Dynamics of large uncontrolled loss networks

Stan Zachary
Dept. of Actuarial Mathematics and Statistics
Heriot-Watt University
Edinburgh EH14 4AS
Scotland, U.K.

Abstract

This paper studies the connection between the dynamical and equilibrium behaviour of large uncontrolled loss networks. We consider the behaviour of the number of calls of each type in the network, and show that, under the limiting regime of Kelly (1986), all trajectories of the limiting dynamics converge to a single fixed point, which is necessarily that on which the limiting stationary distribution is concentrated. The approach uses Lyapunov techniques and involves the evolution of the transition rates of a stationary Markov process in such a way that it tends to reversibility.

Keywords and phrases: loss network, Lyapunov function.
AMS subject classifications: primary 60K20; secondary 60G60, 68M20, 90B12.

1 Introduction

In a loss network calls, or customers, of various classes are accepted for service provided that service can commence immediately; otherwise they are considered lost. Such networks have been widely studied, with applications to telecommunication systems and elsewhere.

Early studies focused on the stationary, or equilibrium, behaviour of loss networks. Motivated by applications, where the physical networks to be modelled are frequently very large, particular attention was paid to two limiting regimes. In the first, studied by Kelly (1986), capacities and offered traffic are allowed to grow in proportion to some scale parameter N with all other features of the network held constant. In the second, considered by Whitt (1985) (who also considered network dynamics) and further by Ziedins and Kelly (1989), the number of distinct resources in the network is allowed to grow while an appropriate measure of the traffic offered to each is held constant.

More recent work has further considered network dynamics, with attention again paid to these two limiting regimes. In particular, for the first regime, Hunt and Kurtz (1994) consider a suitably normalised measure $\boldsymbol{x}^N(t)$ of the number of calls of each type in the N^{th} network at time t. They prove a functional law of large numbers describing the behaviour of the limit $(\boldsymbol{x}(t),\ t \geq 0)$ of the processes $(\boldsymbol{x}^N(t),\ t \geq 0)$. (In fact, for some models, this limit process is not always uniquely defined and the result strictly describes the behaviour of the limit in any convergent subsequence.) Hunt and Laws (1993) establish a similar functional law of large numbers for the second of the above limiting regimes.

A study of dynamical behaviour is important for three reasons. First, in many applications, network descriptions—in particular input rates—may change faster than the networks are able to come to equilibrium. An example is that of a telephone network subjected to a sudden overload. This may effectively break down before achieving its (new) equilibrium distribution, and it may be important to know just how breakdown occurs. Second, under certain circumstances (usually when they are inappropriately controlled) network dynamics may exhibit unstable behaviour. In particular, depending on its initial value, the measure of the number of calls of each type in the network may tend to one of a number of "quasi-stationary" states. Here the formal stationary distribution of this measure (which may be expected to be concentrated on some convex combination of the quasi-stationary states) is an inappropriate measure of network performance. Finally, even where the stationary distribution of the network is of interest, often it may only be obtained through a study of the network dynamics—see the further discussion at the end of this section.

The latter two issues are considered by Bean *et al.* (1997) who further study dynamics under the first of the above limiting regimes. They give a simple example of unstable behaviour (in a single-resource network) and show how, when the limiting dynamics are suitably stable, stationary behaviour may be deduced.

This paper is concerned with the relationship between the dynamical and equilibrium behaviour of loss networks in the very important special case where calls are not subject to acceptance controls. We again consider the first of the above limiting regimes. Kelly (1986) shows that, for this regime, the *stationary* distribution of the number of calls of each type in progress, again suitably normalised, converges to a single point $\bar{\boldsymbol{x}}$. The point $\bar{\boldsymbol{x}}$ maximises a concave function f on the limiting, convex, state space. In Section 2 we study the corresponding limiting dynamics. We discuss the extent to which these are uniquely defined for the current model. We show that for the limit process $(\boldsymbol{x}(t),\ t \geq 0)$ (in any convergent subsequence) $f(\boldsymbol{x}(t))$ is strictly increasing with t, except where $\boldsymbol{x}(t) = \bar{\boldsymbol{x}}$. The function f thus acts as a Lyapunov function for the limiting dynamics and $\bar{\boldsymbol{x}}$ is the unique fixed point of these dynamics. We then give a formal proof of the convergence of $\boldsymbol{x}(t)$ to $\bar{\boldsymbol{x}}$, by showing that the rate of increase of $f(\boldsymbol{x}(t))$ is bounded away from zero while $\boldsymbol{x}(t)$ lies outside any given neighbourhood of $\bar{\boldsymbol{x}}$. We thus establish an important stability property of such networks.

As is frequently the case with loss networks, the results involve an interesting interplay between the limit process $(\boldsymbol{x}(t),\ t \geq 0)$ and the associated family of "free capacity" Markov processes, whose rates are indexed by $\boldsymbol{x}(t)$ and which serve as control processes for the the limit process. (The general relationship between these processes is described by Hunt and Kurtz (1994) and reviewed briefly in Section 2.) For the present model, for each t, the free capacity process is a *stationary* Markov process which is not in general reversible. However, the transition rates of the free capacity process evolve in such a way that it tends to a reversible process (and for $\boldsymbol{x}(t) = \bar{\boldsymbol{x}}$ *is* a reversible process). Indeed the rate of increase of $f(\boldsymbol{x}(t))$ is a measure of the extent to which the free capacity process at time t fails to be reversible.

The result has applications which extend well beyond uncontrolled loss networks. Most controlled loss networks use call admission rules which depend on the current state of the network. For many such states the effect of the admission rule is simply to restrict acceptance of calls to those belonging to some subset of the set of call types. Thus, the *dynamics* of such a network may well be such that, in most states, it behaves as an uncontrolled network, albeit one whose description is state-dependent. An example is given by Zachary and Ziedins (2000), who consider virtual partitioning controls (a form of dynamic trunk reservation). Here, in many cases, results for uncontrolled networks may be used to provide a complete description of network dynamics and so also of stationary behaviour. Similar results hold in many other cases where trunk reservation strategies are used.

For reviews of loss networks, see, in particular, Kelly (1991) and Ross (1995).

