Compactlike Kink Solutions in Reaction Diffusion Systems

J.C. Comte

*Physics Department, University of Crete and Foundation for Research and Technology-Hellas P. O. Box 2208, 71003 Heraklion, Crete, Greece**

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Abstract

We introduce and demonstrate theoretically the existence of discrete compact-like kink solutions in a class of discrete reaction-diffusion systems with a nonlinear coupling. Numerical simulations show that the extent of such solutions is related only to the coupling parameter, and that propagation failure mechanism occurs for non zero critical coupling value D_c like for discrete reaction-diffusion system with linear coupling. We quantify this critical coupling value and propose an electrical device allowing to verify experimentally our theoretical predications.

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*Electronic address: comte@physics.uoc.gr

I. INTRODUCTION

Reaction-diffusion equations arise in many fields of biology, ecology, chemistry and physics [1]. For instance, they are used to describe the dispersive behavior of cell or animal populations as well as chemical concentrations. Indeed, for a single species in three space dimensions, the general conservation of particle density v leads to an equation of the form,

$$\frac{\partial v}{\partial t} + \nabla F = f(v), \tag{1}$$

where, F is a general *flux transport* owing to diffusion or some other processes, and $f(v)$ is a nonlinear continuous function describing the rate of particle creation or a source reaction term. For the specific case, where $F = \kappa \nabla v$, and κ is a constant corresponding to a diffusion coefficient, equation (1) in one dimension reduces to,

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2} + f(v). \tag{2}$$

We can also find equation (2) in cardiophysiology, or neurophysiology to describe the wave propagation in nerve cells, where v represents the membrane potential, and $f(v)$ models the ionic exchange processes between the extracellular and intracellular media. The Burgers equation [2] is an other nonlinear diffusion equation proposed as a model for turbulence.
Today, the qualitative behavior of such scalar reaction-diffusion equations in one dimension is relatively well understood [3, 4], which is not yet the case in more than one dimension and for coupled reaction-diffusion equation systems.
Indeed, in the case of traveling waves of the form $v(\xi)$, where $\xi = x - ct$, and c is the wave velocity, substitution into Eq.(2), yields an ordinary differential equation of type:

$$K\ v_{\xi\xi} + c\ v_{\xi} + f(v) = 0. \tag{3}$$

A fundamental result of [3] is that the only stable bounded stationary or traveling wave solutions with constant profile are on the one hand, the constant solutions $v(x) = v(s) = v_0$, where v_0 is a stable fixed point of $f(v)$, that is, $f(v_0) = 0$, and $f'(v_0) < 0$, and on the other hand the traveling wave fronts $v(s)$, such that $lim_{s\to-\infty} v(s) = v_0$, and $lim_{s\to+\infty} v(s) = v_1$, where v_0, v_1 are two distinct stable fixed points of $f(v)$.
Since recent years, it has become clear that continuous reaction-diffusion equations of the general form (2) provide an inadequate description of the behavior of weakly coupled systems

where the interplay between nonlinearity and discreteness can lead to novel effects not present in the continuum models. Indeed, propagation failure of wave fonts in discrete excitable media, is certainly the most important example. Such systems modellised through a chain of a bistable elements coupled diffusively to its nearest neighbors in a lattice are governed by a discrete analogue version of Eq.(2):

$$\frac{dv_n}{dt} = K(v_{n+1} - 2v_n + v_{n-1}) + f(v_n), \tag{4}$$

where v_n is the state or the position (if we consider a mechanical analogy with a chain of overdamped oscillators lying to a bistable substrate potential) of the nth site of the lattice. Bistability is represented by $f(v_n)$, with a cubic function of the form $f(v_n) = (v_n^2 - 1)(v_n - \alpha)$, with $-1 < \alpha < +1$, which represents a substrate force deriving from a bistable on site potential with two non degenerated minima (excepted for $\alpha = 0$), located respectively at $v_n = -1$ and $v_n = +1$, as represented in Fig.(1). Mathematical solutions of such equations are very extent in space or time when the coupling term is not very small. However, observed patterns in nature whether stationary or traveling are of finite extent. Indeed, recently it has been shown by Rosenau and Hyman [5–7], that solitary-wave solutions may be compactify under the influence of nonlinear dispersion which is capable of causing deep qualitative changes in the nature of nonlinear phenomena. Such robust soliton-like solutions, characterized by the absence of infinite tails or wings and whose width is velocity independent, have been called compactons [7–9].

