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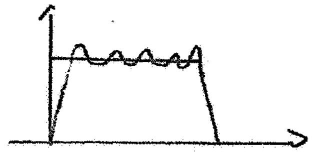
1 (a) $f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx$

where $b_n = \frac{2}{\pi}\int_0^{\pi} \sin nx \, dx$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_{x=0}^{x=\pi} = \frac{2}{\pi n}(1-\cos n\pi) = \begin{cases} \frac{4}{n\pi} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

6

5 Thus $f(x) \sim \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$



1

(b) $-y'' = \lambda y \qquad y(0)=0 \qquad y(\pi)+y'(\pi)=0$

Suppose $\lambda<0$. Then $\lambda = -k^2$ where $k>0$

$y'' = k^2 y$ has general solution $y(x) = A\cosh kx + B\sinh kx$

$y(0)=0 \Rightarrow A=0$ and so $y(x) = B\sinh kx$.

Hence $y(\pi)+y'(\pi)=0 \Leftrightarrow B\sinh k\pi + Bk\cosh k\pi = 0 \Rightarrow B=0$

4 Thus we must have $A=B=0$ and so $\lambda<0$ is not an eigenvalue.

Suppose $\lambda=0$

$y''=0$ has general solution $y = Ax+B$

$y(0)=0 \Rightarrow B=0$ and so $y=Ax$

Hence $y(\pi)+y'(\pi)=0 \Leftrightarrow A\pi + A = 0 \Rightarrow A=0$

2 Thus $\lambda=0$ is not an eigenvalue.

Suppose $\lambda>0$. Then $\lambda=k^2$ where $k>0$

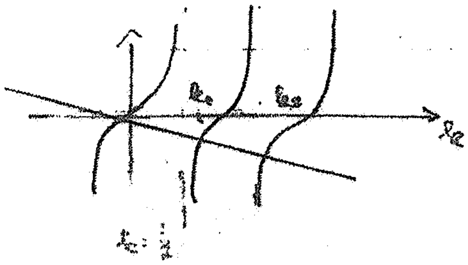
14 $y'' = -k^2 y$ has general solution $y = A\cos kx + B\sin kx$

$y(0)=0 \Leftrightarrow A=0$ and so $y = B\sin kx$

$y(\pi)+y'(\pi)=0 \Leftrightarrow B\sin k\pi + kB\cos k\pi$

$\Leftrightarrow B=0$ or $\sin k\pi + k\cos k\pi = 0 \Leftrightarrow B=0$ or $\tan k\pi = -k$

The graph below shows that $\tan k\pi = -k$ has infinitely many solutions $k_1, k_2, \ldots k_n$.



5

d Hence there are infinitely many eigenvalues $\lambda_n = k_n^2$ with corresponding eigenfunctions $\sin k_n x$.

If f has eigenfunction expansion $f(x) \sim \sum_{n=1}^{\infty} c_n \sin k_n x$

then $c_n = \int_0^{\pi} \sin k_n x \, dx \Big/ \int_0^{\pi} \sin^2 k_n x \, dx$

$\int_0^{\pi} \sin k_n x \, dx = -\frac{1}{k_n} \cos k_n x \Big|_0^{\pi} = \frac{1}{k_n}(1 - \cos k_n \pi)$

$\int_0^{\pi} \sin^2 k_n x \, dx = \frac{1}{2}\int_0^{\pi}(1 - \cos 2k_n x)\,dx = \frac{1}{2}\left(\pi - \frac{1}{2k_n}\sin 2k_n x \Big|_0^{\pi}\right.$

$= \frac{1}{2}\left(\pi - \frac{1}{k_n}\sin k_n\pi \cos k_n \pi\right) = \frac{1}{2}(\pi + \cos^2 k_n \pi)$

3 Hence $c_n = \frac{2}{k_n} \frac{(1 - \cos k_n \pi)}{\pi + \cos^2 k_n \pi)}$.

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2. (a) $u_x + 2x u_y = u^2$; $u(0,y) = y$

Characteristics are solutions of $\frac{dy}{dx} = 2x$ i.e $y = \int 2x \, dx$

2 i.e $y = x^2 + c$.

Let $(x_0, y_0) \in \mathbb{R}^2$. The characteristic through (x_0, y_0) is

1 $y = x^2 + (y_0 - x_0^2)$ which intersects the initial curve $(x=0)$ where

1 $y = y_0 - x_0^2$.

