Asymptotic Analysis of Single Resource Loss Systems in Heavy Traffic, with Applications to Integrated Networks

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Abstract

In this paper we consider the analysis of call blocking at a single resource with differing capacity requirements as well as differing arrival rates and holding times. We include in our analysis trunk reservation parameters which provide an important mechanism for tuning the relative call blockings to desired levels. We base our work on an asymptotic regime where the resource is in heavy traffic. We further derive, from our asymptotic analysis, methods for the analysis of finite systems. Empirical results suggest that these methods perform well for a wide class of examples.

LOSS NETWORKS; INTEGRATED NETWORKS; TIME-SCALE SEPARATION; TRUNK RESERVATION; BLOCKING PROBABILITIES

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1 Introduction

Recent developments in communication networks have lead to much interest in systems where traffic of widely differing characteristics is integrated. In this paper we address one of the probabilistic issues associated with these developments. Formally we study a resource of integer capacity C offered a finite number of traffic streams indexed in a set I. Calls of type $i \in I$ arrive as a Poisson stream of rate ν_i and have exponential holding times of mean μ_i^{-1}; each such call requires an integer e_i units of resource, and is accepted if and only if the subsequent free capacity of the resource

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is at least r_i (r_i integer); otherwise it is lost. All arrival streams and holding times are independent. In order to ensure irreducibility of the stochastic process which records the free capacity of the system at any time, we further assume that the capacity requirements e_i, $i \in I$, have greatest common divisor equal to 1. (There is no loss of generality here: if their greatest common divisor is equal to $d > 1$, then we may simply rescale the unit of resource by a factor of d. Any fractional part of the rescaled total capacity cannot be used.)

The capacity requirements e_i, $i \in I$, correspond to the important concept of an *effective bandwidth* that has arisen from many studies of integrated networks (Hui [10], Gibbens and Hunt [7], Kelly [17]). The effective bandwidth is an accurate assessment of the capacity required by a traffic source at each resource of the network in order to guarantee constraints on cell loss or delay.

The parameters r_i are usually referred to as *trunk reservation parameters* and provide an important and very robust mechanism—much used in applications—for controlling the behaviour of the system (see, for example, Key [19]). They can be used to effect an almost complete prioritization of the different traffic streams, while still utilising the full capacity of the system. (Their effectiveness is illustrated in the examples of Section 5.) It is well-known that, in the special case where $r_i = 0$ for all $i \in I$, the equilibrium distribution of the number of calls of each type in progress (and so the equilibrium blocking probability for each type of call) does not depend on the assumption of exponential holding times (see Burman *et al* [3]). In the case $r_i \geq 0$, this is not in general true, and so this assumption is necessary for the results of this paper. However, it is generally held to be reasonable in applications, at least as a first approximation, even in the case where mean holding times for different traffic streams vary widely.

We consider in detail an asymptotic regime where the resource is in heavy traffic. The analysis takes as its starting point the ideas and results on separation of time scales, discussed informally by Kelly [18], and made rigorous by Hunt [11] and by Hunt and Kurtz [12]. (We report these results below, and give some further discussion.) These very general results permit the current treatment of both the dynamics and equilibrium behaviour associated with the model considered here.

We further derive, from our asymptotic analysis, methods for the analysis of finite systems. Empirical results suggest that these methods perform well for a wide class of examples.

Various authors have considered forms of this model. Kaufman [14] considers the model without trunk reservation and develops an elegant and efficient recursion technique for determining the blocking probabilities. Kelly [16], Gersht and Lee [6] and Tran-Gia and Hübner [22] consider approximations to cope with trunk reservation but do not give an asymptotic justification for their methods.

Although we study a single-resource (or, in the language of circuit-switched networks, a single link) network, we expect the methods of this paper to be generalizable to multi-resource networks.

Let $n(t) = (n_i(t), i \in I)$, where the random variable $n_i(t)$ denotes the number of calls of type i in progress at time t. Let $m(t) = C - \sum_{i \in I} e_i n_i(t)$ denote the free capacity at time t. We are interested in the behaviour through time of both the processes $n(\cdot)$ and $m(\cdot)$, their equilibrium distributions, and any quasi-equilibrium distributions they may possess. By a quasi-equilibrium distribution for a process we here mean simply a distribution which behaves as an equilibrium distribution over a sustained period of time.

We give some exact theory, but concentrate on obtaining approximate results where C and $\nu = (\nu_i, i \in I)$ are large and there is heavy traffic, that is $\sum_i e_i \nu_i / \mu_i > C$, so that $m(t)$ is in general small. In particular we obtain asymptotic results for the limiting scheme due to Kelly [15] in which the capacity and arrival rates are allowed to grow in proportion to each other. Thus, we consider a sequence of models indexed by the capacity C, with $\nu(C)$ replacing ν, such that

$$\nu(C) = C\kappa \quad \text{and} \quad C \to \infty, \tag{1}$$

for some constant vector κ, together with the corresponding heavy traffic condition,

$$\sum_i \frac{e_i \kappa_i}{\mu_i} > 1. \tag{2}$$

(Kelly's original limiting scheme allows $\nu(C)/C \to \kappa$, but in the present context there is no loss of generality with regard to applications in assuming relation (1)—in practice we only deal with one member of the sequence!)

In the case where $r_i = 0$ for all i, that is where there is no trunk reservation, it is well known that the equilibrium distribution π^* of the process $n(\cdot)$ is given by

$$\pi^*(n) = G \prod_i \frac{(\nu_i/\mu_i)^{n_i}}{n_i!},$$

where $\sum_i e_i n_i \le C$, $n_i \ge 0$ for all $i \in I$, and G is the appropriate normalising constant. The equilibrium distribution π $(= (\pi(m), m \ge 0))$ of the process $m(\cdot)$ may be determined directly from this expression, or via the recursion

$$\pi(m)(C - m) = \sum_{i:\ m + e_i \le C} e_i \frac{\nu_i}{\mu_i} \pi(m + e_i), \quad 0 \le m \le C - 1. \tag{3}$$

The recursion (3) (due originally to Kaufman [14], see also Dziong and Roberts [5] and Zachary [23]) determines π up to a multiplicative constant. Kelly [15] shows that under the limiting scheme defined by equations (1) and (2) the equilibrium distribution of $m(\cdot)$ converges weakly to the geometric distribution π given by

$$\pi(m) = (1-p)p^m,$$

where p is the unique positive root of

$$1 = \sum_i \frac{e_i \kappa_i}{\mu_i} p^{e_i},$$

(as suggested by the recursion (3)).

Our interest therefore centres on the general case $r_i \geq 0$. For the model in the sequence of capacity C, let $n^C(\cdot)$ and $m^C(\cdot)$ replace $n(\cdot)$ and $m(\cdot)$ respectively, and define $x^C(\cdot) = n^C(\cdot)/C$. Then $x^C(\cdot)$ is a positive Markov process taking values in the region $\mathcal{X} = \{x : \sum_i e_i x_i \leq 1,\ x_i \geq 0 \text{ for all } i \in I\}$. The corresponding free circuit process $m^C(\cdot)$ is a function of this Markov process with transition rates given by, at time t and for all i,

$$m^C \to \begin{cases} m^C - e_i, & \text{at rate } C\kappa_i I_{\{m^C \geq r_i + e_i\}}, \\ m^C + e_i, & \text{at rate } C\mu_i x_i^C(t). \end{cases} \tag{4}$$

The process $m^C(\cdot)$ is not in general itself Markov, but the argument below shows that for large C it behaves as an approximate Markov process over short periods of time.

