Loss networks and Markov random fields

Stan Zachary
Dept. of Actuarial Mathematics and Statistics
Heriot-Watt University
Edinburgh EH14 4AS
Scotland, U.K.

Ilze Ziedins
Dept. of Statistics
University of Auckland
Private Bag 92019
Auckland
New Zealand

Abstract

This paper examines the connection between loss networks without controls and Markov random field theory. The approach taken yields insight into the structure and computation of network equilibrium distributions, and into the nature of spatial dependence in networks. In addition, it provides further insight into some commonly used approximations, enables the development of more refined approximations, and permits the derivation of some asymptotically exact results.

1 Introduction

Loss networks have been widely studied, primarily as models for telecommunication systems, although other applications have also been discussed. One of the major aims in modelling such networks is to obtain good estimates of performance measures such as blocking probabilities. In this paper we consider simple loss networks without controls (for example on routing or admissions) and examine the connection between such networks and Markov random field theory. The benefits of this approach are several. It gives insight into the structure and computation of network equilibrium distributions, and into the nature of spatial dependence in networks. In addition, it provides further insight into the reduced load approximation (see below) commonly employed when exact computation of equilibrium distributions is impossible, and enables the development of more refined approximations. Finally, for some highly symmetric networks this approach yields asymptotically exact results. For excellent introductions to loss networks see Kelly [12], and also Ross [14].

A general loss network without controls can be described as follows. Denote by J the finite collection of resources in the network and let $\boldsymbol{C} = \{C_j,\ j \in J\}$ where C_j is the capacity of resource j. Let R denote the finite set of possible call (or customer) types in the network. Calls of each type $r \in R$ arrive as a Poisson process with rate ν_r and have identically distributed holding (or service) times, which we assume, without loss of generality, to have mean 1 (see Burman *et al.* [3]). Each call of type r requires (integer) capacity A_{jr} from resource $j \in J$ for the duration of its holding time. If one or more of these resources does not have sufficient free capacity to carry the call then it is *blocked* and considered lost. Otherwise the call is accepted. All arrival processes and holding times are independent of one another. The traditional example is that of a circuit-switched network, in which resources correspond to links in the network, each of a given capacity. In this case a call of any type is assumed to require a single unit of capacity from each link on the route over which it travels, so that call types may effectively be identified with routes. However, there are many other examples and applications, such as cellular radio networks and modern ATM networks. In the latter, capacity (or bandwidth) requirements for different call types (for example, voice and video) may vary very greatly, even over the same routes in the network.

Let n_r be the number of calls of type r in progress, and for any $R' \subseteq R$, let $\boldsymbol{n}_{R'} = (n_r,\ r \in R')$. Let $S_{R'} = \{\boldsymbol{n}_{R'}\colon \sum_{r\in R'} A_{jr} n_r \le C_j \text{ for all } j \in J\}$. Then, under the conditions outlined above, it is now very well-known that the stationary distribution π of $\boldsymbol{n}_R$ is given by

$$\pi(\boldsymbol{n}_R) = G(\boldsymbol{C})^{-1} \prod_{r\in R} \frac{\nu_r^{n_r}}{n_r!}, \qquad \boldsymbol{n}_R \in S_R, \tag{1}$$

where $G(\boldsymbol{C})^{-1}$ is a normalising constant; that is, π has a truncated product form. Unfortunately, for networks of realistic size, this expression is of little help in the calculation of, for example, the stationary blocking probability that a call of any given type is rejected. This is due to the difficulties of calculating the normalising constant. However, a number of relatively fast and efficient methods have been suggested which do permit exact calculations to be made in certain circumstances. Dziong and Roberts [7] (see also Zachary [19]) give an exact recurrence based on consideration of the reduced state space which records only the total occupancy of each resource. Refinements of Buzen's convolution algorithm [4] for closed queueing networks have also been applied to loss networks. Of particular interest are Choudhury *et al.* [5], who invert the generating function of the partition function, and Bean and Stewart [1], who apply refined dimension reduction techniques to Buzen's algorithm, and thus obtain considerable efficiency gains.

Due to the difficulties of calculating the normalising constant, various approximations to quantities of interest, such as blocking probabilities, have been developed. In particular the *(multiservice) reduced load approximation* (Dziong and Roberts [7], see also Ross [14]) may be described briefly as follows. For each $j \in J$, let L_{jr} be the (stationary) probability that resource j has fewer than the A_{jr} units of free capacity needed to accommodate a call of type r at that resource (with $L_{jr} = 0$ whenever $A_{jr} = 0$). Each L_{jr} is calculated by reference to an easily analysed model of resource j in isolation. In this, calls of each type r are assumed to arrive at resource j as a Poisson process with rate $\nu_r \prod_{k\neq j}(1 - L_{kr})$. The capacity of the resource and call holding times are unchanged. The assumption underlying this approximation is that resources block "as if" independently of each other. We are thus led to a set of fixed point equations in the probabilities L_{jr}, for which the existence—but not always the uniqueness, see Chung and Ross [6]—of a solution is guaranteed. In the same spirit, the probability B_r that a call of type r is blocked is then taken to be given by

$$1 - B_r = \prod_{j\in J} (1 - L_{jr}). \tag{2}$$

The reduced load approximation is of course exact in the case of a single-resource network.