Let $v(x) = u(x, y(x))$ where y is the above characteristic

Then

1 $\frac{dv}{dx} = u_x + u_y \frac{dy}{dx} = u_x + 2x u_y = u^2 = v^2$

1 Hence $\frac{dv}{v^2} = dx \Rightarrow -\frac{1}{v} = x + c \Rightarrow v = -\frac{1}{x+c}$

1 Also $v(0) = u(0, y_0 - x_0^2) = y_0 - x_0^2$

1 Hence $-\frac{1}{c} = y_0 - x_0^2$, i.e. $c = \frac{1}{x_0^2 - y_0}$

Thus $u(x_0, y_0) = v(x_0) = -\frac{1}{x_0 + \frac{1}{x_0^2 - y_0}} = \frac{y_0 - x_0^2}{x_0^3 - x_0 y_0 + 1}$

10 Hence $u(x,y) = \frac{y - x^2}{x^3 - xy + 1}$.

(b) $u_x + 2x u_y = u$

We seek a solution of the form $u(x,y) = X(x) Y(y)$

Thus we require

$$X'(x) Y(y) + 2x X(x) Y'(y) = XY$$

i.e

3 $$\frac{X'(x)}{X(x)} + 2x \frac{Y'(y)}{Y(y)} = 1$$

+ Some attempt 4

i.e

$$2 \frac{Y'(y)}{Y(y)} = \frac{1}{x} - \frac{1}{x} \frac{X'(x)}{X(x)}$$

1 Thus it suffices to find X and Y satisfying

$$2 \frac{Y'(y)}{Y(y)} = 1 \quad \text{and} \quad \frac{1}{x} - \frac{1}{x} \frac{X'(x)}{X(x)} = 1$$

3 $2 \frac{Y'}{Y} = 1 \Rightarrow Y' = \frac{1}{2} Y \Rightarrow Y(y) = C e^{\frac{1}{2} y}$

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2ct $\frac{1}{x} - \frac{1}{x}\frac{X'(x)}{X(x)} = 1 \implies \frac{X'(x)}{X(x)} = (1-x)$

$\implies X' = (1-x)X$ - linear/integrating factor type

Integrating factor $= \exp\int(x-1)\,dx = e^{\frac{1}{2}x^2 - x}$

Thus we require $\frac{d}{dx}\left(e^{\frac{1}{2}x^2-x}\right)X = 0 \implies X(x) = C\,e^{x-\frac{1}{2}x^2}$

10 Thus we obtain solution $u(x,y) = e^{x-\frac{1}{2}x^2} \cdot e^{2y} = e^{x+\frac{1}{2}y-\frac{1}{2}x^2}$

3) $u_{tt} = u_{xx} \quad 0 < x < 1, \quad t > 0$

We seek solutions of the form $u(x,t) = X(x)T(t)$

Since ends are fixed at same horizontal level we require $X(0) = 0 = X(1)$

Substituting into the equation we require

$$T''(t)X(x) = T(t)X''(x)$$

Thus it suffices to find X and T such that

$$\frac{X''}{X} = \frac{T''}{T} = -k \quad \text{where } k \text{ is a constant}$$

i.e

$$-X'' = kX \qquad X(0) = 0 = X(1) \qquad (1)$$

5 $$-T'' = kT \qquad (2)$$

(1) has non zero solutions iff $k = n^2\pi^2$ with corresponding solutions

1 $\sin n\pi x$ for $n = 1, 2, \ldots$

1 If $k = n^2\pi^2$, (2) has general solution $T(t) = A_n \cos n\pi t + B_n \sin n\pi t$

Hence any function of the form

1 $$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos n\pi t + B_n \sin n\pi t) \sin n\pi x$$

(4) [illegible] boundary conditions.

If $u(x,0) = \sin 2\pi x$ we require $\sin 2\pi x = \sum_{n=1}^{\infty} A_n \sin n\pi x$

2 Thus [illegible] $A_2 = 1$ and $A_n = 0$ for $n \neq 2$.

Now

$$u_t(x,t) = \sum_{n=1}^{\infty} (-n\pi A_n \sin n\pi t + n\pi B_n \cos n\pi t) \sin n\pi x$$

Since $u_t(x,0) = x$ we require

$$x = \sum_{n=1}^{\infty} n\pi B_n \sin n\pi x$$

i.e

$$n\pi B_n = \frac{2}{1}\int_0^1 x \sin n\pi x \, dx$$

$$= 2\left[-\frac{1}{n\pi} x \cos n\pi x \Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi x \, dx \right]$$

$$= -\frac{2}{n\pi}\cos n\pi = (-1)^{n+1}\frac{2}{n\pi}$$

4 i.e $B_n = (-1)^{n+1}\dfrac{2}{n^2\pi^2}$.