Define the boundary set

$$\mathcal{B} = \{x \in \mathcal{X} : \sum_i e_i x_i = 1\} \tag{5}$$

and also the set

$$\mathcal{L} = \{x \in \mathcal{X} : \sum_i e_i(\kappa_i - \mu_i x_i) > 0\}. \tag{6}$$

For each $x \in \mathcal{B} \cap \mathcal{L}$, let π_x be the equilibrium distribution of the Markov process on $\mathbb{Z}_+ = \{0, 1, 2, \ldots,\}$ with transition rates given by, for all i,

$$m \to \begin{cases} m - e_i, & \text{at rate } \kappa_i I_{\{m \geq r_i + e_i\}}, \\ m + e_i, & \text{at rate } \mu_i x_i. \end{cases} \tag{7}$$

(The condition $x \in \mathcal{B} \cap \mathcal{L}$ ensures that this process is positive recurrent—see Theorem 2.1 below.) For each $x \in \mathcal{X}$ and for each $i \in I$, let

$$P_i(x) = \begin{cases} \sum_{m \geq r_i + e_i} \pi_x(m), & \text{if } x \in \mathcal{B} \cap \mathcal{L}, \\ 1, & \text{otherwise,} \end{cases} \tag{8}$$

and define also

$$v_i(x) = \kappa_i P_i(x) - \mu_i x_i. \tag{9}$$

We give below an informal argument, which is a special case of that given by Kelly [18], to show why we expect that, as $C \to \infty$, the dynamics of the process $x^C(\cdot)$ approach those of a deterministic process $x(\cdot)$ with dynamics given by

$$x_i(t) = x_i(0) + \int_0^t v_i(x(u))\, du. \tag{10}$$

We then state the extent to which these results are made rigorous by the theory of Hunt and Kurtz [12].

Fix $x \in \mathcal{X}$. Suppose that C is large, and that at some time t, $x^C(t)$ is within a distance $O(C^{-1})$ (as $C \to \infty$) of x. Then, since $x^C(\cdot)$ changes at a rate which is $O(1)$, it remains close to x over time periods which are $o(1)$. If $x \in \mathcal{B} \cap \mathcal{L}$, it follows that, over such time periods in the neighbourhood of t, the process $m^C(\cdot)$ behaves approximately as a positive recurrent Markov process with equilibrium distribution π_x (by the limiting relation (1) and the result that the equilibrium distribution of a Markov process is invariant under a scalar multiplication of its rates). Thus, over such periods, calls of each type i arrive and depart at average rates which are approximately $C\kappa_i P_i(x)$ and $C\mu_i x_i$ respectively, so that the rate of increase of $x_i^C(t)$ is approximated by $v_i(x)$. This last result also holds in the case $x \notin \mathcal{B} \cap \mathcal{L}$, since here, over time periods as above, and for sufficiently large C, the average arrival rate of calls of each type i is simply $C\kappa_i$. Thus we expect the limiting dynamics of the process to be as given by equation (10).

These ideas, which involve a separation (in the limit) of the time scales of the processes $x^C(\cdot)$ and $m^C(\cdot)$, are made rigorous by Theorem 3 and Lemma 4 of Hunt and Kurtz [12]. These show that, under the limiting regime defined by equation (1), and provided $x^C(0) \Rightarrow x(0)$, then the sequence of processes $\left\{x^C(\cdot)\right\}$ is relatively compact in $D_{\mathbb{R}^I}[0,\infty)$ and any convergent subsequence has a limit $x(\cdot)$ which satisfies the relation (10).

We define a *fixed point* of the dynamical system $x(\cdot)$ to be any point x such that $v_i(x) = 0$ for all $i \in I$. Now suppose that the heavy traffic condition (2) holds. We show in Section 3 that the process $x(\cdot)$ eventually enters and remains within a subset of $\mathcal{B} \cap \mathcal{L}$ and that any fixed point lies within this set. We also show, in Section 2, that the functions v_i, $i \in I$, are continuous on $\mathcal{B} \cap \mathcal{L}$. Thus, while the process $x^C(\cdot)$ considered above is in the neighbourhood of any fixed point x, the process $m^C(\cdot)$ maintains approximately the distribution π_x, and so π_x is an approximate quasi-equilibrium distribution, in the sense defined earlier, of the process $m^C(\cdot)$.

Now consider the case where there exists a unique fixed point x, to which all trajectories of the limiting system $x(\cdot)$ converge. The above informal arguments suggest that, for sufficiently large C, $x^C(t)$ eventually remains within a neighbourhood of x, implying that the equilibrium distribution of the process $x^C(\cdot)$ converges weakly to x, and further that, under the heavy traffic condition (2), the equilibrium distribution of the process $m^C(\cdot)$ converges weakly to the distribution π_x. Proofs of these assertions (in the more general setting considered by Hunt and Kurtz [12]) are included in a forthcoming paper (Bean *et al* [2]). In independent work, Greenberg et al [8] obtain similar results for the current model in the case where the capacity requirements are all equal.

It follows that in order to study in detail the limiting dynamics and the equilibrium behaviour of the process $x^C(\cdot)$, it is necessary to be able to determine the distribution π_x, $x \in \mathcal{B} \cap \mathcal{L}$, and in particular the associated *passing probabilities* (or call acceptance probabilities) $P_i(x)$, $i \in I$. In Section 2 we show (in Theorem 2.4) how π_x may be determined explicitly by the solution of a finite system of equations. We further show that the tail of this distribution is geometric, and identify the geometric parameter.

Section 3 considers the dynamics of the process $x(\cdot)$ introduced above. In particular we prove the existence of at least one fixed point. In view of the above comments, it is important to know when this fixed point is unique. We show that this is always so in the special case where $e_i = 1$ for all i. We conjecture that in most other cases of practical interest the fixed point will also be unique, but this may be checked numerically in each instance—as in the examples of Section 5.

In Section 4 we show how to improve our asymptotic results to give more accurate descriptions of the behaviour of realistically-sized systems, and in particular more accurate estimates of call-acceptance (or alternatively of *blocking*) probabilities. Section 5 considers some numerical examples to examine the accuracy, in a variety of situations, of both the approximations derived from the asymptotic results and the improved approximations derived in Section 4.

An alternative approach to the derivation of the relationship between the fixed points of the dynamical system $x(\cdot)$ and the asymptotic equilibrium distributions of the processes $x^C(\cdot)$ and $m^C(\cdot)$ can be found in [1], together with further examples.

2 Analysis of the Asymptotic Free Capacity Distribution

In this section we consider further the Markov process introduced in the previous section in connection with the limiting regime (1). This is the process $m(\cdot)$ on $\mathbb{Z}_+$ with transition rates given, for all i, by equation (7), where we now allow $x \in \mathcal{B}$. We show that this process is positive recurrent when $x \in \mathcal{B} \cap \mathcal{L}$, and study the corresponding equilibrium distribution π_x. Define $\hat{e} = \max_{i \in I} e_i$ and $\hat{s} = \max_{i \in I}(e_i + r_i)$.

Theorem 2.1 *The Markov process $m(\cdot)$ is positive recurrent, null recurrent, or transient, according as $\sum_i e_i(\kappa_i - \mu_i x_i)$ is greater than, equal to, or less than zero.*

Proof: The process $m(\cdot)$ may be considered, in the obvious manner, as a Markov chain; the transitions of which occur as a Poisson process of rate $\lambda = \sum_i (\kappa_i + \mu_i x_i)$. (This chain may be regarded as the uniformized jumping chain of the process.) Then on the set $\{m : m \geq \hat{s}\}$ the chain behaves as a random walk with mean increment $\lambda^{-1} \sum_i e_i(\mu_i x_i - \kappa_i)$, so that the result now follows easily by standard arguments for random walks (see, for example, Durrett [4]). ■

Throughout the rest of this section we assume that $x \in \mathcal{B} \cap \mathcal{L}$. The equilibrium distribution π_x of the process $m(\cdot)$ then exists and is the unique solution $\pi = \pi_x$ of the system of *global balance equations*

$$\pi(m) \left(\sum_{i:\ m \geq e_i + r_i} \kappa_i + \sum_i \mu_i x_i \right) = \sum_{i:\ m \geq r_i} \pi(m + e_i)\kappa_i + \sum_{i:\ m \geq e_i} \pi(m - e_i)\mu_i x_i, \quad (11)$$

for all $m \geq 0$, and

$$\sum_{m \geq 0} \pi(m) = 1. \quad (12)$$

Note that, given (12), any one of the equations (11) may, as usual, be omitted (being implied by the remaining equations in the system).