The well-known Erlang fixed point approximation (EFPA) differs from the multiservice reduced load approximation by assuming a simplified model of each resource j in which, in particular,

$$1 - L_{jr} = (1 - L'_j)^{A_{jr}} \tag{3}$$

where L'_j is the probability that the resource j has no free capacity. The approximation (3) reduces the number of variables in the fixed point equations referred to above. However, these may again have multiple solutions, see, for example, Ziedins and Kelly [20]. Blocking probabilities are calculated using (2) as before. In general the EFPA is not exact for a single-resource network. The reduced load and Erlang fixed point approximations coincide in the case where the capacity requirements A_{jr} can only take the values 0 or 1. Here the fixed point equations do have a unique solution, see Kelly [11], and also Ross [14].

The use of the EFPA has been justified by considering two limiting regimes. Kelly [11] (see also Hunt and Kelly [8]) shows that the EFPA is asymptotically exact in the Kelly limiting regime in which the network topology, defined by J, R and the matrix (A_{jr}), is held fixed, while both the arrival rates ν_r, $r \in R$, and the capacities C_j, $j \in J$, increase in proportion. Whitt [16], and Ziedins and Kelly [20] consider what has come to be known as the *diverse routing* limit; here the numbers of resources and call types increase while the capacity of each resource and the total traffic offered to it is held constant. Again, under appropriate conditions, the EFPA is asymptotically exact. (Further, in this case the EFPA may continue to perform well even when controls, such as trunk reservation or alternative routing are added to the network; see, for example, Hunt and Laws [9], and MacPhee and Ziedins [13].) All these results extend easily to the more refined reduced load approximation.

However, for networks with small capacities and/or highly linear topologies (see below), neither of the above approximations performs so well. In this paper we examine the problem from a different perspective. We think of the stationary distribution π defined by (1) as a finite random field on the set R of call types. This is Markov with respect to the relation in which two call types r, r' are considered to be neighbours if they utilise a common resource, that is, there exists $j \in J$ such that $A_{jr} > 0$, $A_{jr'} > 0$. This neighbour relation induces a graph in which R is the set of nodes and two nodes are connected by an edge if and only if they are neighbours. In the special case where this graph is a tree, that is, a connected graph which becomes disconnected when any edge is removed, an exact analysis of the stationary distribution π is relatively straightforward. For the general technique here (in the context of general Markov random fields on trees) see, for example, Zachary [17], and for an application to a simple linear cellular radio network, see Kelly [10].

We now show that this idea may be extended to a considerably more general class of network topologies by considering an appropriate neighbour relation between (possibly overlapping) groups of call types. When each such group is the set of call types which share a given resource and two such groups are defined to be neighbours if and only if they share a common call type, then this generalisation is particularly fruitful. We show in Section 2 that if the corresponding induced graph is here a tree then an exact analysis is again straightforward. In other circumstances we may construct approximations which considerably refine the reduced load approximation described above. (Additionally they provide further insight into this approximation.) We give examples of both these situations in Sections 3 and 4 respectively. Finally, in Section 5 we consider a ring network where, although the induced graph is not a tree, the Markov random field approach once more permits an exact analysis. The expressions obtained in Section 5 for the blocking probabilities are suggested as approximations by Bebbington *et al.* [2]. The analyses we give also illustrate the exponential decay of spatial correlation in such networks.

Although all the examples that we consider here have $A_{jr} \in \{0, 1\}$, there is no difficulty in implementing the method in the more general situation. In addition, for ease of presentation the examples we consider are highly symmetric, and again there is no difficulty in implementing the ideas we consider here for asymmetric networks.

2 A refined approximation

We require some additional notation. For any $R' \subseteq R$, let $\pi_{R'}$ denote the marginal stationary distribution on $S_{R'}$ of $\boldsymbol{n}_{R'}$. For each $j \in J$, let $R_j = \{r \in R \colon A_{jr} > 0\}$, that is, the set of call types which utilise the resource j. For each $K \subseteq J$, let $R_K = \bigcup_{j \in K} R_j$ and let $\partial R_K = R_K \cap R_{J \setminus K}$. Finally, for $j, k \in J$, define $j \sim k$ if and only if $R_j \cap R_k \neq \emptyset$. The relation $\sim$ defines a graph $(J, \sim)$ with the set of resources J as the set of nodes and two nodes j, k connected by an edge if and only if $j \sim k$.