(b) $$F(t) = \frac{1}{2}\int_0^1 [u_x^2(x,t) + u_t^2(x,t)] \, dx$$

Bookwork Hence $\dfrac{dF}{dt} = \displaystyle\int_0^1 [u_x(x,t)\, u_{xt}(x,t) + u_t(x,t)\, u_{tt}(x,t)] \, dx$

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3 ctd $\qquad = \quad u_x(x,t)\, u_t(x,t) \Big|_{x=0}^{x=1} - \int_0^1 u_{xx}(x,t)\, u_t(x,t)\, dx + \int_0^1 u_t(x,t)\, u_{tt}(x,t)\, dx$

Since $u(0,t) \equiv 0$, $u_t(0,t) = 0$; similarly $u_t(1,t) \equiv 0$

Hence $\frac{dE}{dt} = \int_0^1 [u_{tt}(x,t) - u_{xx}(x,t)]\, u_t(x,t)\, dx \equiv 0$ as $u_{tt} \equiv u_{xx}$

Thus $E(t) \equiv E(0) = 0$ as $u(x,0) \equiv u_t(x,0) \equiv 0$

Hence $\int_0^1 [u_x^2(x,t) + u_t^2(x,t)]\, dx \equiv 0$

Thus $u_x(x,t) \equiv u_t(x,t) \equiv 0$ and so $u(x,t) \equiv$ constant

6 Hence as $u(x,0) \equiv 0$, $u(x,t) \equiv 0$

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4) $u_t = u_{xx} \quad 0 < x < \pi \quad t > 0 \ ; \quad u(0,t) = u(\pi,t) = 0$

We seek solutions of the form $u(x,t) = X(x)\,T(t)$

To ensure that $u(0,t) = u(\pi,t) = 0$ we require $X(0) = 0 = X(\pi)$.

$$u_t = u_{xx} \implies X(x)\,T'(t) = X''(x)\,T(t)$$

$$\implies \frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = -k \quad \text{where } k \text{ is a constant}$$

ie
$$-X'' = kX \qquad X(0) = 0 = X(\pi) \qquad (1)$$

5 $$T' = -kT \qquad (2)$$

(1) has non-zero solutions iff $k = n^2$ for $n = 1, 2, \dots$ with solutions $\sin nx$.

1 If $k = n^2$, (2) has general solution $T(t) = A_n e^{-n^2 t}$

Hence any function of the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin nx \, e^{-n^2 t}$$

2. is a solution of the PDE satisfying the boundary conditions

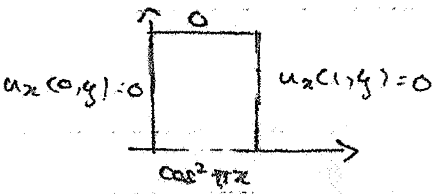
Since $u(x,0) = \sin x \cos x = \frac{1}{2}\sin 2x$ we require

$$\tfrac{1}{2}\sin 2x = \sum_{n=1}^{\infty} A_n \sin nx$$

and so we choose $A_2 = \frac{1}{2}$ and $A_n = 0$ for $n \neq 2$

2. Hence solution is $u(x,t) = \frac{1}{2} e^{-4t} \sin 2x$

(b).



$$u_{xx} + u_{yy} = 0$$

We seek solutions of the form $u(x,y) = X(x)\,Y(y)$.

Since $u_x(0,y) = 0 = u_x(1,y)$, we require $X'(0) = 0 = X'(1)$

Since $u(x,2) = 0$ we require $Y(2) = 0$

$$u_{xx} + u_{yy} = 0 \implies X''(x)\,Y(y) + X(x)\,Y''(y) = 0$$

$$\implies \frac{X''}{X} = -\frac{Y''}{Y} = -k \quad \text{where } k \text{ is a constant}$$

i.e

$$X'' = -kX \qquad X'(0) = 0 = X'(1) \qquad (1)$$

5 $$Y'' = kY \qquad Y(2) = 0 \qquad (2)$$

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4 ctd ① has non zero solutions iff $k = 0, \pi^2, 4\pi^2, \dots$ and the corresponding