The solution of this system of equations is clearly related to the roots of the polynomial $f : \mathbb{C} \to \mathbb{C}$ of degree $2\hat{e}$, defined by

$$f(z) = z^{\hat{e}} \sum_i (\kappa_i + \mu_i x_i) - \sum_i \left(z^{\hat{e}+e_i} \kappa_i + z^{\hat{e}-e_i} \mu_i x_i \right). \quad (13)$$

The exact relationship is clarified in Theorem 2.4. In order to do this we first require a careful characterization of the $2\hat{e}$ roots in the complex plane $\mathbb{C}$ of this polynomial. This is given by Lemma 2.2. The proof of this lemma is somewhat technical and is deferred to the Appendix.

Lemma 2.2 *The polynomial f has exactly two positive real roots, one of which is unity and the other, p_1 say, satisfies $p_1 \in (0,1)$. Of the remaining roots, $\hat{e}-1$ have modulus strictly less than p_1 and the remaining $\hat{e}-1$ have modulus strictly greater than unity.*

Now let $p_2, \ldots, p_{\hat{e}}$ denote those roots of the polynomial f with modulus less than p_1. Let $p_{\hat{e}+1}, \ldots, p_{2\hat{e}}$ denote the remaining roots. For $k = 2, \ldots, \hat{e}$, define $\beta(k)$ to be the number of indices $j < k$ such that $p_j = p_k$. (In particular if $p_2, \ldots, p_{\hat{e}}$ are distinct then $\beta(k) = 0$ for all k.) Further, for each such k, define the function $h_k : \mathbb{Z}_+ \to \mathbb{C}$ by $h_k(m) = m^{\beta(k)} p_k^m$. In order to prove Theorem 2.4 we also require the following lemma, the proof of which is again deferred to the Appendix.

Lemma 2.3 *Let Π be the set of finite nonzero measures π on $\mathbb{Z}_+$ satisfying the recurrence relations*

$$\pi(m) \sum_i (\kappa_i + \mu_i x_i) = \sum_i \left[\pi(m+e_i)\kappa_i + \pi(m-e_i)\mu_i x_i \right], \quad m \geq \hat{e}. \tag{14}$$

Then any $\pi \in \Pi$ has a unique spectral representation

$$\pi(m) = a_1(\pi) p_1^m + \sum_{k=2}^{\hat{e}} a_k(\pi) h_k(m), \quad m \geq 0, \tag{15}$$

where $a_1(\pi)$ is a strictly positive real number, and $a_k(\pi) \in \mathbb{C}$ for $k = 2, \ldots, \hat{e}$. Conversely, if π is a finite nonzero measure on $\mathbb{Z}_+$ having a representation of the form (15) with $a_1(\pi), \ldots, a_{\hat{e}}(\pi)$ as above, then $\pi \in \Pi$.

We are now in a position to state and prove the main result of this section.

Theorem 2.4 *Consider the system of equations in $\pi = (\pi(m), m \in \mathbb{Z}_+)$, with each $\pi(m) \in [0,1]$, and $a = (a_1, \ldots, a_{\hat{e}})$ with $a_1 > 0$, and $a_2, \ldots, a_{\hat{e}} \in \mathbb{C}$, defined by the global balance equations (11) for $m = 0, 1, \ldots, \hat{s}-2$, the normalization equation (12), and the equations*

$$\pi(m) = a_1 p_1^m + \sum_{k=2}^{\hat{e}} a_k h_k(m), \quad m \geq \hat{s} - \hat{e}. \tag{16}$$

Then these equations have a unique solution (π, a) and $\pi = \pi_x$. In particular π_x has a geometric tail with parameter p_1.

Proof: By Lemma 2.3, with the shift $\pi'(m) = \pi(m+\hat{e}-\hat{s})$, a nonzero finite measure π satisfies the equations (16), for some a as given, if and only if it satisfies the global balance equations (11) for $m \geq \hat{s}$. Since any one of the global balance equations (for all $m \geq 0$) is implied by the remainder, the requirement that π satisfy the system of equations in the statement of the theorem is equivalent to the requirement that π satisfy the equations (11) for all $m \geq 0$ together with the equation (12), and then $\pi = \pi_x$. The uniqueness of a follows by Lemma 2.3 again. Finally, since $|p_k| < p_1$ for $k = 2, \ldots, \hat{e}$, π_x has a geometric tail with parameter p_1. ∎

The system of equations in Theorem 2.4 may be solved as follows. Consider the global balance equations (11) for $m \leq \hat{s}-2$ (which involve $\pi(0), \ldots, \pi(\hat{s}+\hat{e}-2)$) and the equations (16) for $\hat{s}-\hat{e} \leq m \leq \hat{s}+\hat{e}-2$. The equations (16) may also be summed over all $m \geq \hat{s}-\hat{e}$ and the result substituted into the normalization equation (12). This gives a set of $\hat{s}+2\hat{e}-1$ linear equations which determine $\pi(0), \ldots, \pi(\hat{s}+\hat{e}-2)$ and $a_1, \ldots, a_{\hat{e}}$.

Note also that the distribution π_x is (pointwise) continuous in $x \in \mathcal{B} \cap \mathcal{L}$. To see this, observe that the roots p_k, $1 \leq k \leq \hat{e}$, of the polynomial f may be taken to be continuous functions of x. Then, in the case where $x = x_0$ is such that these roots are distinct, the $\hat{s}+2\hat{e}-1$ equations considered above have a continuous solution which implies that π_x is continuous in x at x_0. In the case where x_0 is such that f has repeated roots, the usual elementary modifications are required.

It follows that, for each $i \in I$, the functions P_i and v_i, defined by equations (8) and (9) respectively, are continuous when restricted to the set $\mathcal{B} \cap \mathcal{L}$ (though they are not of course continuous on $\mathcal{X}$).

3 Limiting Dynamics

In this section we consider the behaviour of the limiting process $x(\cdot)$, defined in Section 1, and assuming the heavy traffic condition (2). Recall that the dynamics of $x(\cdot)$ are given by equation (10). We investigate the transient behaviour of the process and show also that it always possesses at least one fixed point. We consider further the special case in which $e_i = 1$ for all $i \in I$, where we prove uniqueness of the fixed point. (In other cases this may readily be investigated numerically.)

Define the sets

$$\mathcal{R} = \{x \in \mathcal{X} \,:\, x_i < \kappa_i/\mu_i \text{ for all } i\}, \qquad \mathcal{H} = \mathcal{B} \cap \mathcal{R},$$

(where $\mathcal{B}$ is as defined by equation (5)). Note that $\mathcal{R}$ is a subset of the set $\mathcal{L}$ defined by equation (6), and hence that $\mathcal{H}$ is a subset of $\mathcal{B} \cap \mathcal{L}$. Note also that the condition (2) implies that there is some constant $\delta > 0$ such that, for all $x \in \mathcal{X}$,

$\sum_i e_i(\kappa_i/\mu_i - x_i) \geq \delta$ and so

$$\kappa_i - \mu_i x_i \geq \mu_i \delta/(e_i|I|), \quad \text{for some } i \in I, \text{ for all } x \in \mathcal{X}. \tag{17}$$

Thus in particular the closure $\overline{\mathcal{H}}$ of $\mathcal{H}$ is also a subset of $\mathcal{B} \cap \mathcal{L}$.

Figure 1 illustrates the sets $\mathcal{X}$, $\mathcal{L}$, $\mathcal{B}$ and $\mathcal{R}$ in an example in which there are two call types ($|I| = 2$), type 1 calls have the parameters $e_1 = 1$, $\kappa_1 = 0.5$, $\mu_1 = 1$, $r_1 = 0$ and type 2 calls have the parameters $e_2 = 2$, $\kappa_2 = 0.6$, $\mu_2 = 2$, $r_2 = 0$. The shaded region is $\mathcal{L}$ and the region $\mathcal{R}$ is the subset of $\mathcal{L}$ defined by $x_1 < 0.5$, $x_2 < 0.3$.

In the usual terminology of dynamical systems we say that a set $\mathcal{A} \subset \mathcal{X}$ is *attracting* if, for all $x(0)$, there exists a finite t such that $x(t) \in \mathcal{A}$. We say that it is *invariant* if, whenever $x(0) \in \mathcal{A}$, then $x(t) \in \mathcal{A}$ for all $t \geq 0$. Since the process $x(\cdot)$ has time-homogeneous dynamics, it follows that, if $\mathcal{A}$ is an attracting invariant set, then, for all $x(0)$, the process $x(\cdot)$ eventually enters and then remains within $\mathcal{A}$.