2 Uncontrolled loss networks

Consider now the standard limiting regime introduced by Kelly (1986). This consists of a sequence of networks, indexed by a scale parameter N, in which all members of the sequence are identical except in respect of capacities and call arrival rates, and are identically controlled. Resources (or links) are indexed in a finite set J and call types in a finite set R. For the Nth member of the sequence, each resource $j \in J$ has integer capacity $C_j(N)$, and calls of each type $r \in R$ arrive as a Poisson process of rate $\kappa_r(N)$. Each such call simultaneously requires an integer A_{jr} units of the capacity of each resource j for the duration of its holding time, which is exponentially distributed with mean $1/\mu_r$. The call is accepted if and only if this capacity is available. All arrival streams and holding times are independent. Finally we suppose that, as $N \to \infty$, for all $j \in J$, $r \in R$,

$$\frac{1}{N} C_j(N) \to C_j, \qquad \frac{1}{N} \kappa_r(N) \to \kappa_r, \tag{2.1}$$

where we take $C_j > 0$ and $\kappa_r > 0$.

Let $\boldsymbol{n}^N(t) = (n_r^N(t),\ r \in R)$, where $n_r^N(t)$ is the number of calls of type r in progress at time t, and let $\boldsymbol{x}^N(t) = \boldsymbol{n}^N(t)/N$. We are interested in both the dynamic and the stationary behaviour of the sequence of processes $\boldsymbol{x}^N(\cdot)$.

Any limit of the above sequence necessarily takes values in the space $X = \{\boldsymbol{x} \in \mathbb{R}_+^R \colon \sum_r A_{jr} x_r \le C_j \text{ for all } j \in J\}$. For each $K \subseteq J$, define also $X_K = \{\boldsymbol{x} \in X \colon \sum_r A_{jr} x_r = C_j \text{ for all } j \in K\}$; write X_j for each $X_{\{j\}}$, $j \in J$. Note in particular that $X_\emptyset = X$ and that $X_K \subset X_{K'}$ whenever $K' \subset K$. We assume that the matrix (A_{jr}) of capacity requirements is such that, for each K with $X_K \neq \emptyset$ and for each $(m_j,\ j \in K)$, there exists $\boldsymbol{n} \in \mathbb{Z}^R$ with $\sum_{r \in R} A_{jr} n_r = m_j$ for all $j \in K$. This implies in particular that the matrix $(A_{jr},\ j \in K, r \in R)$ has rank $|K|$. This assumption is without any real loss of generality—see the discussion following Theorem 3 of Hunt and Kurtz (1994). We do not, however, assume that the matrix $(A_{jr},\ j \in J, r \in R)$ necessarily has rank $|J|$ as this would exclude many interesting models. An example is that considered by Mitra (1987) in which each call type requires capacity from both a dedicated and a common resource.

Define the (real-valued) concave function f on X by

$$f(\boldsymbol{x}) = \sum_{r\in R}(x_r \log\kappa_r - x_r\log\mu_r x_r + x_r) \tag{2.2}$$

and let $\bar{\boldsymbol{x}}$ be the value of $\boldsymbol{x}$ which maximizes $f(\boldsymbol{x})$ in the set X. Then Kelly (1986) shows that $\bar{\boldsymbol{x}}$ is the unique solution in X of

$$\mu_r\bar{x}_r = \kappa_r\prod_j \bar{p}_j^{A_{jr}}, \quad r\in R, \tag{2.3}$$

for some, unique, $(\bar{p}_j,\ j\in J)$ with

$$0<\bar{p}_j\le 1, \quad j\in J, \tag{2.4}$$

and

$$\bar{\boldsymbol{x}}\in X_j \quad \text{if} \quad \bar{p}_j<1, \quad j\in J. \tag{2.5}$$

Kelly further shows that, as $N\to\infty$, the stationary distribution of the process $\boldsymbol{x}^N(\cdot)$ converges to that concentrated on the single point $\bar{\boldsymbol{x}}$.

We now consider the dynamics of the sequence of processes $\boldsymbol{x}^N(\cdot)$. Let $E=(\mathbb{Z}_+\cup\{\infty\})^J$. For each $r\in R$, let $\mathcal{A}_r=\{\boldsymbol{m}\in E\colon m_j\ge A_{jr} \text{ for all } j\in J\}$ (where $\infty\ge A_{jr}$ for all j and for all r). For each $\boldsymbol{x}\in X$, let $\boldsymbol{m}_{\boldsymbol{x}}(\cdot)$ be the Markov process on E with transition rates given by, for each $r\in R$,

$$\boldsymbol{m}\to\begin{cases}\boldsymbol{m}-\boldsymbol{A}_r & \text{at rate } \kappa_r I_{\{\boldsymbol{m}\in\mathcal{A}_r\}}\\ \boldsymbol{m}+\boldsymbol{A}_r & \text{at rate } \mu_r x_r,\end{cases} \tag{2.6}$$

where $\boldsymbol{A}_r$ denotes the vector $(A_{jr}, j\in J)$, I is the indicator function, and $\infty\pm a=\infty$ for any $a\in\mathbb{Z}_+$. Note that the process $\boldsymbol{m}_{\boldsymbol{x}}(\cdot)$ is reducible, and so does not always have a unique stationary distribution. Hunt and Kurtz (1994, Theorem 3) show that, provided the distribution of $\boldsymbol{x}^N(0)$ converges weakly to that of $\boldsymbol{x}(0)$, the sequence of processes $\boldsymbol{x}^N(\cdot)$ is relatively compact in $D_{\mathbb{R}^R}[0,\infty)$ and any weakly convergent subsequence has a limit $\boldsymbol{x}(\cdot)$ which obeys the relation

$$x_r(t) = x_r(0) + \int_0^t \Big(\kappa_r\pi_u(\mathcal{A}_r) - \mu_r x_r(u)\Big)\,du, \tag{2.7}$$

where, for each t, π_t is *some* stationary distribution of the Markov process $\boldsymbol{m}_{\boldsymbol{x}(t)}(\cdot)$ and additionally satisfies, for all j,

$$\pi_t\{\boldsymbol{m}\colon\ m_j=\infty\} = 1 \text{ if } \boldsymbol{x}(t)\notin X_j. \tag{2.8}$$

Thus, at each time t, the stationary distribution π_t acts as a control for the limit process $\boldsymbol{x}(\cdot)$, corresponding to a limiting acceptance rate for calls of each type. For a discussion of this result, which involves a separation of the time scales of the process $\boldsymbol{x}(\cdot)$ and each of the processes $\boldsymbol{m}_{\boldsymbol{x}}(\cdot)$, see Hunt and Kurtz (1994) and Bean *et al.* (1995).