1 solutions are $1, \cos \pi x, \cos 2\pi x, \dots$

If $k = 0$. ② has general solution $Y(y) = A(2-y)$

1 If $k = 1, 2 \dots$ ② has general solution $Y(y) = A \sinh n\pi (2-y)$

Thus we have the general solution

3 $$u(x,y) = A_0 (2-y) + \sum_{n=1}^{\infty} A_n \cos n\pi x \sinh n\pi (2-y)$$

1) If $u(x,0) = \cos^2 \pi x = \frac{1}{2}(\cos 2\pi x + 1)$ we require

$$\tfrac{1}{2}(\cos 2\pi x + 1) = A_0 2 + \sum_{n=1}^{\infty} A_n \sinh 2n\pi \cos n\pi x$$

Thus we choose $A_0 = \frac{1}{4}$, $A_2 = \frac{1}{2 \sinh 4\pi}$, $A_n = 0$, $n \neq 0, 2$

Thus we have solution

3 $$u(x,y) = \frac{1}{4}(2-y) + \frac{1}{2 \sinh 4\pi} \cos 2\pi x \sinh 2\pi (2-y)$$

5(a) $x'(t) = x(t) - 2y(t)$; $y'(t) = 2x(t) - 3y(t)$; $x(0) = 1$; $y(0) = 0$

Taking Laplace transforms we obtain

$s\bar{x}(s) - x(0) = \bar{x}(s) - 2\bar{y}(s)$ i.e $(s-1)\bar{x}(s) + 2\bar{y}(s) = 1$ (1)

4 $s\bar{y}(s) - y(0) = 2\bar{x}(s) - 3\bar{y}(s)$ i.e. $2\bar{x}(s) - (s+3)\bar{y}(s) = 0$ (2)

$2 \times (1) - (s-1) \times (2)$ gives $(4 + (s-1)(s+3))\,\bar{y}(s) = 2$

8 i.e $(s^2 + 2s + 1)\,\bar{y}(s) = 2$

i.e $\bar{y}(s) = \dfrac{2}{(s+1)^2}$

2 Hence $y(t) = 2te^{-t}$

Also $(s+3) \times (1) + 2 \times (2)$ gives $((s+3)(s-1) + 4)\,\bar{x}(s) = s+3$

i.e $(s^2 + 2s + 1)\,\bar{x}(s) = s + 3$

i.e $\bar{x}(s) = \dfrac{s+3}{(s+1)^2} = \dfrac{s+1}{(s+1)^2} + \dfrac{2}{(s+1)^2} = \dfrac{1}{s+1} + \dfrac{2}{(s+1)^2}$

2 Hence $x(t) = e^{-t} + 2te^{-t}$

(b) $x'(t) = x(t) - 3y(t)$; $y'(t) = 2x(t) - 4y(t)$

System may be written as

$$\underline{x}' = A\underline{x} \quad \text{where } A = \begin{pmatrix} 1 & -3 \\ 2 & -4 \end{pmatrix}$$

λ is an eigenvalue of A iff $\begin{vmatrix} 1-\lambda & -3 \\ 2 & -4-\lambda \end{vmatrix} = 0$

i.e, $(1-\lambda)(-4-\lambda) + 6 = 0$ i.e $\lambda^2 + 3\lambda + 2 = 0$

3 i.e $(\lambda+1)(\lambda+2) = 0$ i.e $\lambda = -1, -2$

(12) $\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -1$ if

$x - 3y = -x$ i.e $2x - 3y = 0$

$2x - 4y = -y$

2 Thus $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -1$

$\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -2$ if

$x - 3y = -2x$ i.e $3x - 3y = 0$ i.e $x = y$

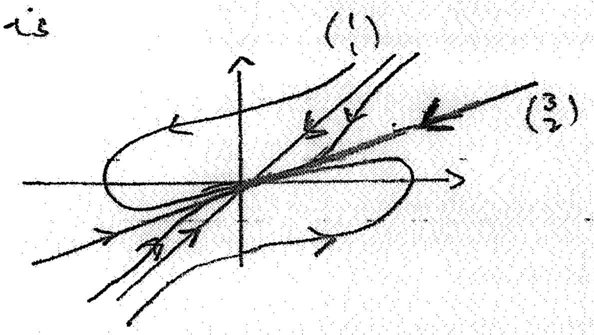
$2x - 4y = -2y$ $\quad$ $2x - 2y = 0$

2 Hence $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -2$

7(b) ctd General solution is

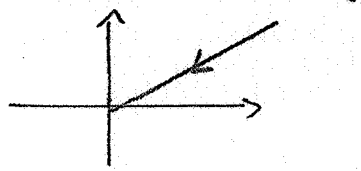
7 $$\underline{x}(t) = c_1 e^{-t}\begin{pmatrix}3\\2\end{pmatrix} + c_2 e^{-2t}\begin{pmatrix}1\\1\end{pmatrix}$$