It follows from (1) that, for each $K \subseteq J$,

$$\pi_{R_K}(\boldsymbol{n}_{R_K}) = \theta_K(\boldsymbol{n}_{\partial R_K}) \prod_{r \in R_K} \frac{\nu_r^{n_r}}{n_r!}, \quad \boldsymbol{n}_{R_K} \in S_{R_K}, \tag{4}$$

for some function θ_K on $S_{\partial R_K}$. This result is easily established by induction: the result is trivially true when K is replaced by J, and we may successively eliminate resources $j \in J \setminus K$ to obtain the general result. In particular the function θ_K determines $\theta_{K'}$ for all $K' \subset K$. For many network topologies this forms the basis of an efficient recursion, enabling easy determination of exact blocking probabilities.

Consider first the case where the graph $(J, \sim)$ is a tree. Note that this can only be the case when each call type requires capacity from at most two resources. If, in the inductive argument which establishes (4), resources are successively eliminated so that at each step the graph associated with those remaining is connected, then it further follows that, for any connected subset K of J, θ_K has a product form, so that

$$\pi_{R_K}(\boldsymbol{n}_{R_K}) \propto \prod_{\substack{j \in K, l \notin K \\ j \sim l}} \lambda_{jl}(\boldsymbol{n}_{R_j \cap R_l}) \prod_{r \in R_K} \frac{\nu_r^{n_r}}{n_r!}, \quad \boldsymbol{n}_{R_K} \in S_{R_K}. \tag{5}$$

Here, for each ordered pair (j, k) such that $j \sim k$, λ_{jk} is a function on $S_{R_j \cap R_k}$ which we take to be defined only up to a multiplicative constant, and we therefore, here and elsewhere, use the proportionality symbol $\propto$ to denote equality up to such a constant. With this convention the functions λ_{jk} are uniquely determined by (5) and conversely, given these functions, the relation (5) determines π_{R_K} since the latter is a probability measure. Further, comparison of (5) for $K = \{j, k\}$, where $j \sim k$, and for $K = \{j\}$ shows that the functions λ_{jk} satisfy the recursion

$$\lambda_{jk}(\boldsymbol{n}_{R_j \cap R_k}) \propto \sum_{\substack{\boldsymbol{n}'_{R_k} \in S_{R_k}: \\ \boldsymbol{n}'_{R_j \cap R_k} = \boldsymbol{n}_{R_j \cap R_k}}} \prod_{\substack{l \sim k \\ l \neq j}} \lambda_{kl}(\boldsymbol{n}'_{R_k \cap R_l}) \prod_{r \in R_k \setminus R_j} \frac{\nu_r^{n'_r}}{n'_r!}, \quad \boldsymbol{n}_{R_j \cap R_k} \in S_{R_j \cap R_k}, \tag{6}$$

where, for ordered pairs (j, k) such that $l \sim k$ implies $l = j$, the first product in (6) is taken over the empty set. Hence the recursion (6) provides an efficient method of determining the functions λ_{jk}, and thus also marginal distributions and blocking probabilities (see the examples below).

For networks where the graph $(J, \sim)$ is not a tree, the relation (5) fails to hold precisely. In some cases exact computation remains tractable by employing the general relation (4) and its associated recursion—see Section 5 for an example. However, in the case where each call type requires capacity from at most two resources the relation (5) may be regarded as an attractive approximation which, in particular, is very much less restrictive in its assumptions than the reduced load approximation described in Section 1. The latter assumes the relation (5) for sets of the form $K = \{j\}$, with the functions λ_{jk} being given by

$$\lambda_{jk}(\boldsymbol{n}_{R_j \cap R_k}) = \prod_{r \in R_j \cap R_k} (1 - L_{kr})^{n_r}$$

and with the probabilities L_{kr} calculated as described in that section. Note also that in this case the application of (5) to the sets $K = \{j, k\}$ and $K = \{j\}$ continues to imply precisely the recursion (6) as before. Once this recursion has been solved, which may now require repeated substitution, the functions λ_{jk} can be estimated, and quantities of interest can once more be calculated. The asymptotic results of Section 5 provide further insight into the validity of this approximation.

For more general networks, where call types may require capacity from more than two resources, the relation (5) requires some modification, the straightforwardness of which depends on the topology of the network.

3 A linear network

Consider a simple network in which the set $J = \{1, \dots, l\}$ of resources may be thought of as being arranged linearly and in which $C_j = C$ for all $j \in J$. Calls require capacity from either a single or two adjacent resources. *Single-resource* calls of type j, $j = 1, \dots, l$ arrive at rate ν_1 and each requires a single unit of the capacity of resource j. *Two-resource* calls of type $(j, j+1)$, $j = 1, \dots, l-1$, arrive at rate ν_2 and each requires a single unit of the capacity of each of resources j and $j+1$.

