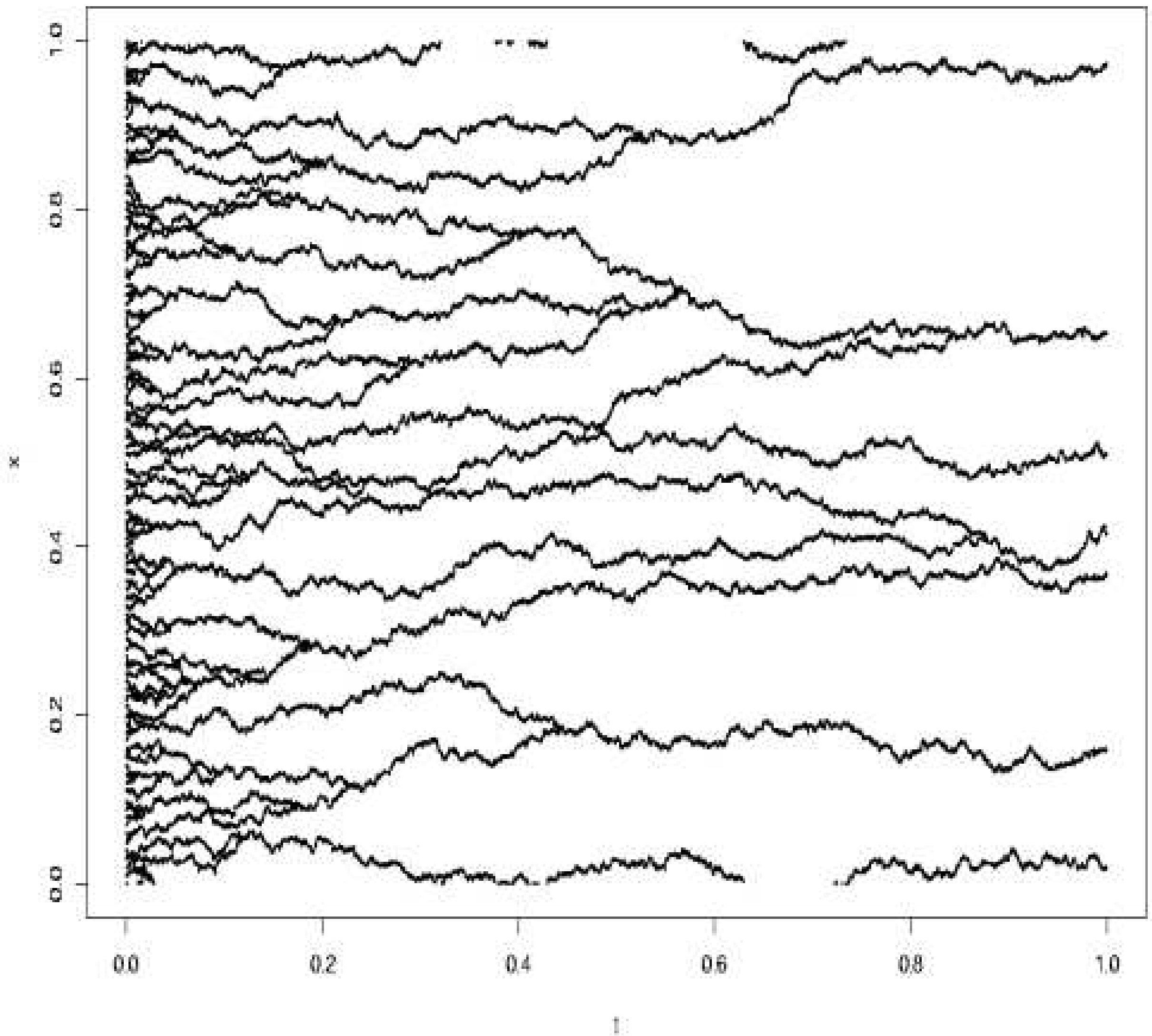
Let K_n be the cluster after n arrivals and set $K_t = K_{\lfloor \rho(P)t \rfloor}$.

For $k = 1, \ldots, n$, write H_t^k for the harmonic measure in K_t of the boundary segment of all fingers in K_t attached between x_{k-1} and x_k.

Let $(B_t^1, \ldots, B_t^n)_{t \geq 0}$ be a family of coalescing Brownian motions in $\mathbb{R} \setminus \mathbb{Z}$ starting from $(x_1, \ldots, x_n)$.

In the limit $\delta(P) \rightarrow 0$, $(H_t^1, \ldots, H_t^n)_{t \geq 0}$ converges weakly in $D([0, \infty), [0,1]^n)$ to $(B_t^1 - B_t^0, \ldots, B_t^n - B_t^{n-1})_{t \geq 0}$.

The slit model flow on the circle for $\delta = 1/50$



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