Phase plane is



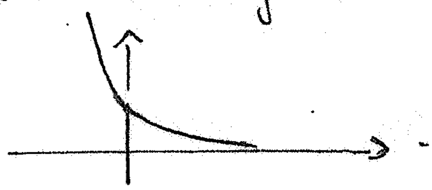
3

The point $(1,1)$ lies on trajectory



and so if $x(0) = y(0) = 1$, $x(t) = y(t)$ for all t

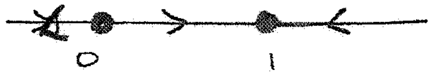
Hence graph of both x and y is



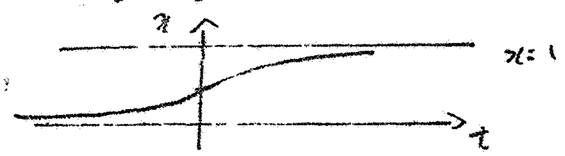
2

6 (a) $x' = x - x^2 = x(1-x)$

Since $x - x^2 > 0$ for $0 < x < 1$ and $x - x^2 < 0$ for $x < 0$ and $x > 1$ we have phase plane



Hence solution satisfying $x(0) = \frac{1}{2}$ has graph



5

(b) Equation may written as the system $x' = y$, $y' = x - x^2$

Then any trajectory of the form $y = y(x)$ must satisfy

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{x - x^2}{y}$$

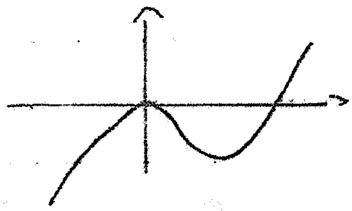
i.e. $y\,dy = (x - x^2)\,dx$ i.e. $\frac{1}{2}y^2 + \frac{1}{3}x^3 - \frac{1}{2}x^2 = c$

5 i.e. $y^2 + \frac{2}{3}x^3 - x^2 = c$

Let $f(x) = \frac{2}{3}x^3 - x^2$; then $f'(x) = 2x^2 - 2x = 2x(x-1)$

Hence f is increasing for $x < 0$, decreasing for $0 < x < 1$ and increasing again for $x > 1$.

Also $f(0) = f(\frac{3}{2}) = 0$ and so f has graph



2

Clearly system has equilibrium points at $(0,0)$ and $(1,0)$

Consider the trajectory in the first quadrant through $(0, a)$ where $a \geq 0$

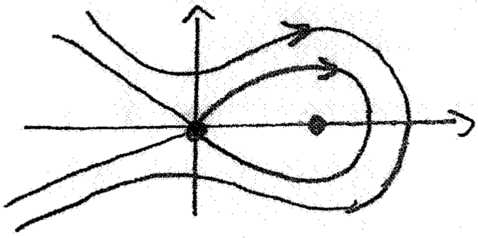
i.e. $y^2 + f(x) = a^2$

As x increases from 0 to 1, $f(x)$ decreases and so y^2 increases

As x increases beyond 1, $f(x)$ increases and y^2 decreases until $y^2 = 0$ where $f(x) = a$.

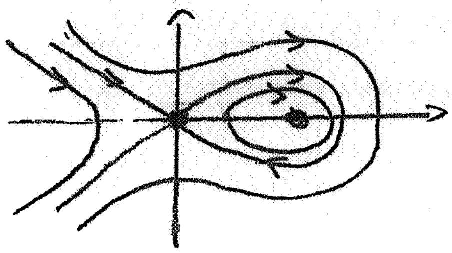
As x becomes negative in third quadrant, $f(x)$ decreases and so y^2 increases

6(c)(i) As equation of trajectory is even in y we can reflect trajectories in x axis to obtain



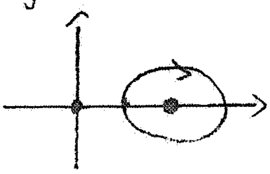
Using similar arguments for the trajectories through points $(a,0)$ we obtain phase plane