Theorem 3.1 *The sets $\overline{\mathcal{H}}$ and $\mathcal{H}$ are attracting and invariant.*

Proof: Define the sets

$$\mathcal{R}_\epsilon = \{x \in \mathcal{X} : x_i \leq \kappa_i/\mu_i + \epsilon \ \text{ for all } i\} \quad \text{and} \quad \mathcal{H}_\epsilon = \mathcal{R}_\epsilon \cap \mathcal{B}$$

where $\epsilon > 0$ is chosen sufficiently small that $\mathcal{R}_\epsilon \subset \mathcal{L}$. (That this is possible follows from the relation (17).) Note that $\overline{\mathcal{H}} \subset \mathcal{H}_\epsilon$. For each $i \in I$, on the set $\{x \in \mathcal{X} : x_i > \kappa_i/\mu_i + \epsilon\}$, we have $v_i(x) \leq -\mu_i \epsilon$. Thus the set $\mathcal{R}_\epsilon$ is attracting and invariant. For all $x \in \mathcal{R}_\epsilon \setminus \mathcal{B}$, $\sum_i e_i v_i(x) = \sum_i e_i (\kappa_i - \mu_i x_i)$, which, since $\mathcal{R}_\epsilon$ is closed and contained in $\mathcal{L}$, is bounded below by a positive constant. Therefore, using also the continuity of the process $x(\cdot)$, the set $\mathcal{H}_\epsilon$ is attracting and invariant. Finally, for all i, we have that P_i is continuous and strictly less than 1 on $\mathcal{H}_\epsilon$ and, since $\mathcal{H}_\epsilon$ is closed, it follows that P_i is bounded away from 1 on this set. Hence the sets $\overline{\mathcal{H}}$ and $\mathcal{H}$ are attracting and invariant. ∎

We now consider fixed points of the process $x(\cdot)$. The result below follows from Theorem 3.1, but we give an alternative simple proof which does not depend on consideration of the dynamics of the process.

Theorem 3.2 *Any fixed point must lie on the bounded hyperplane $\mathcal{H}$.*

Proof: If $x \in \mathcal{X} \setminus (\mathcal{B} \cap \mathcal{L})$, then $P_i(x) = 1$ for all $i \in I$, and so

$$\sum_i e_i v_i(x)/\mu_i = \sum_i e_i (\kappa_i/\mu_i - x_i) \geq \sum_i e_i \kappa_i/\mu_i - 1 > 0,$$

by the heavy traffic condition (2). Thus any fixed point belongs to the set $\mathcal{B} \cap \mathcal{L}$. Also, since $P_i(x) < 1$ for all $x \in \mathcal{B} \cap \mathcal{L}$, it follows that any fixed point belongs to the set $\mathcal{R}$. ■

Theorem 3.3 *There exists at least one fixed point.*

Proof: By Theorem 3.1, if $x(0) \in \overline{\mathcal{H}}$ then $x(t) \in \overline{\mathcal{H}}$ for all $t > 0$. Define the mapping $\theta : \mathbb{R}_+ \times \overline{\mathcal{H}} \to \overline{\mathcal{H}}$ by,

$$\theta(t, x(0)) = x(t).$$

Define the sequence of sets

$$A_n = \left\{ y \in \overline{\mathcal{H}} : y = \theta(2^{-n}, y) \right\},$$

for $n = 0, 1, \ldots$. Brouwer's fixed point theorem (Heuser [9, Lemma 106.2]) implies that the set A_n is non-empty for all $n \geq 0$, as $\overline{\mathcal{H}}$ is a compact and convex set and the functions v_i are continuous on $\overline{\mathcal{H}}$. Note also that, for all $n \geq 1$, $A_n \subseteq A_{n-1}$, by the time invariance of the dynamics of $x(\cdot)$. Therefore, the intersection of all finite collections is non-empty. The finite intersection principle (Johnsonbaugh and Pfaffenberger [13, Ex 42.5]) shows that

$$\bigcap_{n=0}^{\infty} A_n \neq \emptyset,$$

since $\overline{\mathcal{H}}$ is a compact set and the A_n are closed. Hence, there exists a $y \in \overline{\mathcal{H}}$ such that $y = \theta(2^{-n}, y)$ for all $n \geq 0$. Since the function $x(\cdot)$ is continuous it follows that there exists at least one fixed point $y \in \overline{\mathcal{H}}$, which by Theorem 3.2 is necessarily in $\mathcal{H}$. ■

Consideration of Theorem 3.1 shows that it is usually a simple matter to find fixed points of the limiting process by following trajectories of the process $x(\cdot)$, starting from any point in the bounded hyperplane $\mathcal{H}$. When there are only two call types it is also very simple to find the fixed points using a bi-section search method along the line defined by $\mathcal{H}$.

Figure 1 shows some trajectories for the example considered earlier, together with the single fixed point for this example, which is marked by an asterix at $(0.46995, 0.26503)$.

In general we are unable to make statements about the stability or uniqueness of fixed points. However, in the important special case where $e_i = 1$ for all i we can say more.

Theorem 3.4 *Let $e_i = 1$ for all $i \in I$. Then there is a unique fixed point. Further in the case $|I| = 2$ all trajectories of the process $x(\cdot)$ converge to this fixed point.*

Proof: Suppose that y and z are fixed points (by Theorem 3.2 necessarily in $\mathcal{H}$) and, without loss of generality, that $\sum_i \mu_i y_i \geq \sum_i \mu_i z_i$. Since the Markov process $m(\cdot)$ is skip free, and hence reversible, it follows easily that $P_i(y) \geq P_i(z)$ and so $y_i \geq z_i$, for all $i \in I$. Since $y, z \in \mathcal{B}$, we therefore have $y = z$.

In the case $|I| = 2$, the region $\mathcal{H}$ is one-dimensional and so the continuity of the functions v_i on $\mathcal{H}$ ensure that all trajectories of the process $x(\cdot)$ converge to the unique fixed point. ■

4 Approximations for Finite Systems

In this section we consider the determination of practical approximations to the passing probabilities associated with the model introduced in Section 1. We again assume the heavy traffic condition, and further that the capacity C is reasonably large. In order to use the asymptotic theory of the preceding sections to justify and interpret these approximations, we assume the model to be embedded in a sequence of models satisfying the conditions (1) and (2). To emphasize our concern with a particular member of this sequence we shall drop the notational dependence on C.

Recall that the free capacity process $m(\cdot)$ is a function of the Markov process $n(\cdot)$ with transition rates, for all $i \in I$ and at time t, given by,

$$m \to \begin{cases} m - e_i & \text{at rate } \nu_i I_{\{m \geq r_i + e_i\}} \\ m + e_i & \text{at rate } \mu_i n_i(t). \end{cases} \tag{18}$$

Now suppose that the limiting process $x(\cdot)$ introduced in Section 1 has a unique fixed point $\bar{x}$. Then we expect that, for all sufficiently large t, $m(t)/C$ will remain close to 0 and $n(t)/C$ will remain close to $\bar{x}$. Therefore, we may approximately model the process $m(\cdot)$ as a Markov process on $\mathbb{Z}_+$ with transition rates given by (18) but with $n(t)$ replaced by a constant $\bar{n} = (\bar{n}_i, i \in I)$, which in particular satisfies

$$\sum_i e_i(\nu_i - \mu_i \bar{n}_i) > 0 \tag{19}$$

(analogously to the requirement that $\bar{x} \in \mathcal{L}$). For given $\bar{n}$, the equilibrium distribution $\bar{\pi}_{\bar{n}}$ of this process may be exactly determined as in Section 2. (The condition $\sum_{i \in I} e_i x_i = 1$, although natural in the context of that section, is not at all necessary to the analysis there.) We also require that, for all i, $\bar{n}_i$ is the expected number of calls of type i in the system under the equilibrium distribution $\bar{\pi}_{\bar{n}}$, hence

$$\nu_i \bar{P}_i(\bar{n}) = \mu_i \bar{n}_i, \quad \text{for all } i \in I, \tag{20}$$

where

$$\bar{P}_i(\bar{n}) = \sum_{m \geq e_i + r_i} \bar{\pi}_{\bar{n}}(m).$$

Finally, we require that

$$\sum_i e_i \bar{n}_i + \bar{m} = C, \tag{21}$$

where

$$\bar{m} = \sum_{m=0}^{C} m \bar{\pi}_{\bar{n}}(m)$$

is the expected free capacity under the equilibrium distribution $\bar{\pi}_{\bar{n}}$.