For each $K \subseteq J$, and for each $\boldsymbol{x} \in X_K$, let $\pi_{\boldsymbol{x}}^K$ be the stationary distribution, where it exists, of the Markov process $\boldsymbol{m}_{\boldsymbol{x}}(\cdot)$ on E which assigns probability one to the set $E_K = \{\boldsymbol{m} \in E \colon m_j < \infty \text{ if and only if } j \in K\}$. Our earlier assumption about the matrix of capacity requirements (A_{jr}) implies that the restriction of the process $\boldsymbol{m}_{\boldsymbol{x}}(\cdot)$ to E_K is irreducible, so that the stationary distribution $\pi_{\boldsymbol{x}}^K$ is unique. Define also $X'_K = \{\boldsymbol{x} \in X_K \colon \pi_{\boldsymbol{x}}^K \text{ exists}\}$. Note in particular that the distribution $\pi_{\boldsymbol{x}}^{\emptyset}$ exists for all $\boldsymbol{x} \in X$, assigning probability one to the single point $(\infty, \dots, \infty)$ (so that $\pi_{\boldsymbol{x}}^{\emptyset}(\mathcal{A}_r) = 1$ for all $r \in R$), and thus $X'_{\emptyset} = X_{\emptyset} = X$. It now follows, using (2.8), that, for each t,

$$\pi_t = \sum_{K \subseteq J \colon \boldsymbol{x}(t) \in X'_K} \lambda^K(t) \pi_{\boldsymbol{x}(t)}^K,$$

where

$$\lambda^K(t) \geq 0 \text{ for all } K \text{ with } \boldsymbol{x}(t) \in X'_K, \qquad \sum_{K \subseteq J \colon \boldsymbol{x}(t) \in X'_K} \lambda^K(t) = 1. \tag{2.9}$$

Thus the equation (2.7) above may be rewritten as

$$x_r(t) = x_r(0) + \int_0^t \sum_{K \subseteq J \colon \boldsymbol{x}(u) \in X'_K} \lambda^K(u) \Big(\kappa_r \pi_{\boldsymbol{x}(u)}^K(\mathcal{A}_r) - \mu_r x_r(u) \Big)\, du \tag{2.10}$$

(see Bean *et al.*, 1997, for some further discussion here).

Note that, by identifying E_K with $\mathbb{Z}_+^K$, the distribution $\pi_{\boldsymbol{x}}^K$ may also be thought of as the stationary distribution of the obvious projection of the process $\boldsymbol{m}_{\boldsymbol{x}}(\cdot)$ onto $\mathbb{Z}_+^K$. Thus, using also the definition of the sets $\mathcal{A}_r$, we see that the stationary distribution $\pi_{\boldsymbol{x}}^K$ in fact depends only on the subset K of the set J of resource constraints. Our results depend on an analysis of $\pi_{\boldsymbol{x}}^K$ separately for each $K \subseteq J$, and for each $\boldsymbol{x} \in X'_K$. However, it is more convenient to continue to work with E_K rather than $\mathbb{Z}_+^K$, though the coordinates $j \notin K$ play no real part in the analysis.

There now arises the question of whether the functions $\lambda^K(\cdot)$, $K \subseteq J$, (with the convention that $\lambda^K(t) = 0$ whenever $\boldsymbol{x}(t) \notin X'_K$), and so the limiting dynamics $\boldsymbol{x}(\cdot)$, are uniquely determined. If so, we then have convergence of $\boldsymbol{x}^N(\cdot)$ to $\boldsymbol{x}(\cdot)$ in the entire sequence of networks defined above. In many examples it is indeed possible to determine the functions $\lambda^K(\cdot)$ uniquely (for almost all t), often using no more than the additional observation that any limit process $\boldsymbol{x}(\cdot)$ must remain within the set X. In particular Hunt and Kurtz (1994, Lemma 4) show uniqueness of the limiting dynamics for all single resource networks (with the acceptance controls of the present model). Similarly, Zachary (1996, Theorem 3.1) generalises a result of Moretta (1995) to show uniqueness for two-resource networks in the case where $A_{1r} = A_{2r}$ for those call types r such that $A_{1r} \wedge A_{2r} > 0$. Particular examples of models with more than two resources may be similarly analysed, but general results appear more difficult to obtain. While there exist examples of nonuniqueness in networks with acceptance controls more complex than those considered here (see Hunt, 1995), we conjecture that, for the present model, we always do have uniqueness of the limiting dynamics—see also Hunt and Kurtz (1994, Conjecture 5).

Theorem 2.3 below is independent of these uniqueness considerations, in that it uses a Lyapunov technique to show that, in any subsequence of the above sequence of networks such that $\boldsymbol{x}^N(\cdot)$ converges to a limit process $\boldsymbol{x}(\cdot)$, all trajectories of the latter process converge to the fixed point $\bar{\boldsymbol{x}}$ identified above. (This result of course lends further support to the uniqueness conjecture above.) The result establishes an important stability property of uncontrolled networks. It may also be used to identify limiting dynamics in certain networks which are more generally controlled than those considered here (see the discussion of Section 1).

The formal convergence result is established, in complete generality, in Theorem 2.3. Theorem 2.2, which is considerably simpler to prove, gives a slightly weaker version of the result, which shows that $\bar{\boldsymbol{x}}$ is the unique fixed point of the process $\boldsymbol{x}(\cdot)$ and which is sufficient to establish the full result in the single-resource case $|J| = 1$.

We require first Lemma 2.1 below. For each $K \subseteq J$ with $X_K \neq \emptyset$, let $\boldsymbol{x}^K$ be the value of $\boldsymbol{x}$ which maximises $f(\boldsymbol{x})$ in the set $Y_K = \{\boldsymbol{x} \in \mathbb{R}_+^R\colon \sum_r A_{jr} x_r = C_j \text{ for all } j \in K\}$ (note that $X_K = Y_K \cap X$). Then, since $(A_{jr},\ j \in K, r \in R)$ has rank $|K|$, it follows easily that $\boldsymbol{x}^K$ is the unique solution in Y_K of

$$\mu_r x_r^K = \kappa_r \prod_{j \in K} (p_j^K)^{A_{jr}}, \quad r \in R, \tag{2.11}$$

for some, unique, $(p_j^K,\ j \in K)$ with

$$p_j^K > 0, \quad j \in K. \tag{2.12}$$

Lemma 2.1. *For any $K \subseteq J$ with $X_K \neq \emptyset$, $\boldsymbol{x}^K = \bar{\boldsymbol{x}}$ if and only if $\boldsymbol{x}^K \in X$ and $p_j^K \leq 1$ for all $j \in K$. Further, if $\boldsymbol{x}^K \in X'_K$, then $\boldsymbol{x}^K = \bar{\boldsymbol{x}}$.*