Let Λ be the set of non-negative functions on $N_C = \{0, 1, \dots, C\}$ which are not identically zero and which are defined up to a multiplicative constant. We define convergence in Λ to correspond to pointwise convergence under the normalisation, for example, $\sum_{n \in N_C} \lambda'(n) = 1$ for all $\lambda' \in \Lambda$. Exponentially fast convergence is similarly defined. Let the matrix $Q = (Q(m, n),\ m, n \in N_C)$ be given by

$$Q(m,n) = \frac{\nu_2^n}{n!} \sum_{p=0}^{C-m-n} \frac{\nu_1^p}{p!}, \tag{7}$$

with $Q(m, n) = 0$ whenever $m + n > C$. Hence, regarding Q as a transformation $Q\colon \Lambda \to \Lambda$, we have

$$(Q\lambda)(m) \propto \sum_{\substack{p,n \geq 0,\\ m+p+n \leq C}} \frac{\nu_1^p}{p!} \frac{\nu_2^n}{n!} \lambda(n), \quad m \in N_C. \tag{8}$$

For $j \geq 0$, define $\lambda_j \in \Lambda$ by

$$\begin{aligned} \lambda_0(m) &\propto 1, \quad m \in N_C, \\ \lambda_1(m) &\propto \sum_{p=0}^{C-m} \frac{\nu_1^p}{p!}, \quad m \in N_C, \\ \lambda_j &\propto Q\lambda_{j-1}, \quad j \geq 2 \end{aligned} \tag{9}$$

(note that we do not have $\lambda_1 \propto Q\lambda_0$ here). Let $R_{ik} = \bigcup_{j=i}^{k} R_{\{j\}}$ denote the set of call types requiring capacity from any of resources $i, i+1, \dots, k$ where $1 \leq i \leq k \leq l$. It now follows from (5) and the recursion (6) that

$$\pi_{R_{ik}}(\boldsymbol{n}_{R_{ik}}) \propto \lambda_{i-1}(n_{i-1,i})\lambda_{l-k}(n_{k,k+1}) \prod_{j=\max(1,i-1)}^{\min(k,l-1)} \frac{\nu_2^{n_{j,j+1}}}{n_{j,j+1}!} \prod_{j=i}^{k} \frac{\nu_1^{n_j}}{n_j!}, \quad \boldsymbol{n}_{R_{ik}} \in S_{R_{ik}}, \tag{10}$$

where, for all j, $n_j = n_{\{j\}}$, $n_{j,j+1} = n_{\{j,j+1\}}$. Hence marginal distributions and blocking probabilities are readily computed.

In particular let B_i, $i = 1, \ldots, l$, and $B_{i,i+1}$, $i = 1, \ldots, l-1$, be the blocking probabilities associated with the single- and two-resource call types. It follows from (10) and further use of (8) that, for $2 \le i \le l-1$, the probability that the i^{th} resource has occupancy less than or equal to m is given by

$$\sum_{\substack{n_{i-1,i}, n_i, n_{i,i+1} \ge 0, \\ n_{i-1,i}+n_i+n_{i,i+1} \le m}} \pi_{R_i}(n_{i-1,i}, n_i, n_{i,i+1})$$

$$\propto \sum_{n_{i-1,i}=0}^{m} \lambda_{i-1}(n_{i-1,i})(Q\lambda_{l-i})(C-m+n_{i-1,i})\frac{\nu_2^{n_{i-1,i}}}{n_{i-1,i}!}$$

$$\propto \sum_{n_{i-1,i}=0}^{m} \lambda_{i-1}(n_{i-1,i})\lambda_{l-i+1}(C-m+n_{i-1,i})\frac{\nu_2^{n_{i-1,i}}}{n_{i-1,i}!}$$

(the constant of proportionality being independent of m), and so

$$B_i = 1 - \frac{\sum_{n=0}^{C-1} \lambda_{i-1}(n)\lambda_{l-i+1}(n+1)\nu_2^n/n!}{\sum_{n=0}^{C} \lambda_{i-1}(n)\lambda_{l-i+1}(n)\nu_2^n/n!}. \tag{11}$$

It is straightforward to verify that this result also holds for $i = l$. However, in this case we also have, more simply and again by elementary calculation, that $B_l = B_1 = 1 - \lambda_l(1)/\lambda_l(0)$. Similar calculations show that the two-resource call blocking probabilities are given, for $i = 1, \ldots, l-1$, by

$$B_{i,i+1} = 1 - \frac{\sum_{n=0}^{C-1} \lambda_i(n+1)\lambda_{l-i}(n+1)\nu_2^n/n!}{\sum_{n=0}^{C} \lambda_i(n)\lambda_{l-i}(n)\nu_2^n/n!}. \tag{12}$$

We now consider what happens as the length l of the network tends to infinity.