If the term $\bar{m}$ in equation (21) were replaced by 0, then the equations (19), (20) and (21) would imply that $\bar{n}/C$ was equal to the fixed point $\bar{x}$ of the limit process. The term $\bar{m}$ thus represents a correction reflecting the fact that, even in heavy traffic, the equilibrium proportion of unused capacity in the system is non-zero, and merely tends to zero under the limiting regime.

We expect that, at least for sufficiently large C, the equations (19), (20) and (21) will have a unique solution $\bar{n}$. We further expect, under the limiting regime where $C \to \infty$, that $\bar{m}/C \to 0$, $\bar{P}_i(\bar{n}) \to P_i(\bar{x})$ (where P_i is as defined by equation (8)), and $\bar{n}/C \to \bar{x}$. The distribution $\bar{\pi}_{\bar{n}}$ may be taken as approximating the equilibrium distribution of the process $m(\cdot)$, and, for each $i \in I$, $\bar{P}_i(\bar{n})$ may be taken as approximating the passing probability associated with the call type i.

In any case where the limiting process $x(\cdot)$ has more than one fixed point, we similarly expect that, for sufficiently large C, the equations (19), (20) and (21) will have multiple solutions $\bar{n}$ converging, as $C \to \infty$, to these fixed points. The corresponding distributions $\bar{\pi}_{\bar{n}}$ will then be quasi-equilibrium distributions in the sense of Section 1.

We refer to the above approximation scheme, which replaces each 'death rate' $\mu_i n_i(t)$ in equation (18) by $\mu_i \bar{n}_i$, as the *constant death rate approximation* (CDRA).

The process of solving equations (19), (20) and (21) is very similar to that of finding the fixed points of the limiting process. However, because of the presence of the strictly positive term $\bar{m}$, the passing probabilities are lower than those associated with the limiting process.

It is natural to attempt to improve the above approximation by replacing each death rate $\mu_i n_i(t)$ in equation (18) by $\mu_i \hat{n}_i(m)$ where $\hat{n}_i(m)$ is a function of m (rather than simply a constant) and where we require the obvious condition,

$$\sum_i e_i \hat{n}_i(m) + m = C, \quad \text{for all } m \geq 0, \tag{22}$$

to be satisfied. The simplest such approximation is given by taking

$$\hat{n}_i(m) = k(m)\bar{n}_i, \quad \text{for all } \; i \in I, \tag{23}$$

for some function k on $\mathbb{Z}_+$ which, given $\bar{n}$, is determined by equation (22). (This approximation is less than ideal and may certainly be improved, but appears to work well in practice.) We here require each $\bar{n}_i$ to be the equilibrium expected value of $n_i(\cdot)$ under the resulting equilibrium distribution $\hat{\pi}_{\bar{n}}$ of the process $m(\cdot)$, determined by the modified transition rates (18). More precisely we require,

$$\nu_i \hat{P}_i(\bar{n}) = \mu_i \bar{n}_i, \quad \text{for all } i \in I, \tag{24}$$

where

$$\hat{P}_i(\bar{n}) = \sum_{m \geq e_i + r_i} \hat{\pi}_{\bar{n}}.$$

We refer to this approximation, defined by equations (22), (23) and (24), as the *variable death rate approximation* (VDRA). When the limit process $x(\cdot)$ has a unique fixed point, we expect this approximation to yield a unique solution $\bar{n}$, that $\hat{\pi}_{\bar{n}}$ will then approximate the equilibrium distribution of the free capacity process $m(\cdot)$, and that, for each $i \in I$, $\hat{P}_i(\bar{n})$ will then approximate the corresponding passing probability. In the case where $x(\cdot)$ has several fixed points, we expect the approximation to yield multiple solutions corresponding to the quasi-equilibrium distributions of the processes $n(\cdot)$ and $m(\cdot)$.

Solving for the approximate equilibrium distribution $\hat{\pi}_{\bar{n}}$ of the process $m(\cdot)$ is not trivial, as this process is not necessarily skip free and has a large state space. However, Theorem 2.4 indicates that the asymptotic equilibrium distribution of the process $m(\cdot)$ has a geometric tail. Therefore, by approximating the tail of the distribution $\hat{\pi}_{\bar{n}}$ as geometric, we can reduce the complexity of finding this distribution. Choose a threshold M; for $m < M$, $\hat{\pi}_{\bar{n}}(m)$ is found by solving the global balance equations and for $m \geq M$ we assume that $\hat{\pi}_{\bar{n}}(m)$ has a geometric distribution. Theorem 2.4 suggests that the parameter of this distribution may reasonably be taken as the unique positive real root less than unity of the polynomial

$$f(p) = p^{\hat{e}} \left[\sum_i (\nu_i + \mu_i \hat{n}_i(M)) \right] - \sum_i \left[p^{\hat{e}+e_i} \nu_i + p^{\hat{e}-e_i} \mu_i \hat{n}_i(M) \right]. \tag{25}$$

A similar approach is followed in Tijms and Van de Coevering [21].

Note that the use of the geometric tail and the threshold value M is just an artifice required by finite computing power and is not an essential element of the VDRA. In practice we are usually able to choose M so large that the effect of this additional approximation is negligible.

5 Numerical Results

In this section we report some of our numerical results. Throughout our examples we consider two types of offered traffic, write $s = r_1 - r_2$ and hold at least one of r_1 or r_2 equal to zero. In all figures where simulations are used we also plot the relevant 99% confidence intervals. Figure 2 shows the case of a link in heavy traffic with parameters $\mu_1 = 1$, $\mu_2 = 2$, $e_1 = 1$, $e_2 = 2$, $\kappa_1 = 0.5$, $\kappa_2 = 0.6$ and $s = 0$, where, as usual, $\kappa = \nu/C$. (All parameter values are summarised in Table 1). The limiting dynamics for this example are those discussed in Section 3. We show the exact equilibrium call blocking probability for each traffic type as the link capacity C increases (a logarithmic scale is used). Since $s = 0$ this is readily computed as described in Section 1. Additionally, we show the asymptotic equilibrium blocking probabilities and the exact results can be seen to converge to their asymptotic values. We also show the results of our approximate methods. The CDRA is seen to give good accuracy only when C is large, whereas the more refined procedures of the VDRA give highly accurate results over the full range of capacities considered.

In Figure 3, we again consider a case where $s = 0$ and consequently where exact results are readily computed, but reduce the level of offered load, subject still to (2) holding. The results here are similar to those of the preceding example, except that the CDRA has poor accuracy for a greater range of capacities. In Figure 4 we look at the heavy load case of the example of Figure 2 but take $s = 2$. We therefore show simulations instead of the exact results. Notice that now the relative levels of the blocking of the two traffic types have been interchanged. This reflects the fact that acceptance of calls belonging to the first traffic stream is now restricted by the trunk reservation $r_1 = 2$. (The effect of further increasing this trunk reservation parameter would be to reduce towards zero the blocking probability for calls of type 2, at the expense of a further increase in the blocking probability for calls of type 1.) The behaviour of the two approximations is very similar to that of the example of Figure 2.

In Figures 5, 6 and 7 we look at three examples where C is held fixed and s is allowed to vary, in order to show clearly the effect of varying the trunk reservation

Figure	e_1	μ_1	κ_1	e_2	μ_2	κ_2	C	s
1, 2	1	1	0.5	2	2	0.6	$< 10^5$	0
3	1	1	0.5	2	2	0.501	$< 10^5$	0
4	1	1	0.5	2	2	0.6	$< 10^5$	2
5	1	1	0.5	2	2	0.6	1000	$[-20, 20]$
6	1	1	0.5	2	2	0.501	1000	$[-20, 20]$
7	1	1	0.2	30	1/30	0.001	10000	$[-20, 50]$

Table 1: Parameter values used for the examples.

parameters. Figures 5 and 6 show the results for the same cases as in Figures 2 and 3 respectively except that the capacity C is held fixed at 1000 and s is allowed to vary in the range -20 to $+20$. We see that in the heavy load case of Figure 5 both the CDRA and VDRA procedures are accurate over the full range of values of s. However, when the load is reduced, as shown in Figure 6, only the VDRA is accurate over the full range of values of s.