Proof. The first assertion follows by defining $p_j^K = 1$ for $j \notin K$ and comparing (2.3)–(2.5) with (2.11) and (2.12). Now suppose $\boldsymbol{x}^K \in X'_K$. From (2.6) and (2.11), the (unnormalised) measure π' on E_K given by $\pi'(\boldsymbol{m}) = \prod_{j \in K} (p_j^K)^{m_j}$, $\boldsymbol{m} \in E_K$, is invariant for the restriction to E_K of the free capacity process $\boldsymbol{m}_{\boldsymbol{x}^K}(\cdot)$. The condition $\boldsymbol{x}^K \in X'_K$ implies in particular that $\pi_{\boldsymbol{x}}^K$ exists, and so $p_j^K < 1$ for all $j \in K$. The result now follows from the first assertion of the lemma. □

Remark. Observe in particular that we have $\boldsymbol{x}^{\bar{K}} \in X'_{\bar{K}}$ (and so $\boldsymbol{x}^{\bar{K}} = \bar{\boldsymbol{x}}$) for $\bar{K} = \{j \in J\colon \bar{p}_j < 1\}$.

Throughout the rest of this section we take $\boldsymbol{x}(\cdot)$ to be the limit of the processes $\boldsymbol{x}^N(\cdot)$ in any (fixed) convergent subsequence. Note that elementary arguments show that there exists some $p > 0$ such that

$$\pi_{\boldsymbol{x}}^K(\mathcal{A}_r) \geq p \quad \text{for all } K \subseteq J,\ \boldsymbol{x} \in X'_K,\ r \in R. \tag{2.13}$$

It follows straightforwardly from (2.10) that, for all $t > 0$,

$$f(\boldsymbol{x}(t)) = f(\boldsymbol{x}(0)) + \int_0^t \sum_{K\subseteq J:\ \boldsymbol{x}(u)\in X'_K} \lambda^K(u) g_K(\boldsymbol{x}(u))\, du, \tag{2.14}$$

where, for each $K \subseteq J$, the function g_K on X'_K is given by

$$g_K(\boldsymbol{x}) = \sum_r \log(\kappa_r/\mu_r x_r)\big(\kappa_r \pi^K_{\boldsymbol{x}}(\mathcal{A}_r) - \mu_r x_r\big). \tag{2.15}$$

For $\boldsymbol{x} \in X'_K$ such that $x_r = 0$ for some r, we define $g_K(\boldsymbol{x}) = \infty$, so that, with respect to the usual topology on $\mathbb{R} \cup \infty$, the function g_K is continuous on X'_K. This definition is primarily a matter of convenience, since, from (2.13), we have $x_r(t) > 0$ for all $t > 0$ and for all r.

We now show that f is a Lyapunov function for the process $\boldsymbol{x}(\cdot)$. Recall that $\bar{\boldsymbol{x}}$ maximises the function f in X. For each $K \subseteq J$, define the set $Z_K = \{\boldsymbol{n} \in \mathbb{Z}^R\colon \sum_r A_{jr} n_r = 0 \text{ for all } j \in K\}$.

Theorem 2.2. *The function $f(\boldsymbol{x}(t))$ is strictly increasing in t whenever $\boldsymbol{x}(t) \neq \bar{\boldsymbol{x}}$, and thus $\bar{\boldsymbol{x}}$ is the unique fixed point of the limit process $\boldsymbol{x}(\cdot)$.*

Proof. From (2.9) and (2.14), it is sufficient to show that, for each $K \subseteq J$ with $X'_K \neq \emptyset$, $g_K(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in X'_K$ with $\boldsymbol{x} \neq \bar{\boldsymbol{x}}$. Thus fix both K and $\boldsymbol{x} \in X'_K$ (with $x_r > 0$ for all $r \in R$). Recall that $\pi^K_{\boldsymbol{x}}$ is then the stationary distribution on E_K of the process $\boldsymbol{m}_{\boldsymbol{x}}(\cdot)$. Now, for any bounded function ϕ on E_K,

$$\sum_{\boldsymbol{m}\in E_K} \pi^K_{\boldsymbol{x}}(\boldsymbol{m}) \sum_r \big[\kappa_r I_{\{\boldsymbol{m}\in\mathcal{A}_r\}} \{\phi(\boldsymbol{m} - \boldsymbol{A}_r) - \phi(\boldsymbol{m})\} + \mu_r x_r \{\phi(\boldsymbol{m} + \boldsymbol{A}_r) - \phi(\boldsymbol{m})\}\big] = 0. \tag{2.16}$$

(In the case where ϕ is the indicator function associated with any given state $\boldsymbol{m} \in E_K$, the result (2.16) is just the balance equation associated with that state for the stationary distribution $\pi^K_{\boldsymbol{x}}$, and so the general result follows easily.) The result (2.16) further extends to any function ϕ such that, for all r, $\phi(\boldsymbol{m} + \boldsymbol{A}_r) - \phi(\boldsymbol{m})$ is bounded over $\boldsymbol{m} \in E_K$: this follows by considering a sequence of truncated functions converging to ϕ and using the dominated convergence theorem (with the above bound as the dominating function). Now consideration of the stationary balance equations for $\pi^K_{\boldsymbol{x}}$ shows that $\log \pi^K_{\boldsymbol{x}}(\boldsymbol{m} + \boldsymbol{A}_r) - \log \pi^K_{\boldsymbol{x}}(\boldsymbol{m})$ is bounded over $\boldsymbol{m} \in E_K$. Hence, from the above results,

$$\sum_{\boldsymbol{m}\in E_K} \pi^K_{\boldsymbol{x}}(\boldsymbol{m}) \sum_r \left[\kappa_r I_{\{\boldsymbol{m}\in\mathcal{A}_r\}} \log \frac{\pi^K_{\boldsymbol{x}}(\boldsymbol{m} - \boldsymbol{A}_r)}{\pi^K_{\boldsymbol{x}}(\boldsymbol{m})} + \mu_r x_r \log \frac{\pi^K_{\boldsymbol{x}}(\boldsymbol{m} + \boldsymbol{A}_r)}{\pi^K_{\boldsymbol{x}}(\boldsymbol{m})}\right] = 0.$$