Theorem 3.1. *We have $\lambda_i \to \lambda$ in Λ as $i \to \infty$, where λ is the unique solution in Λ of the equation $\lambda \propto Q\lambda$. Further the convergence is exponentially fast.*

Proof. The matrix $Q = (Q(m,n); m,n \in N_C)$ has entries which are strictly positive for m,n such that $m+n \le C$ and are zero otherwise. It follows that the matrix Q^2 has entries which are all strictly positive. The result is now immediate from, for example, Theorem 1.2 of Seneta[15]. □

It follows that the blocking probabilities exhibit similar exponential convergence. In particular, as l, i and $l-i$ tend to infinity, the single- and two-resource call blocking probabilities B_i and $B_{i,i+1}$ converge to $B^{(1)}$ and $B^{(2)}$ respectively, where

$$B^{(1)} = 1 - \frac{\sum_{n=0}^{C-1} \lambda(n)\lambda(n+1)\nu_2^n/n!}{\sum_{n=0}^{C} \lambda(n)^2\nu_2^n/n!} \tag{13}$$

and

$$B^{(2)} = 1 - \frac{\sum_{n=0}^{C-1} \lambda(n+1)^2\nu_2^n/n!}{\sum_{n=0}^{C} \lambda(n)^2\nu_2^n/n!}. \tag{14}$$

Some of the expressions found here have been seen before. Zachary[18] obtains the result (6) for the Markov random field which models the above network in the case $\nu_1 = 0$—corresponding to the presence of two-resource calls only. This latter model is also considered by Kelly[10] for a linear cellular radio network. He obtains an expression for the blocking probability—as the length of the network tends to infinity—which is the same as $B^{(2)}$ above with (again) $\nu_1 = 0$. (Indeed, in that particular case, the argument of this section is essentially the same as his.) More recently, Bebbington *et al.* [2] propose the above expressions for $B^{(1)}$ and $B^{(2)}$ as *approximations* to the blocking probabilities in a ring network (see also Section 5)—these are obtained from a two-resource approximation with state-dependent arrival rates.

Note that a probability model exists directly for the infinite network corresponding to $l = \infty$, for which the single- and two-resource call blocking probabilities are given exactly by $B^{(1)}$ and $B^{(2)}$. Here the reduced load approximation yields a single-resource call blocking probability $\tilde{B}^{(1)}$ which is equivalent to that obtained by applying the result (10) to any of the sets $R_{\{i\}}$, but with $\lambda_{i-1}(n) = \lambda_{l-i}(n) \propto \alpha^n$, where $\alpha = 1 - \tilde{B}^{(1)}$. The two-resource call blocking probability $\tilde{B}^{(2)}$ is then given by $\tilde{B}^{(2)} = 1 - (1 - \tilde{B}^{(1)})^2$.

4 A symmetric network

Consider a network where, as in Section 3, each resource $j \in J$ has capacity $C_j = C$ and each call type requires capacity from at most two resources. Again *single-resource* calls of each type arrive at rate ν_1 and each requires a single unit of the capacity of a single resource, while *two-resource* calls of each type arrive at rate ν_2 and each requires a single unit of the capacity of each of two resources. Each resource j is offered one stream of single-resource traffic and $a + 1$ streams of two-resource traffic. No two streams of two-resource traffic utilise the same pair of resources. Examples are the fully-connected circuit-switched network (see Kelly [11]—this example possesses a higher degree of symmetry than is actually required by the present model) and the ring network of which we give an exact analysis in Section 5. The latter analysis is also applicable to the infinite linear network considered briefly in the preceding section.

We apply the approximation of Section 2. As in Section 3, let Λ be the set of non-negative functions on $N_C = \{0, 1, \ldots, C\}$, not identically zero and defined up to a multiplicative constant. The equations (6) are satisfied by $\lambda_{jk} = \lambda$ for all j, k such that $j \sim k$ where λ is the solution in Λ of

$$\lambda(m) \propto \sum_{\substack{n, n_1, \ldots, n_a \geq 0, \\ m+n+n_1+\ldots n_a \leq C}} \frac{\nu_1^n}{n!} \prod_{l=1}^{a} \frac{\nu_2^{n_l}}{n_l!} \lambda(n_l), \qquad m \in N_C. \tag{15}$$

Since (under the normalisation $\sum_{n \in N_C} \lambda'(n) = 1$ for all $\lambda' \in \Lambda$) the space Λ is compact and convex, Brouwer's fixed point theorem implies that the equation (15) does have a solution.

The case $a = 1$ corresponds to the ring network that we study in more detail in Section 5. We will see that in this case the recursion of the equations (6) is that given by the matrix Q defined in Section 3 and the equation (15) reduces to the equation $\lambda \propto Q\lambda$ of that section. Hence not only is λ unique in this case, but the convergence established in Theorem 3.1 also shows that $\lambda_{jk} = \lambda$ for all j, k is the unique solution of the equations (6).