Finally, in Figure 7 we consider an example where the two traffic streams have very different characteristics, given by $\nu_1 = 2000$, $\mu_1 = 1$, $e_1 = 1$, and by $\nu_2 = 10$, $\mu_2 = 1/30$ and $e_2 = 30$. Thus calls of the second type have both much greater capacity requirements and much longer holding times, corresponding to the very varied traffic mix which may be found in integrated networks. We also take $C = 10000$, so that the link is in heavy traffic. Figure 7 shows that our approximation procedures continue to provide accurate estimates for the call blocking probabilities.

In summary, we have observed that the CDRA procedure is accurate for a wide range of trunk reservation parameters when the load is high and the link capacity is large. The VDRA procedure has been found to give accurate results in all the cases considered, even when the load is reduced and link capacities are small. The approximate methods have also been found to be appropriate even when the traffic streams have quite widely differing parameter values, which is expected to be the case in future integrated communication networks.

The CDRA procedure has an appealing simplicity, which was justified in Section 4 under the assumptions of reasonably large capacity C and heavy traffic, and where we argued for its asymptotic correctness. (In the case where $r_i = 0$ for all $i \in I$, it follows from, for example, Lemma 4.5 of Zachary [23], that under the conditions (1) and (2), the error in the blocking probabilities as determined by the CDRA is $o(C^\alpha)$ for all $\alpha > -2$; we conjecture that a similar result holds in the more general case $r_i \geq 0$.) However, the assumption of a constant 'death rate' is only appropriate under the above conditions, and is not at all justified in other circumstances. By contrast, the basic approximation underlying the VDRA, namely the relation (23), still appears reasonable (in the absence of a full I-dimensional analysis) even when the heavy traffic condition does not hold, and for networks of moderate to large size (say $C \geq 50$), provided only that the trunk reservation parameters are small in relation to the capacity of the system. Our above numerical results are therefore not unexpected, and in general the VDRA procedure is to be preferred.

Appendix

Proof of Lemma 2.2: Trivially $f(1) = 0$. Define the function $g : (0, \infty) \to \mathbb{R}$ by $g(y) = y^{-\hat{e}} f(y)$. Then g is strictly concave and satisfies $g(1) = 0$, $g'(1) < 0$, since $x \in \mathcal{L}$. Further $g(y) \to -\infty$ as $y \to 0$ or $y \to \infty$. Hence the equation $g(y) = 0$, and so also the equation $f(y) = 0$, has exactly one further strictly positive real root p_1, and further $p_1 < 1$.

If $z \in \mathbb{C} \setminus \mathbb{R}_+$ ($\mathbb{R}_+$ the set of nonnegative real numbers) satisfies $|z| = p_1$ or $|z| = 1$ then, since the greatest common divisor of the e_i, $i \in I$, is 1, it follows that $\mathcal{R}e(g(z)) > 0$. Hence z cannot be a root of the polynomial f. Now define polynomials $f_1 : \mathbb{C} \to \mathbb{C}$ and $f_2 : \mathbb{C} \to \mathbb{C}$ by

$$f_1(z) = z^{\hat{e}} \sum_i (\kappa_i + \mu_i x_i), \quad f_2(z) = \sum_i \left(z^{\hat{e}+e_i} \kappa_i + z^{\hat{e}-e_i} \mu_i x_i \right).$$

Note that $f = f_1 - f_2$, and that, for all $z \in \mathbb{C}$, $|f_1(z)| = f_1(|z|)$ and $|f_2(z)| \leq f_2(|z|)$. For any real $a \in (p_1, 1)$, define $\gamma(a) = \{z \in \mathbb{C} : \ |z| = a\}$. Then

$$|f_1(z)| - |f_2(z)| \geq f(a) > 0 \quad \text{for all } z \in \gamma(a),$$

by the above results for f_1, f_2 and the concavity of the function g. It now follows that if $b \in \mathbb{R}_+$ is chosen such that $0 < b < 2f(a)$,

$$|f_1(z) - f_2(z) - bz^{\hat{e}}| < |f_1(z) - f_2(z)| \quad \text{for all } z \in \gamma(a).$$

(This inequality is geometrically clear, and may formally be established, in squared form, by recalling the definition of $f_1(z)$, expressing $z^{\hat{e}}$ and $f_2(z)$ in terms of real and imaginary parts, and elementary manipulations.) Thus, by Rouché's Theorem (Rudin [20, Theorem 10.36, p218]) f has exactly $\hat{e}$ roots with modulus less than a. Since this is true for all real $a \in (p_1, 1)$ the result follows. ∎

Proof of Lemma 2.3: Since, by definition, any $\pi \in \Pi$ satisfies the recurrence relations (14), it has the usual spectral representation in terms of the roots of the polynomial f. Since $|p_k| \geq 1$ for $k = \hat{e}+1, \ldots, 2\hat{e}$, the finiteness of the measure π implies that this representation reduces to the form (15). Further, since

$$a_1(\pi) = \lim_{m\to\infty} p_1^{-m}\pi(m), \tag{26}$$

it follows that $a_1(\pi) \geq 0$.

We now show that $a_1(\pi) > 0$ for all $\pi \in \Pi$. Define $E = \{m \in \mathbb{Z}_+ : \ m < \hat{e}\}$. Observe that there exists a unique $\pi \in \Pi$ with any given (non-negative) values of $\pi(m)$, $m \in E$, and that $\pi(m) > 0$ for all $m \geq \hat{e}$. (This follows from the result that, if (q_{ij}) is the matrix of transition rates associated with an ergodic Markov process on a discrete state space S, and A is a finite subset of S, then there exists a unique finite measure π on S satisfying the equations $\sum_{i\in S}\pi_i q_{ij} = \pi_j \sum_{i\in S} q_{ji}$ for all $j \in S \setminus A$ and having prescribed values $\pi_j \geq 0$ for all $j \in A$. This result is proved by considering the obvious modified transition matrix on the modified state space given by replacing A by a single state.) In particular, for each $k \in E$, let $\pi_k \in \Pi$ be defined by $\pi_k(k) = 1$, $\pi_k(m) = 0$ for $m \in E \setminus \{k\}$. Then any $\pi \in \Pi$ has a unique representation $\pi = \sum_{k\in E}\xi_k\pi_k$, and

$$a_1(\pi) = \sum_{k\in E}\xi_k a_1(\pi_k). \tag{27}$$

Let $\hat{\pi}$ be defined by $\hat{\pi}(m) = p_1^m$ for all $m \in \mathbb{Z}_+$. Then $\hat{\pi} \in \Pi$ and $\hat{\pi} = \sum_{k\in E} p_1^k \pi_k$. Since $a_1(\hat{\pi}) = 1$, it follows from (27) that $a_1(\pi_k) > 0$ for at least one $k \in E$.

Now consider again any $\pi \in \Pi$. If $\pi(m) > 0$ for all $m \in E$, it follows that, in the above representation of π, $\xi_k > 0$ for all $k \in E$ and so, from the above result and (27), that $a_1(\pi) > 0$. Otherwise, the measure π' defined by $\pi'(m) = \pi(m+\hat{e})$ for all $m \geq 0$ belongs to Π and takes strictly positive values everywhere. Thus $a_1(\pi') > 0$ and so by (26) it follows again that $a_1(\pi) > 0$.

The final statement of the lemma follows trivially by the general theory of recurrence relations. ■

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References

[1] Bean, N. G., Gibbens, R.J. and Zachary, S. (1994) The Performance of Single Resource Loss Systems in Multiservice Networks. In *Proceedings of 14th International Teletraffic Congress.*

[2] Bean, N. G., Gibbens, R. J. and Zachary, S. (1994) In preparation.

[3] Burman, D. Y., Lehoczky, J. P. and Lim, Y. (1984) Insensitivity of Blocking Probabilities in a Circuit-Switched Network. *J. Appl. Prob.* **21,** 850–859.

[4] Durrett, R. (1991) *Probability: Theory and Examples.* Wadsworth & Brooks/Cole Statistics/Probability Series, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, California.