Rearranging, we obtain

$$\sum_r \sum_{\boldsymbol{m}\in E_K} \big[\pi^K_{\boldsymbol{x}}(\boldsymbol{m} + \boldsymbol{A}_r)\kappa_r - \pi^K_{\boldsymbol{x}}(\boldsymbol{m})\mu_r x_r\big] \log \frac{\pi^K_{\boldsymbol{x}}(\boldsymbol{m} + \boldsymbol{A}_r)}{\pi^K_{\boldsymbol{x}}(\boldsymbol{m})} = 0. \tag{2.17}$$

Now, from (2.15) and (2.17),

$$g_K(\boldsymbol{x}) = \sum_r \sum_{\boldsymbol{m}\in E_K} \left[\pi_{\boldsymbol{x}}^K(\boldsymbol{m}+\boldsymbol{A}_r)\kappa_r - \pi_{\boldsymbol{x}}^K(\boldsymbol{m})\mu_r x_r\right] \log \frac{\pi_{\boldsymbol{x}}^K(\boldsymbol{m}+\boldsymbol{A}_r)\kappa_r}{\pi_{\boldsymbol{x}}^K(\boldsymbol{m})\mu_r x_r} \tag{2.18}$$
$$\geq 0,$$

with equality if and only if

$$\pi_{\boldsymbol{x}}^K(\boldsymbol{m}+\boldsymbol{A}_r)\kappa_r = \pi_{\boldsymbol{x}}^K(\boldsymbol{m})\mu_r x_r \quad \text{for all } r\in R,\ \boldsymbol{m}\in E_K. \tag{2.19}$$

Define a *loop* l in E_K to be a sequence of jumps $\delta_1\boldsymbol{A}_{r_1},\ldots,\delta_d\boldsymbol{A}_{r_d}$ in E_K such that, for $1\leq i\leq d$, $r_i\in R$, $\delta_i=\pm1$, and also

$$\sum_{i=1}^d \delta_i A_{jr_i} = 0 \quad \text{for all } j\in K, \tag{2.20}$$

$$B_j^{(l,i)} \geq 0 \quad \text{for all } j\in K, \tag{2.21}$$

where

$$\boldsymbol{B}^{(l,i)} = \sum_{h=1}^i \delta_h \boldsymbol{A}_{r_h}. \tag{2.22}$$

For each $r\in R$, let $n_r(l) = -\sum_{i:\, r_i=r}\delta_i$, and observe that the condition (2.20) is then equivalent to

$$\sum_r A_{jr} n_r(l) = 0 \quad \text{for all } j\in K.$$

Now, for each $\boldsymbol{n}\in Z_K$, clearly we can find a loop l such that $\boldsymbol{n}=\boldsymbol{n}(l)=(n_r(l), r\in R)$. If $g_K(\boldsymbol{x})=0$, then, from (2.19), by considering $\boldsymbol{m}=\boldsymbol{B}^{(l,i)}$ for successive i, it follows that

$$\sum_r n_r \log\frac{\kappa_r}{\mu_r x_r} = 0 \quad \text{for all } \boldsymbol{n}\in Z_K, \tag{2.23}$$

and so, from the definition of Z_K, $\log(\kappa_r/\mu_r x_r) = \sum_{j\in K} y_j A_{jr}$ for all $r\in R$ and some $(y_j,\ j\in K)$. This implies that $\boldsymbol{x}=\boldsymbol{x}^K$ (with $p_j^K=\exp(-y_j)$, $j\in K$). (Conversely, if $\boldsymbol{x}^K\in X_K'$, then $\pi_{\boldsymbol{x}^K}^K$ is as given in the proof of Lemma 2.1 and it is easily verified from (2.11) that (2.19) holds with $\boldsymbol{x}=\boldsymbol{x}^K$, and so $g_K(\boldsymbol{x}^K)=0$.) The required result now follows on using the last assertion of Lemma 2.1. □

The above result does not quite guarantee the convergence of all trajectories of the limit process $\boldsymbol{x}(\cdot)$ to $\bar{\boldsymbol{x}}$. This convergence is straightforward to show in the single-resource case $|J|=1$. Here, from (2.14),

$$f(\boldsymbol{x}(t)) = f(\boldsymbol{x}(0)) + \int_0^t g(\boldsymbol{x}(u))\,du, \tag{2.24}$$

where, for each $\boldsymbol{x}\in X$,

$$g(\boldsymbol{x}) = \begin{cases} g_1(\boldsymbol{x}) & \text{if } \boldsymbol{x}\in X_1', \\ g_\emptyset(\boldsymbol{x}) & \text{otherwise} \end{cases}$$

—see, for example, Hunt and Kurtz (1994, Lemma 4) or Bean *et al.* (1997). The function $g_\emptyset$ is continuous on the closed set $X'_\emptyset = X$. Further, it is straightforward to show that restriction of the function g to the closed set X_1 is continuous, including on the boundary in this set of the (not necessarily closed) set X'_1—see Bean *et al.* (1995). The proof of Theorem 2.2 shows that $g_\emptyset(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in X$ with $\boldsymbol{x} \neq \bar{\boldsymbol{x}}$, and also that $g(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in X_1$ with $\boldsymbol{x} \neq \bar{\boldsymbol{x}}$. The above continuity results now imply that, for any given neighbourhood N of $\bar{\boldsymbol{x}}$, we have $\inf_{\boldsymbol{x} \notin X \setminus N} g(\boldsymbol{x}) > 0$.

It seems likely that such continuity arguments can be extended to the case where $|J| > 1$, but it also seems difficult to make this approach rigorous. We use an alternative argument to prove the general result (Theorem 2.3). The proof of the theorem is in effect an extension of that of Theorem 2.2.

Theorem 2.3. *All trajectories of the the limit process $\boldsymbol{x}(\cdot)$ converge to $\bar{\boldsymbol{x}}$.*

Proof. Again from (2.9) and (2.14), it is sufficient to show that, for each fixed $K \subseteq J$ with $X'_K \neq \emptyset$, there exists a strictly positive lower bound for the function g_K on the set $X'_K \setminus N$, where N is again any given neighbourhood of $\bar{\boldsymbol{x}}$. To do this we show that (i) for any given neighbourhood N_K of $\boldsymbol{x}^K$, the function g_K is bounded away from zero on the set $X'_K \setminus N_K$; (ii) if $\boldsymbol{x}^K \neq \bar{\boldsymbol{x}}$, then there is some neighbourhood of $\boldsymbol{x}^K$ which lies wholly outside the set X'_K. In the case $\boldsymbol{x}^K = \bar{\boldsymbol{x}}$, the required result then follows from (i), while in the case $\boldsymbol{x}^K \neq \bar{\boldsymbol{x}}$ it is immediate from (i) and (ii). We assume also that $|K| < |R|$: if $|K| = |R|$ then, by our earlier assumption about the matrix (A_{jr}) and Lemma 2.1, $X'_K = \{\boldsymbol{x}^K\} = \{\bar{\boldsymbol{x}}\}$ and there is nothing to prove.