In the case $a \geq 2$ we conjecture that the equation (15) again always has a unique solution. The methods described by Zachary [18] gives a sufficient condition for this to be the case: attempt to solve (15) by repeated substitution, starting with $\lambda \in \Lambda$ given by $\lambda(m) = 0$ for all $m \geq 1$; convergence to a fixed point ensures uniqueness of the solution, and indeed—at least for most network topologies—uniqueness of the solution of the equations (6).

We now assume that the equation (15) does have a unique solution λ and apply the approximation (5) to sets of the form $K = \{j\}$ and $K = \{j, k\}$ with $j \sim k$ (and with λ replacing each of the functions λ_{jk}) to calculate blocking probabilities. Calculations entirely analogous to those used to obtain equations (11) and (12) show that the corresponding single- and two-resource call blocking probabilities $B^{(1)}$ and $B^{(2)}$ respectively are again as given by (13) and (14) where the function λ is now the solution of (15).

Note that these blocking probabilities are independent of the number of resources in the network. However, the exact blocking probabilities (and so the quality of these approximations) do in general depend on this number, and more generally on the details of the network topology. We show below that for the ring network these approximations become asymptotically exact as the number of resources is allowed to tend to infinity.

We now compare the above blocking probabilities with those obtained by the use of the reduced load approximation (here also the EFPA). For the stated model, the latter are once more independent of the number of resources and more detailed topology of the network. The example that we consider is given by a fully-connected circuit-switched network with four nodes and a link of capacity C between each pair of nodes, so that there are six links or resources in the network. Single-resource calls arrive at each link at rate ν_1, and two-resource calls arrive at each pair of links which possess a common node at rate ν_2. All calls require a single unit of capacity from each link utilised. We thus have the model of this section with $a = 3$. We look in detail at the case $C = 5$, and consider a sequence of increasing arrival rates with $\nu_2 = \nu_1/4$, so that the proportion of the load on each resource due to each of the two types of call remains the same as the arrival rates increase. Let B_1, B_2 denote the *exact* blocking probabilities for single- and two-resource calls respectively (the network considered here is sufficiently small to permit these to be calculated), and let $\tilde{B}^{(1)}$, $\tilde{B}^{(2)}$ be the corresponding probabilities as determined by the reduced load approximation. The left panel of Figure 1 relates to the single-resource call blocking probability and plots, as ν_1 increases from 0 to 10, the relative error $(B^{(1)} - B_1)/B_1$ of the random field approximation of this section (given by (13)), together with the relative error $(\tilde{B}^{(1)} - B_1)/B_1$ of the reduced load approximation. The right panel similarly plots the relative errors of the two approximations for two-resource calls. In both cases the random field approximation is seen to be very much more accurate than the reduced load approximation (although the latter is of course already good). Similar behaviour, in particular in relation to the quality of the two approximations, is observed for other values of C.

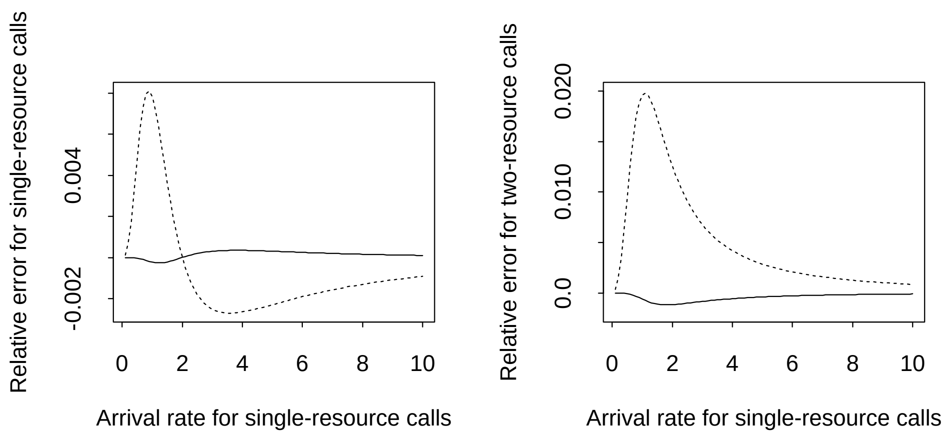

Figure 1: $C = 5$: relative errors in blocking probabilities using the random field approximation (solid line) and the reduced load approximation (dashed line).

5 A ring network

The case $a = 1$ of the model of Section 4 above corresponds to a ring network, and for this case an exact analysis is again possible. The model is identical to that in Section 3, except that there is an additional class of two-resource calls—of type $(l, 1)$—which also arrive at rate ν_2 and for which each call requires one unit of capacity from each of the resources l and 1. Thus the model is cyclically symmetric. Although the graph $(J, \sim)$ defined in Section 2 is here not a tree, it turns out that similar techniques to those used in Section 3 can be used to give an analysis which is both exact and tractable. We also show that the approximate blocking probabilities of Section 4 are asymptotically exact as l tends to infinity.