The purpose of the present paper is to introduce the concept of compactification in nonlinear diffusive media or reaction-diffusion systems by showing that compact solutions exist in such systems with a nonlinear coupling and that the characteristics of the linear coupling model are conserved. Such models may find their place for instance in the neural context as advanced nerves models, which take into account the real finite extent (in time and space) of the action potential, and consequently explain any unresolved phenomena . The outline of this paper is as follow: First, we present our specific lattice model and show analytically that it can admit exact compact-like kink solutions in the static regime. Then, in Sec. III, we study numerically the propagation and the collisions of such compact-like kinks (K) and antikinks ($\bar{K}$) in the discrete case. In Sec. IV, we show that, like for linear coupling, their exists a coupling limit under which propagation stops, and we quantify analytically this value. Section V present an electrical device which can support such solutions. Finally, Sect. VI is devoted to concluding remarks.

II. MODEL AND EQUATION

We consider, a lattice model of overdamped oscillators coupled anharmonically to their nearest neighbors and interacting with a nonlinear substrate potential $V(u_n)$. The equation of the system is then given by:

$$\frac{dv_n}{dt} = D\Big[(v_{n+1} - v_n)^\lambda - (v_n - v_{n-1})^\lambda\Big] + f(v_n). \tag{5}$$

Where $f(v_n) = (v_n^2 - 1)(v_n - \alpha)$ corresponds to the force deriving from the substrate potential $V(v_n)$. D and λ are two constant parameters corresponding respectively to a strength control of the nonlinear coupling, and the strength interaction law between nearest neighbors cells. In the present paper, we will only consider the case $\lambda = 3$, which conserve the traveling symmetry (left to right and vice versa). For $D >> 1$, one approaches the continuum limit, in which v_n varies slowly from one site to another. Thus, using the continuum limit approximation, Eq.(5) can be reduced to the following partial differential equation:

$$\frac{\partial v}{\partial t} = 3\ D \left(\frac{\partial v}{\partial x}\right)^2 \frac{\partial^2 v}{\partial x^2} - (v^2 - 1)(v - \alpha), \tag{6}$$

where the site index n is replaced by the continuous position variable x. Seeking a propagative solution, we substitute $v(s) = v(x - ct)$ in (6), and get the following ordinary differential equation:

$$3D\ y^3\ \frac{dy}{du} + c\ y - (u^2 - 1)(u - \alpha) = 0. \tag{7}$$

Note that, in the particular case $c = 0$ which corresponds to $\alpha = 0$, that is to the symmetric case of the potential $V(v)$, one obtains:

$$\begin{aligned} v(x) &= \pm\sin(\Omega(x - x_0)),\ for|x - x_0| < \frac{\pi}{2}\ \Omega \\ v(x) &= \pm 1\ \ otherwise. \end{aligned} \tag{8}$$

The discrete solution of equation (5) in the static case may be also write as a discrete version of (8), that is:

$$\begin{aligned} v_n &= \pm\sin(\Omega(n - n_0)),\ for|n - n_0| < \frac{\pi}{2}\ \Omega \\ v_n &= \pm 1\ \ otherwise. \end{aligned} \tag{9}$$

Solutions (8) and (9) confirm that solutions with compact support exist in nonlinear diffusive or reaction-diffusion systems. Now in the following section, we are going to show numerically the existence of such compact solutions for $\alpha \neq 0$, with equation (5) which treat intrinsically the lattice discreteness.

III. NUMERICAL RESULTS: KINK DYNAMICS

In this section, we show numerically the existence of discrete kink solutions with compact support for $\alpha \neq 0$.

For this, we have performed numerical simulations for different values of parameters α and D. A spatial Heaviside structure has been chosen as initial condition. These simulations reveal that the extent of the kink solution is independent of α, but follow a power law of D as represented in Fig.(2). One can remark on this figure that, the particular case of coupling corresponding to the propagation failure effect leads to a solution structure with only 2 lattice spacings (minimum case). This important remark, will be used farther in this paper to determine the critical coupling value of $D = D_c$, under which propagation failure occurs.