To show (i), for any loop $l = \delta_1 \boldsymbol{A}_{r_1}, \dots, \delta_d \boldsymbol{A}_{r_d}$ in E_K (defined as in the proof of Theorem 2.2) define the function θ_l on X_K by

$$\theta_l(\boldsymbol{x}) = \min_{\substack{\rho_0, \dots, \rho_d > 0 \\ \rho_0 = \rho_d = 1}} \frac{1}{d} \sum_{i=1}^{d} [\rho_{i-1} q_{r_i}(-\delta_i) - \rho_i q_{r_i}(\delta_i)] \log \frac{\rho_{i-1} q_{r_i}(-\delta_i)}{\rho_i q_{r_i}(\delta_i)}, \tag{2.25}$$

where, for each r,

$$q_r(\delta) = \begin{cases} \mu_r x_r & \text{if } \delta = 1, \\ \kappa_r & \text{if } \delta = -1; \end{cases}$$

for $\boldsymbol{x} \in X_K$ such that $x_r = 0$ for some r, take $\theta_l(\boldsymbol{x}) = \infty$, so that θ_l is then continuous—again with respect to the usual topology on $\mathbb{R} \cup \infty$. Clearly $\theta_l(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in X_K$ and an elementary calculation (analogous to that of the derivation of Kolmogorov's criterion for reversibility—see, for example, Kelly, 1979) shows that $\theta_l(\boldsymbol{x}) = 0$ if and only if $\sum_{i=1}^{d} [\log q_{r_i}(-\delta_i) - \log q_{r_i}(\delta_i)] = 0$, that is,

$$\sum_r n_r(l) \log \frac{\kappa_r}{\mu_r x_r} = 0. \tag{2.26}$$

Now, for $\boldsymbol{x} \in X'_K$ and for each i, $1 \le i \le d$, it follows from (2.18) that,

$$g_K(\boldsymbol{x}) \ge \sum_{\boldsymbol{m} \in E_K} \left[\pi^K_{\boldsymbol{x}}(\boldsymbol{m} + \boldsymbol{B}^{(l,i)}) q_{r_i}(-\delta_i) - \pi^K_{\boldsymbol{x}}(\boldsymbol{m} + \boldsymbol{B}^{(l,i-1)}) q_{r_i}(\delta_i) \right] \times \log \frac{\pi^K_{\boldsymbol{x}}(\boldsymbol{m} + \boldsymbol{B}^{(l,i)}) q_{r_i}(-\delta_i)}{\pi^K_{\boldsymbol{x}}(\boldsymbol{m} + \boldsymbol{B}^{(l,i-1)}) q_{r_i}(\delta_i)}$$

(where $\boldsymbol{B}^{(l,i)}$ is as given by (2.22)). Summing over i and using (2.25), we obtain

$$g_K(\boldsymbol{x}) \ge \sum_{\boldsymbol{m} \in E_K} \pi^K_{\boldsymbol{x}}(\boldsymbol{m}) \theta_l(\boldsymbol{x}) = \theta_l(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in X_K. \tag{2.27}$$

Now consider loops $l_1, \dots, l_a$ (where $a = |R| - |K|$), such that $\{\boldsymbol{n}(l_i),\ 1 \le i \le a\}$ form a basis for Z_K. Then, from (2.27),

$$g_K(\boldsymbol{x}) \ge \max_{1 \le i \le a} \theta_{l_i}(\boldsymbol{x}), \quad \text{for all } \boldsymbol{x} \in X'. \tag{2.28}$$

However, on the closed set X_K, the function $\max_{1 \le i \le a} \theta_{l_i}(\boldsymbol{x})$ is continuous, nonnegative, and, by (2.26), is equal to zero if and only if the condition (2.23) holds, that is, if and only if $\boldsymbol{x} = \boldsymbol{x}^K$. We thus finally obtain that, for any neighbourhood N_K of $\boldsymbol{x}^K$, g_K is bounded away from zero on $X'_K \setminus N_K$, establishing the result (i) as required.

To show (ii), suppose now that $\boldsymbol{x}^K \ne \bar{\boldsymbol{x}}$. In the case $\boldsymbol{x}^K \notin X_K$ the result (ii) follows immediately since the set X_K is closed. Hence suppose also $\boldsymbol{x}^K \in X_K$. By Lemma 2.1, $\boldsymbol{x}^K \notin X'_K$ and also $p^K_j > 1$ for some $j \in K$. Thus not only does $\pi^K_{\boldsymbol{x}^K}$ fail to exist, but we might intuitively expect $\pi^K_{\boldsymbol{x}}$ to fail to exist for all $\boldsymbol{x}$ in some neighbourhood of $\boldsymbol{x}^K$ in X_K. To make this rigorous, we re-express the non-ergodicity of the process $\boldsymbol{m}_{\boldsymbol{x}^K}(\cdot)$ on E_K in terms of the properties of a suitable Lyapunov function $\bar{h}_K$ on E_K, and use simple continuity arguments to show that $\bar{h}_K$ also serves as a Lyapunov function to establish the non-ergodicity of $\boldsymbol{m}_{\boldsymbol{x}}(\cdot)$ on E_K for all $\boldsymbol{x}$ in some neighbourhood of $\boldsymbol{x}^K$ as required.