Let Θ be the set of non-negative functions on $N_C^2 = N_C \times N_C$, again defined up to a multiplicative constant. The matrix Q given by (7) can now be regarded as defining a transformation $Q\colon \Theta \to \Theta$, so that, for $\theta \in \Theta$,

$$(Q\theta)(m, n) \propto \sum_{\substack{p,q:\\ m+p+q\le C}} \frac{\nu_1^p}{p!}\frac{\nu_2^q}{q!}\theta(q, n), \quad (m, n) \in N_C^2 \tag{16}$$

(where the constant of proportionality in this expression is independent of both m and n and where the product $Q\theta$ is given by matrix multiplication). The result (16) follows on recalling that $Q(m, q) = 0$ whenever $m + q > C$.

For $j \ge 1$, define $\theta_j \in \Theta$ by

$$\begin{aligned} \theta_1(m, n) &\propto \sum_{\substack{p:\\ m+p+n\le C}} \frac{\nu_1^p}{p!}, \quad (m, n) \in N_C^2,\\ \theta_j &\propto Q\theta_{j-1}, \quad j \ge 2. \end{aligned} \tag{17}$$

As in Section 3, let $R_{ik} = \bigcup_{j=i}^k R_{\{j\}}$ where $1 \le i \le k \le l$. Then, it follows straightforwardly by induction on i that, for all $i \ge 2$,

$$\pi_{R_{il}}(\boldsymbol{n}_{R_{il}}) \propto \theta_{i-1}(n_{i-1,i}, n_{l,1}) \prod_{j=i-1}^{l} \frac{\nu_2^{n_{j,j+1}}}{n_{j,j+1}!} \prod_{j=i}^{l} \frac{\nu_1^{n_j}}{n_j!}, \quad \boldsymbol{n}_{R_{il}} \in S_{R_{il}} \tag{18}$$

(where $n_{l,l+1} = n_{l,1}$). For other sets of the form R_{ik} the marginal stationary distribution of $\boldsymbol{n}_{R_{ik}}$ follows from the cyclical symmetry of the model. It also follows from (18) and the above symmetry that, for all $j \ge 1$, θ_j is invariant under interchange of its arguments. Hence marginal distributions and blocking probabilities can again be easily calculated. Note that, for this model, the single-resource call blocking probabilities are all the same, as are the two-resource call blocking probabilities.

We again examine the behaviour of the network as $l \to \infty$.

Theorem 5.1. *We have $\theta_i \to \theta$ in Θ as $i \to \infty$, where $\theta(m, n) \propto \lambda(m)\lambda(n)$. Here, as in Section 3, λ is the unique solution in Λ of the equation $\lambda \propto Q\lambda$. Further the convergence is exponentially fast.*

Proof. It follows as in the proof of Theorem 3.1 that, for each fixed $n \in N_C$, the function $\theta_i(\cdot, n)$ converges exponentially fast in the space Λ to the function λ. Hence θ_i similarly converges in the space Θ to θ where $\theta(m, n) \propto \lambda(m)\phi(n)$ for some function $\phi \in \Lambda$. The result now follows from the symmetry of the functions θ_j. □

It follows in particular that we once more have exponential convergence of the single- and two-resource call blocking probabilities to $B^{(1)}$ and $B^{(2)}$ respectively, where these are again as given by equations (13) and (14) and λ is as above.

Now note that, since the equation (15) here reduces to $\lambda \propto Q\lambda$, for any l the single- and two-resource call blocking probabilities as determined by the random field approximation of Section 4 are also $B^{(1)}$ and $B^{(2)}$ as above. Hence this approximation is here asymptotically exact as $l \to \infty$.

We now present a numerical example with $l = 5$ and $C = 5$. As in Section 4 we compare relative errors in blocking probabilities as determined by the random field approximation of that section and by the reduced load approximation. As in Section 4, we let the arrival rates increase with ν_2 held proportional to ν_1, but in this case we have $\nu_2 = \nu_1/2$. Figure 2 gives plots of these relative errors: the left and right panels once more correspond respectively to the single-resource and two-resource call blocking probabilities. We see that the errors associated with the random field approximation are negligible on the scale of these plots, and in particular are again very much less than those of the reduced load approximation (which is again good). If instead we consider the worst-case example $l = 3$ (not shown here) the relative errors of the random field approximation are detectably non-zero, but always considerably less than those of the reduced load approximation. Thus, the rate of convergence to the correct blocking probability as l increases is very fast.