Numerical solutions represented on Fig.(3) for $\alpha = 0.3$ and for two different coupling parameters $D = 0.1$ and $D = 10$ show, curve (a) a solution extent on 8 lattice spacings corresponding to $D = 10$ (continuous limit), while curve (b) extent on 4 (discrete regime) lattice spacings corresponds to $D = 0.1$. As one can see on this figure, our numerical results confirm the compact character of the solutions. Moreover, one can remark on Fig.(3) that, in the continuum limit that is for $D >> 1$ (which is the case for $D = 10$, curve (a)), that the localized solution structure tends to a sine shape, as found in the previous section in the static case.

These kink solutions propagate with constant profile and constant velocity while $D > D_c$. Note that, our numerical simulations confirm the exact character of solution (9) obtained in the static regime (not represented here) since the initial condition given by (9) stays invariant with time as well as in discrete than continuous regime.

Also to complete the kink dynamic analysis, we have investigated the front wave velocity versus the coupling term and parameter α. Our numerical simulations show that, like for a linear coupling the velocity of the solution presents a square root shape, as represented on figure (4). Note that, this beam of curves shows the effect of propagation failure. Indeed, each curve presents this phenomenon at its beginning by a zero velocity for different coupling values $D \neq 0$, and this for all the worths of $\alpha \neq 0$.

Finally, we have considered the collision between a kink K and an anti-kink $\bar{K}$, traveling in opposite direction at the same velocity. The dynamics of this collision which leads to a mutual annihilation of the two entities is represented on Fig.(5) for parameters $D = 0.1$ (discrete regime), $\alpha = 0.3$. The interval between two time has been willfully chosen no constant to describe clearly the dynamics of the collision. Note that, the final state of such collisions in the continuous regime (not represented here) is the same.

IV. PROPAGATION FAILURE: QUANTIFICATION.

In the previous section, we have shown that the kink propagation fails if the coupling term D is lower than a critical value D_c. This effect owing to the discrete character of the system is also present for a linear coupling. Lot of peoples have devoted to determine this critical coupling value in the linear coupling case [11–14]. Note that, under certain conditions it has been shown that some such systems are not liable to this effect [10].

Like one said in the previous section when the coupling term D tends to the critical coupling value D_c, the spatial front wave extent is defined only on 2 lattice spacings, that is just one cell v_n of the chain is located between -1 and $+1$.

Thus, under this conditions equation (5) reduces to:

$$\frac{dv_n}{dt} = D\left[(1-v_n)^3 - (1+v_n)^3\right] - (v_n^2-1)(v_n-\alpha). \qquad (10)$$

Now, following the same procedure exposed in [14], we determine easily the critical coupling D_c given by (11) under which propagation fails or stops.

$$D_c = -\frac{1+\alpha}{2} + \frac{2}{3}\left[1 + \sqrt{1 - \frac{3}{2}(1+\alpha) + \frac{3}{4}(1+\alpha^2)}\right] \qquad (11)$$

This critical value only function of the threshold parameter α, is represented Fig.(6) (continuous curve) and compared to numerical simulations (crosses). One can observe a good agreement between theoretical and numerical results almost all around the interval of $\alpha \in [-1, +1]$.

V. EXPERIMENTAL DEVICE PROPOSITION.

In this section, we propose an experimental device construct around an electrical chain composed of cells coupled anharmonically with two head to foot diodes in parallel. Each cell is realized with a capacitor C in parallel with a nonlinear resistor R_{NL} synthetized with analog multiplieurs [15]. Note that, the R_{NL} resistor may be also realized with a tunnel diode. The current-voltage characteristic of the two beche heads diodes is represented at the left-top of figure (7). Indeed, a standard diode has a current-voltage characteristic described by:

$$i_d = i_0 \ (-1 + e^{v/v_0}), \quad with \ v_0 = kT/q, \tag{12}$$

where k is the Boltzman constant, q the elementary charge and T the surround temperature. Usually, v_0 is taken equal to $v_0 = 25 \ mV$. Under these conditions, setting $u = (v_{n-1} - v_n)/v_0$, the current which cross the element defined by C_{NL} writes:

$$\begin{aligned} i_n &= i_0 \ (e^u - e^{-u}) = 2 \ i_0 \ \sinh(u) \\ &\simeq 2 \ i_0 \ (u + (1/3!)u^3 + ...). \end{aligned} \tag{13}$$