Thus consider the process $(\boldsymbol{m}_{\boldsymbol{x}^K}(t),\ t \ge 0)$. Regard this as being defined on a probability space $(\Omega, \mathcal{F}, P)$. Let the filtration $(\mathcal{F}_t,\ t \ge 0)$ be that generated by $\boldsymbol{m}_{\boldsymbol{x}^K}(0)$ and the obvious $2|R|$ Poisson processes of total rate $\alpha = \sum_r (\kappa_r + \mu_r x^K_r)$ on $[0, \infty)$ (which together are sufficient to generate $(\boldsymbol{m}_{\boldsymbol{x}^K}(t),\ t \ge 0)$). For each $i > 0$, let τ_i be the time of the i^{th} Poisson event. Let $\mathbb{P}_{\boldsymbol{m}}$ and $\mathbb{E}_{\boldsymbol{m}}$ denote probability and expectation conditional on $\boldsymbol{m}_{\boldsymbol{x}^K}(0) = \boldsymbol{m}$. For each t, define

$$h_K(t) = \sum_r n_r(t) \log(\kappa_r / \mu_r x^K_r), \tag{2.29}$$

where, for each r, $n_r(t)$ is the number of (type r) jumps $\boldsymbol{m} \to \boldsymbol{m} - \boldsymbol{A}_r$ minus the number of (type r) jumps $\boldsymbol{m} \to \boldsymbol{m} + \boldsymbol{A}_r$ which have occurred by time t. Recall that $\boldsymbol{x}^K \in X_{K'}$ for all $K' \subseteq K$. Observe that, from (2.15), for any $K' \subseteq K$ such that $\boldsymbol{x}^K \in X'_{K'}$,

$$\mathbb{E}_{\pi^{K'}_{\boldsymbol{x}^K}}[h_K(t)] = g_{K'}(\boldsymbol{x}^K) t > 0 \quad \text{for all } t > 0, \tag{2.30}$$

where $\mathbb{E}_{\pi^{K'}_{\boldsymbol{x}^K}}$ denotes expectation under the stationary distribution $\pi^{K'}_{\boldsymbol{x}^K}$ of the restriction to $E_{K'}$ of the process $\boldsymbol{m}_{\boldsymbol{x}^K}(\cdot)$, and where the inequality in (2.30) follows as in the proof of

Theorem 2.2 (since $\boldsymbol{x}^K \neq \bar{\boldsymbol{x}}$). We show that there exists a stopping time T (with respect to $(\mathcal{F}_t,\ t \geq 0)$) defined on $\bigcup_{K' \subseteq K}\{\boldsymbol{m}_{\boldsymbol{x}^K}(0) \in E_{K'}\}$, which is the time of some Poisson event and is such that, for all $K' \subseteq K$,

$$T \leq \tau_{M_{K'}} \text{ on } \{\boldsymbol{m}_{\boldsymbol{x}^K}(0) \in E_{K'}\}, \text{ for some constant } M_{K'} > 0, \tag{2.31}$$

and, for all $\boldsymbol{m} \in E_{K'}$,

$$\mathbb{E}_{\boldsymbol{m}}[h_K(T)] \geq \alpha^{-1} g_\emptyset(\boldsymbol{x}^K). \tag{2.32}$$

We establish these results by induction on $|K'|$. Recall that if $\boldsymbol{m}_{\boldsymbol{x}^K}(0) \in E_{K'}$, then $\boldsymbol{m}_{\boldsymbol{x}^K}(t) \in E_{K'}$ for all $t \geq 0$. Since $E_\emptyset$ contains the sole element $\boldsymbol{m} = \{\infty, \dots, \infty\}$, the observation (2.30) shows that (2.31) and (2.32) are trivially true, respectively for $K' = \emptyset$ and $\boldsymbol{m} \in E_\emptyset$ (with $M_\emptyset = 1$ and equality in (2.32)), provided we here take T to be the time of the first Poisson event. For general $K' \subseteq K$, assume the results (2.31) and (2.32) to be established for all $K'' \subset K'$, $K'' \neq K'$ and $\boldsymbol{m} \in E_{K''}$ (where in each case T is the time of some Poisson event). We now construct T on the set $\{\boldsymbol{m}_{\boldsymbol{x}^K}(0) \in E_{K'}\}$ so that (2.32) holds for all $\boldsymbol{m} \in E_{K'}$, the result (2.31) for K' also following directly from the construction. Note that the jumps of the process $\boldsymbol{m}_{\boldsymbol{x}^K}(\cdot)$ are bounded. It follows that, for each $j \in K'$, the result (2.32) holds for all $\boldsymbol{m} \in E_{K'}$ such that m_j is sufficiently large, by defining T as for the initial state $\boldsymbol{m}' \in E_{K' \setminus \{j\}}$ where $\boldsymbol{m}'$ is obtained from $\boldsymbol{m}$ by setting $m'_j = \infty$. Thus it remains to establish (2.32) only for $\boldsymbol{m}$ belonging to a finite set $D_{K'} \subset E_{K'}$. In the case where $\boldsymbol{x}^K \in X'_{K'}$ (that is, the stationary distribution $\pi^{K'}_{\boldsymbol{x}^K}$ exists), the result (2.32) follows easily, for $\boldsymbol{m} \in D_{K'}$ and $T = \tau_{M'}$ for some sufficiently large M', from (2.30), the convergence of the initial distribution of the process $\boldsymbol{m}_{\boldsymbol{x}^K}(\cdot)$ to its stationary distribution, and the boundedness of the increments of the process $h_K(\cdot)$. In the case where $\boldsymbol{x}^K \notin X'_{K'}$, define a sequence of stopping times $0 = U_0 \leq U_1 \leq \dots$ as follows: if $\boldsymbol{m}_{\boldsymbol{x}^K}(U_i) \in D_{K'}$, take U_{i+1} to be the time of the first Poisson event subsequent to that at time U_i; if $\boldsymbol{m}_{\boldsymbol{x}^K}(U_i) = \boldsymbol{m}' \notin D_{K'}$ take $U_{i+1} = U_i + T'$ where T' is defined analogously to T but in terms of the history of the process subsequent to time U_i, so that then, by the already established result (2.32) for such $\boldsymbol{m}'$,

$$\mathbb{E}[h_K(U_{i+1}) - h_K(U_i) | \boldsymbol{m}_{\boldsymbol{x}^K}(U_i) = \boldsymbol{m}'] \geq \alpha^{-1} g_\emptyset(\boldsymbol{x}^K).$$

Note also that, for each i, $U_{i+1} - U_i$ corresponds to a bounded number of Poisson events. Hence, for $\boldsymbol{m} \in D_{K'}$, and since also $\mathbb{P}_{\boldsymbol{m}}(\boldsymbol{m}_{\boldsymbol{x}^K}(t) \in D_{K'}) \to 0$ as $t \to \infty$, the result (2.32) follows straightforwardly by taking $T = U_i$ for sufficiently large i. Thus, finally, by the above construction, (2.31) and (2.32) follow for K' and all $\boldsymbol{m} \in E_{K'}$ and the induction is complete.