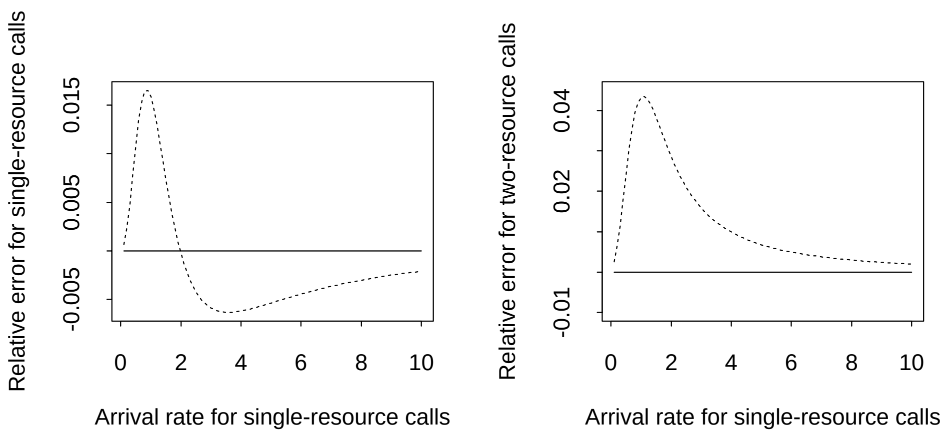

Figure 2: $C = 5$, $l = 5$: relative error in blocking probabilities using the random field approximation (solid line) and the reduced load approximation (dashed line).

Acknowledgement

The authors are very grateful to Richard Gibbens and to the referee for their detailed and helpful comments on an earlier draft of this paper.

References

[1] Bean, N.G. and Stewart, M.A. (1998) A new perspective on the normalisation of invariant measures for loss networks and other product form systems. *Mathematical and Computer Modelling*, to appear.

[2] Bebbington, M.S., Pollett, P.K. and Ziedins, I. (1997) Improved fixed point methods for loss networks with linear structure. In (ed. W.J. Lavery) *Proceedings of the 4th International Conference on Telecommunications, Melbourne, Australia* **3**, 1411–1416.

[3] Burman, D.Y, Lehoczky, J.P. and Lim, Y. (1984) Insensitivity of blocking probabilities in a circuit-switched network. *J. Appl. Prob.* **21**, 850–859.

[4] Buzen, J.P. (1973) Computational algorithms for closed queueing networks with exponential servers. *Commun. ACM* **16**, 527–531.

[5] Choudhury, K.K., Leung, G.L. and Whitt, W. (1995) An algorithm to compute blocking probabilities in multi-rate multi-class multi-resource loss models. *Adv. Appl. Prob.* **27**, 1104–1143.

[6] Chung, S.P. and Ross, K.W. (1991) Reduced load approximations for multirate loss networks. *IEEE Trans. Commun.* **41**, 726–736.

[7] Dziong, Z. and Roberts, J.W. (1987) Congestion probabilities in a circuit-switched integrated services network. *Performance Evaluation* **7**, 267–284.

[8] Hunt, P.J. and Kelly, F.P. (1989) On critically loaded loss networks. *Adv. Appl. Prob.* **21**, 831–841.

[9] Hunt, P.J. and Laws, C.N. (1991) Asymptotically optimal loss network control. *Math. Operat. Res.* **18**, 880–900.

[10] Kelly, F.P. (1985) Stochastic models of computer communication systems. *J. R. Statist. Soc., B* **47**, 379–395.

[11] Kelly, F.P. (1986) Blocking probabilities in large circuit-switched networks. *Adv. Appl. Prob.* **18**, 473–505.

[12] Kelly, F.P. (1991) Loss networks. *Ann. Appl. Prob.* **1**, 319–378.

[13] MacPhee, I.M. and Ziedins, I. (1996) Admission controls for loss networks with diverse routing. In F P Kelly, S Zachary and I Ziedins (ed.) *Stochastic Networks: Theory and Applications.* Oxford University Press, Oxford, 205–214.

[14] Ross, K.W. (1995) *Multiservice loss models for broadband telecommunication networks.* Springer, New York.

[15] Seneta, E. (1981) *Non-negative matrices and Markov chains*, 2nd. ed. Springer.

[16] Whitt, W. (1985) Blocking when service is required from several facilities simultaneously. *A.T. & T. Technical Journal* **64**, 1807–1856.

[17] Zachary, S. (1983) Countable state space Markov random fields and Markov chains on trees. *Ann. Prob.* **11**, 894–903.

[18] Zachary, S. (1985) Bounded, attractive and repulsive Markov specifications on trees and on the one-dimensional lattice. *Stoch. Proc. Appl.* **20**, 247–256.

[19] Zachary, S. (1991) On blocking in loss networks. *Adv. Appl. Prob.* **23**, 355–372.

[20] Ziedins, I.B. and Kelly, F.P. (1989) Limit theorems for loss networks with diverse routing. *Adv. Appl. Prob.* **21**, 804–830.