[5] Dziong, Z. and Roberts, J. W. (1987) Congestion Probabilities in a Circuit Switched Integrated Services Network. *Performance Evaluation* **7,** 267–284.

[6] Gersht, A. and Lee, K. J. (1990) A Bandwidth Management Strategy in ATM Networks. Pre-print, GTE Laboratories, 40 Sylvan Road, Waltham, MA 02254, USA.

[7] Gibbens, R. J. and Hunt, P. J. (1991) Effective Bandwidths for the Multi-type UAS Channel. *Queueing Systems* **9,** 17–28.

[8] Greenberg, A., Mitrani I. and Stolyar A. (1992) Preliminary draft.

[9] Heuser, H. G. (1982) *Functional Analysis.* Wiley, New York.

[10] Hui, J. Y. (1988) Resource Allocation for Broadband Networks. *IEEE J.S.A.C.* **SAC-6,** 1598–1608.

[11] Hunt, P. J. (1990) *Limit Theorems for Stochastic Loss Networks.* Ph.D. Dissertation, University of Cambridge.

[12] Hunt, P. J. and Kurtz, T. G. (1992) *Large Loss Networks.* Research Report No. 92-16, Statistical Laboratory, The University of Cambridge, 16 Mill Lane, Cambridge, CB2 1SB, U.K.

[13] Johnsonbaugh, R. and Pfaffenberger, W. E. (1981) *Foundations of Mathematical Analysis.* Pure and Applied Mathematics, A series of Monographs and Textbooks., Vol. 62, Marcel Dekker, Inc, New York.

[14] Kaufman, J. S. (1981) Blocking in a Shared Resource Environment. *IEEE Trans. Comm.* **COM-29,** 1474–1481.

[15] Kelly, F. P. (1986) Blocking Probabilities in Large Circuit Switched Networks. *Adv. Appl. Prob.* **18,** 473–505.

[16] Kelly, F. P. (1989) Fixed Point Models of Loss Networks. *J. Austral. Math. Soc. Ser. B* **31,** 204–218.

[17] Kelly, F. P. (1991) Effective Bandwidths at Multi-Class Queues. *Queueing Systems* **9,** 5–16.

[18] Kelly, F. P. (1991) Loss Networks. *Ann. Appl. Prob.* **1,** 319–378.

[19] Key, P. B. (1990) Optimal Control and Trunk Reservation in Loss Networks. *Prob. Eng. Inf. Sci.* **4,** 203–242.

[20] Rudin, W. (1970) *Real and Complex Analysis.* McGraw-Hill Series in Higher Mathematics, McGraw-Hill, London.

[21] Tijms, H. C. and Van de Coevering, M. C. T. (1991) A Simple Numerical Approach for Infinite-State Markov Chains. *Prob. Eng. Inf. Sci.* **5,** 285–296.

[22] Tran-Gia, P. and Hübner, F. (1993) An Analysis of Trunk Reservation and Grade of Service Balancing Mechanisms in Multiservice Broadband Networks. In *IFIP Workshop TC6, Modelling and Performance Evaluation of ATM Technology*, La Martinique, French Carribean Island. 25-27 January.

[23] Zachary, S. (1991) On Blocking in Loss Networks. *Adv. Appl. Prob.* **23,** 355–372.

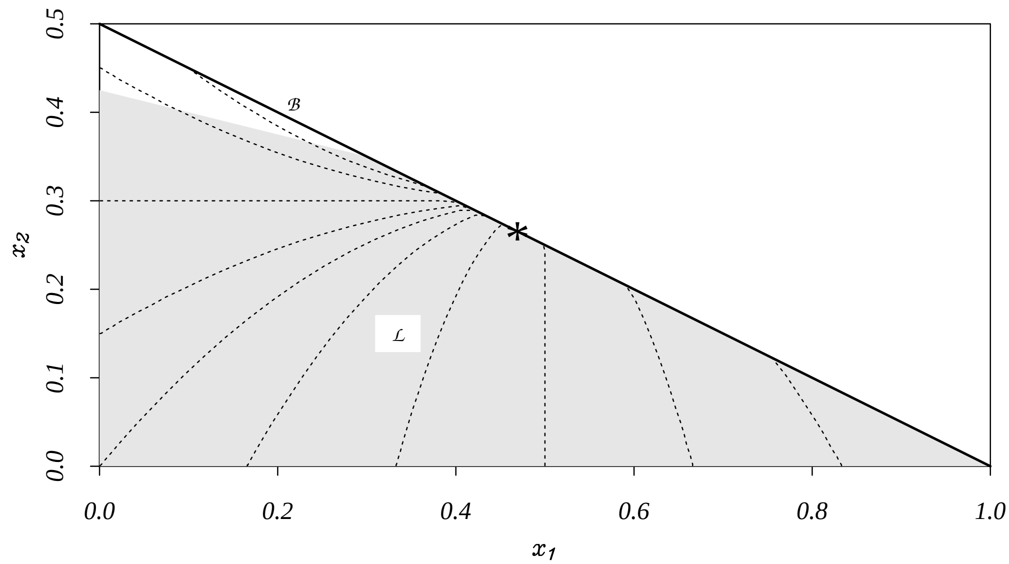

Figure 1: Sample Trajectories for the Limiting Process.

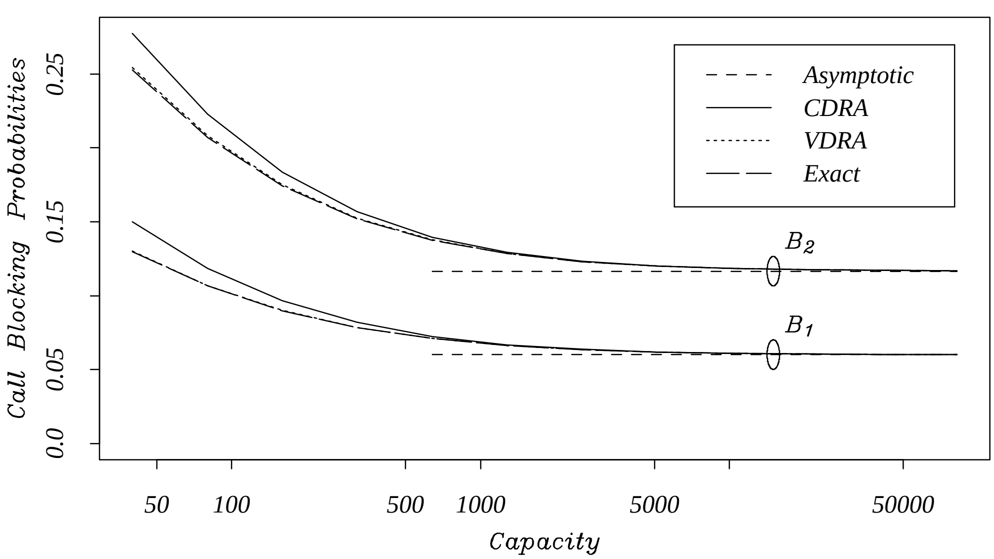

Figure 2: Comparison of techniques as C increases.

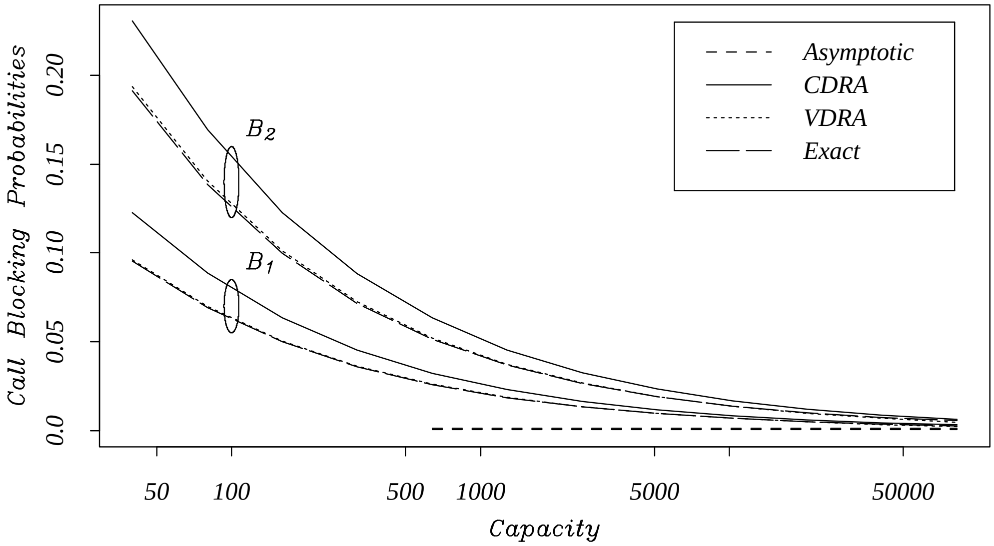

Figure 3: Comparison of techniques as C increases, in lighter traffic.