(15)

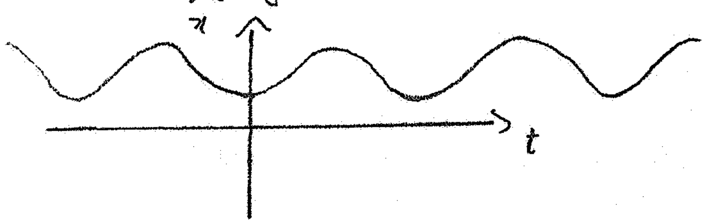


7

Trajectory containing $(\frac{1}{2},0)$ is



Hence solution satisfying $x(0)=\frac{1}{2}$ $\dot{x}(0)=0$ has graph



3

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7(a) $x' = x - y \qquad y' = 1 - 4xy$.

(x,y) is an equilibrium point if

$$x - y = 0 \quad (1)$$
$$1 - 4xy = 0 \quad (2)$$

(1) is satisfied iff $x = y$; hence (2) is also satisfied if $1 = 4x^2$

4 i.e $x = \pm\frac{1}{2}$. Hence equilibrium points are $(-\frac{1}{2}, -\frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$

If $f(x,y) = x - y$, $\frac{\partial f}{\partial x}(x,y) = 1$, $\frac{\partial f}{\partial y}(x,y) = -1$

If $g(x,y) = 1 - 4xy$, $\frac{\partial g}{\partial x}(x,y) = -4y$, $\frac{\partial g}{\partial y}(x,y) = -4x$

Hence linearized equation at $\frac{1}{2}, \frac{1}{2}$ is $\underline{x}' = \begin{pmatrix} 1 & -1 \\ -2 & -2 \end{pmatrix} \underline{x} = A\underline{x}$

λ is an eigenvalue of A iff $\begin{vmatrix} 1-\lambda & -1 \\ -2 & -2-\lambda \end{vmatrix} = 0$

i.e iff $(1-\lambda)(-2-\lambda) - 2 = 0$ i.e $\lambda^2 + \lambda - 4 = 0$

i.e $\lambda = \frac{-1 \pm \sqrt{1+16}}{2} = -\frac{1}{2} \pm \frac{\sqrt{17}}{2}$

3 Hence $(\frac{1}{2}, \frac{1}{2})$ is a saddle point

Linearized equation at $(-\frac{1}{2}, -\frac{1}{2})$ is $x' = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \underline{x} = A\underline{x}$

λ is an eigenvalue of A if $\begin{vmatrix} 1-\lambda & -1 \\ 2 & 2-\lambda \end{vmatrix} = 0$

i.e $(1-\lambda)(2-\lambda) + 2 = 0$ i.e $\lambda^2 - 3\lambda + 4 = 0$

i.e $\lambda = \frac{3 \pm \sqrt{9-16}}{2} = \frac{3}{2} \pm \frac{\sqrt{7}}{2} i$

3 Hence $(\frac{1}{2}, \frac{1}{2})$ is an unstable spiral point

(b) $x' = y - xy^2 \qquad y' = -2x - x^2 y$

Let $V(x,y) = ax^2 + by^2$

$$\frac{d}{dt} V(x(t), y(t)) = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}$$

$$= 2ax(y - xy^2) + 2by(-2x - x^2 y)$$

$$= (2a - 4b)xy - 2ax^2y^2 - 2bx^2y^2$$

4

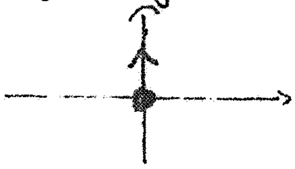
Hence choosing $a = 2$, $b = 1$ we have $V(x,y) = 2x^2 + y^2$ and

7(b) ctd $\frac{d}{dt} V(x(t), y(t)) = -6x^2y^2 \leq 0.$

6 Hence $(0,0)$ is a stable equilibrium point

(c) $x' = x - xy^2 \qquad y' = -2x + y^2$

Consider the solution satisfying the initial condition $x(0)=0$, $y(0)=\varepsilon>$
Clearly this solution is given by $x(t) \equiv 0$ and y satisfies
$y' = y^2 \quad y(0)=\varepsilon$. i.e y is increasing and $y \to \infty$ as $t \to \infty$
Thus we have trajectory



4 and so $(0,0)$ is an unstable equilibrium point.