Finally, define the function $\bar{h}_K$ on the set E_K by $\bar{h}_K(\boldsymbol{m}) = \sum_{j \in K} m_j \log p^K_j$. Observe that, when $\boldsymbol{m}_{\boldsymbol{x}^K}(\cdot) \in E_K$, then, from (2.29) and (2.11), $h_K(t) = \bar{h}_K(\boldsymbol{m}_{\boldsymbol{x}^K}(t)) - \bar{h}_K(\boldsymbol{m}_{\boldsymbol{x}^K}(0))$ for all $t \geq 0$. Hence, from (2.32),

$$\mathbb{E}_{\boldsymbol{m}}[\bar{h}_K(\boldsymbol{m}_{\boldsymbol{x}^K}(T)) - \bar{h}_K(\boldsymbol{m}_{\boldsymbol{x}^K}(0))] \geq \alpha^{-1} g_\emptyset(\boldsymbol{x}^K) \quad \text{for all } \boldsymbol{m} \in E_K. \tag{2.33}$$

Now consider any ϵ such that $0 < \epsilon < \alpha^{-1} g_{\emptyset}(\boldsymbol{x}^K)$. It follows from the continuity in $\boldsymbol{x}$ of the transition probabilities of the process $\boldsymbol{m}_{\boldsymbol{x}}(\cdot)$, their spatial homogeneity, and the condition (2.31), that there exists some neighbourhood N_K of $\boldsymbol{x}^K$ such that, for all $\boldsymbol{x} \in N_K$, the result (2.33) also holds when the process $\boldsymbol{m}_{\boldsymbol{x}^K}(\cdot)$ is replaced by the process $\boldsymbol{m}_{\boldsymbol{x}}(\cdot)$ and $\alpha^{-1} g_{\emptyset}(\boldsymbol{x}^K)$ is replaced by ϵ. Here $\mathbb{E}_{\boldsymbol{m}}$ now denotes expectation, conditional on the initial state $\boldsymbol{m}$ in the probability space appropriate to the process $\boldsymbol{m}_{\boldsymbol{x}}(\cdot)$, and the stopping time T is the same function as previously of the history of the generating processes. Thus, for $\boldsymbol{x} \in N_K$, by the strong Markov property, and using again (2.31), we can define a sequence of bounded stopping times $0 = T_0 \leq T_1 \leq \ldots$ such that, for some M, for all $i \geq 0$, and for all $\boldsymbol{m} \in E_K$,

$$\mathbb{E}[\bar{h}_K(\boldsymbol{m}_{\boldsymbol{x}}(T_{i+1})) - \bar{h}_K(\boldsymbol{m}_{\boldsymbol{x}}(T_i)) | \boldsymbol{m}_{\boldsymbol{x}}(T_i) = \boldsymbol{m}] \geq \epsilon,$$
$$\mathbb{E}[|\bar{h}_K(\boldsymbol{m}_{\boldsymbol{x}}(T_{i+1})) - \bar{h}_K(\boldsymbol{m}_{\boldsymbol{x}}(T_i))| \, | \boldsymbol{m}_{\boldsymbol{x}}(T_i) = \boldsymbol{m}] < M.$$

The function $\bar{h}_K$ thus serves as a Lyapunov function which establishes the non-ergodicity of the process $\boldsymbol{m}_{\boldsymbol{x}}(\cdot)$ on E_K, for all $\boldsymbol{x} \in N_K$. (See, for example, Theorem 2.1.3 of Fayolle *et al.* (1995), which is immediately applicable to the process sampled at the above stopping times.) Hence $N_K \cap X'_K = \emptyset$ as required. □

Acknowledgement

The author is grateful to Florin Avram, Richard Gibbens, Frank Kelly, Ilze Ziedins, and also the referee, for very helpful discussions and comments.

References

[1] Bean, N.G., Gibbens, R.J., and Zachary, S. (1995). Asymptotic analysis of large single resource loss systems under heavy traffic, with applications to integrated networks. *Adv. Appl. Probab.*, **27**, 273–292.

[2] Bean, N.G., Gibbens, R.J., and Zachary, S. (1997). Dynamic and equilibrium behaviour of controlled loss networks. *Ann. Appl. Probab.*, **7**, 873–885.

[3] Fayolle, G., Malyshev, V.A., and Menshikov, M.V. (1995). *Topics in the Constructive Theory of Countable Markov Chains.* Cambridge University Press.

[4] Hunt, P.J. (1995). Pathological behaviour in loss networks. *J. Appl. Probab.*, **32**, 519–533.

[5] Hunt, P.J. and Kurtz, T.G. (1994). Large loss networks. *Stochastic Process. Appl.*, **53**, 363–378.

[6] Hunt, P.J. and Laws, C.N. (1993). Asymptotically optimal loss network control. *Math. Oper. Res.* **18**, 880–900.

[7] Kelly, F.P. (1979). *Reversibility and stochastic networks.* Wiley, Chichester.

[8] Kelly, F.P. (1986). Blocking probabilities in large circuit-switched networks. *Adv. Appl. Prob.*, **18**, 473–505.

[9] Kelly, F.P. (1991). Loss networks. *Ann. Appl. Probab.* **1**, 319–378.

[10] Mitra, D. (1987). Asymptotic analysis and computational methods for a class of simple, circuit-switched networks with blocking. *Adv. Appl. Prob.*, **19**, 219–239.

[11] Moretta, B. (1995). *Behaviour and control of single and two resource loss networks.* Ph.D. dissertation, Heriot-Watt University.

[12] Ross, K.W. (1995). *Multiservice loss models for broadband telecommunication networks.* Springer, New York.

[13] Whitt, W. (1985). Blocking when service is required from several facilities simultaneously. *A.T. & T. Tech. J.* **64**, 1807–1856.

[14] Zachary, S. (1996). The asymptotic behaviour of large loss networks. In F. P. Kelly, S. Zachary, and I. Ziedins (Eds.), *Stochastic Networks: Theory and Applications*, Number 4 in Royal Statistical Society Lecture Note Series, pp. 193–203. Oxford University Press.

[15] Zachary, S. and Ziedins, I. (2000). A refinement of the Hunt-Kurtz theory of large loss networks, with an application to virtual partitioning. In preparation.

[16] Ziedins, I.B. and Kelly, F.P. (1989). Limit theorems for loss networks with diverse routing. *Adv. Appl. Prob.* **21**, 804–830.