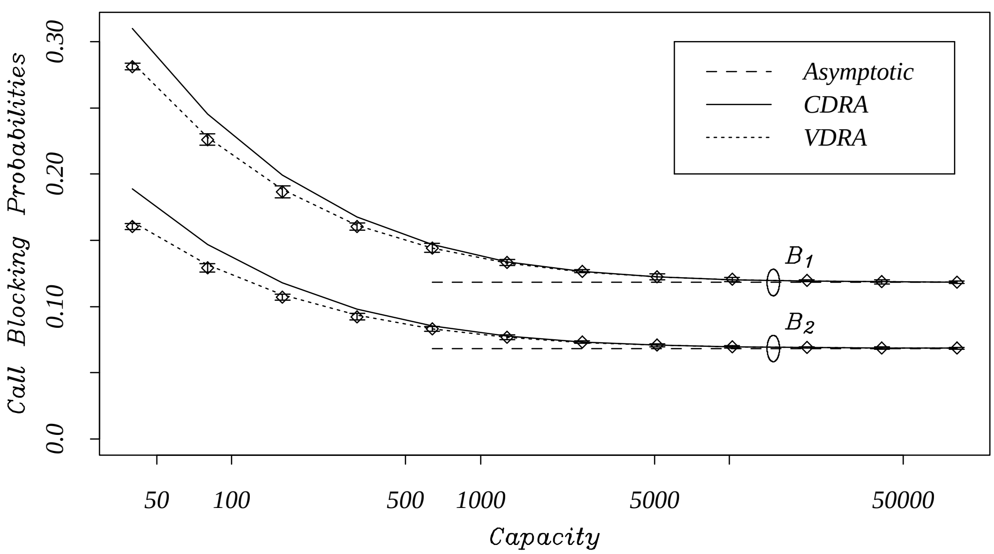

Figure 4: Comparison of techniques as C increases, for non-zero trunk reservation.

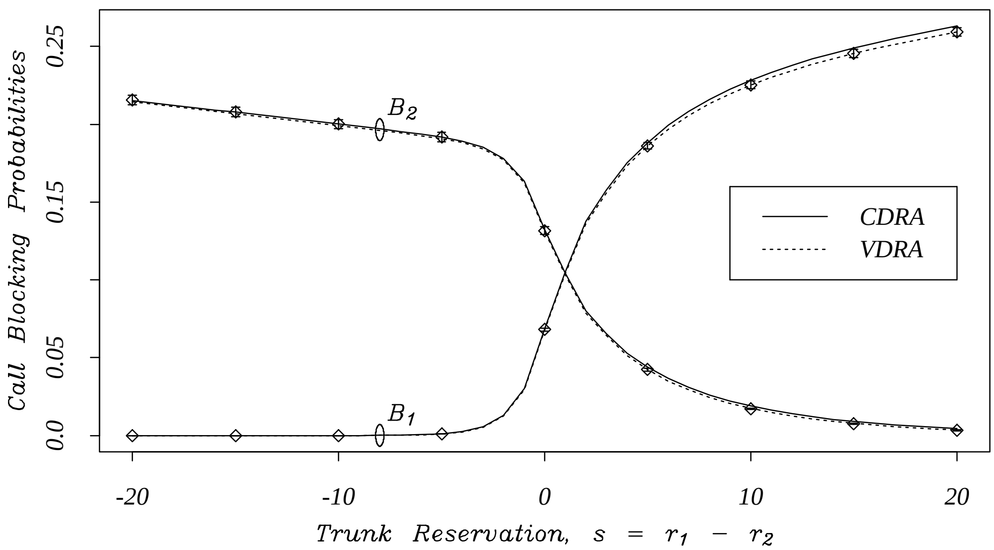

Figure 5: Comparison of techniques as the trunk reservation varies

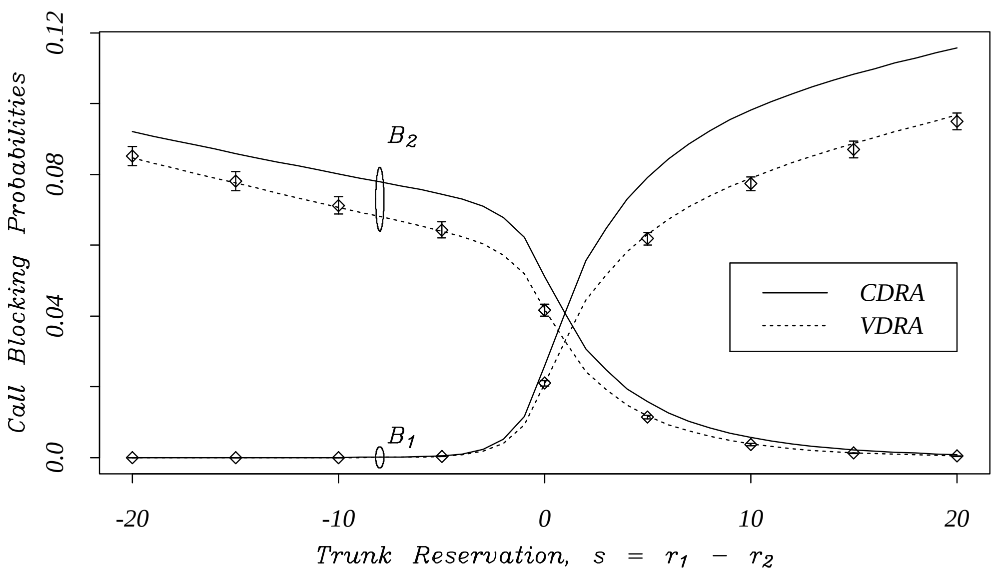

Figure 6: Comparison of techniques as the trunk reservation varies

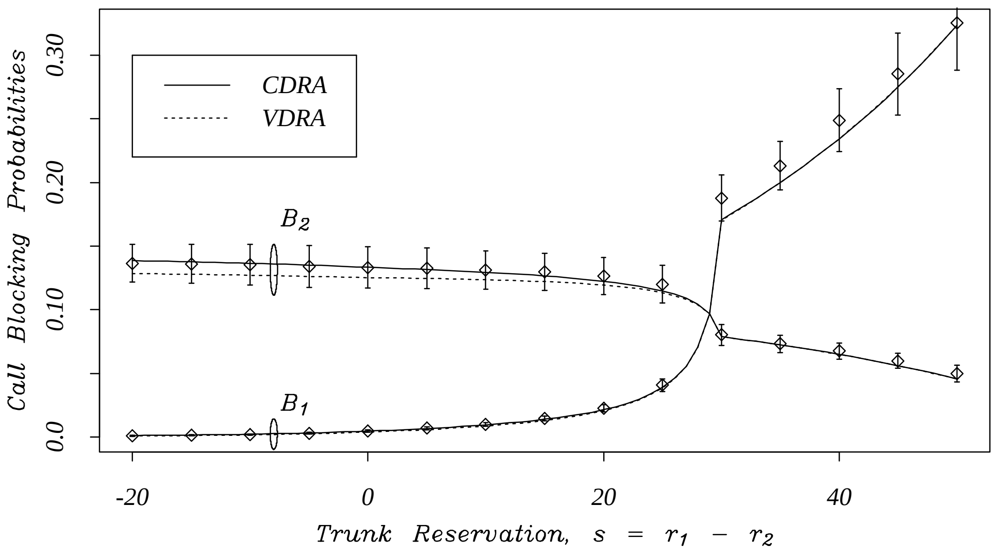

Figure 7: Comparison of techniques as the trunk reservation varies