Neglecting the first term of the serial development, the current i_n, may write under the form:

$$i_n = D \ (v_{n-1} - v_n)^3, \tag{14}$$

with $D = 2i_0/v_0^3$. Now applying the Kirchoff laws on the chain represented Fig.(7), we get:

$$C\frac{dv_n}{dt} = D[(v_{n-1} - v_n)^3 - (v_n - v_{n+1})^3] + f(v_n), \tag{15}$$

where $f(v_n) = -\Gamma \ (v_n^2 - v_\beta^2)(v_n - v_\alpha)$ play the role of R_{NL}. Γ is constant expressed in $[V^{-2} \ \Omega^{-1}]$, v_β, and v_α are the voltages for which $f(v_n) = 0$.

Thus, it is now possible to verify the above theoretical predictions, and extend them to the case of systems with recovery variable (by adding in parallel of each cell v_n a resistor associated to a serial self-inductor) for which solutions have a pulse structure with compact support also. A such device constitutes a pseudo-nerve fiber analog simulator.

VI. CONCLUSION

In summary, we have introduced the concept of compactification in a class a reaction-diffusion systems and showed analytically that exact compact-like solutions exist in the discrete and continuous regime for the static case. Numerical simulations have shown that the extent of such solutions is related only to the coupling parameter D, and that in the discrete case propagation fails or stops when the coupling value is lower than a critical coupling value D_c like for reaction-diffusion systems with a linear coupling. Following the procedure given by [10] we determined easily this critical coupling. Numerical simulations and theoretical results present a very good agreement. Our investigations concerning the collision of such entities reveal a total mutual annihilation of them. Moreover, we proposed a electrical device allowing to verify the theoretical and numerical predictions exposed in the present paper.

Finally, it emerge from this article that this new model of reaction-diffusion system (with nonlinear coupling) which take into account the finite extent character of natural structures is more appropriated than classical models with linear coupling for which solutions present a infinite extent. Without being too speculative we suggest the use of such models or systems to describe the natural structures as patterns in chemistry, ecology, biology and physics or the signal behavior in the neural context. Indeed, until now the well known nerves models such that Hodgkin-Huxkley and FitzHugh-Nagumo do not take into account the finite character (in time and space) of the action-potentials, with a linear coupling chosen arbitrary. We hope introduce through this paper this new concept of compactification in reaction-diffusion systems in order to improve the nature comprehension and its mechanisms.

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FIG. 1: (a) Nonlinear bistable function $f(v_n)$, deriving from the substrate potential $V(v_n)$ (b). α corresponds to an unstable point, while β and γ correspond to stable points.

FIG. 2: Spatial extent of the compactlike kink solutions versus the coupling parameter D.

FIG. 3: Numerical compactlike kink solutions represented for two different coupling paramters D, and a common threshold paramter $\alpha = 0.3$. (a) Kink solution extent on 8 lattice spacings corresponds to $D = 10$. (b) Kink solution extent on 4 lattice spacings corresponds to $D = 0.1$

FIG. 4: Kink or front wave velocity versus the coupling parameter D, for $\alpha = 0.1$, $\alpha = 0.3$, $\alpha = 0.5$ and $\alpha = 0.9$.

FIG. 5: Collision dynamics between a kink K, and an anti-kink $\bar{K}$, for paramters $D = 0.1$ and $\alpha = 0.3$. The interval between to successive time is willfully irregular to describe clearly the collision dynamics.

FIG. 6: Critical coupling D_c versus threshold α under which propagation stops. (+) Crosses represent the simulation results, while the continuous line represents the theoretical predictions.

FIG. 7: Structure of an electrical chain governed by eq.(5) when $\lambda = 3$. The nonlinear coupling is obtained with two head to foot diodes in parallel. The current-voltage characteristic of a such structure is given at left top of the figure. C is a capacitor and R_{NL} is a nonlinear resistor.