Question 1

(a) Find the Fourier sine series for the function $f(x) = 1, \quad 0 < x < \pi$.

Sketch the graph of the function corresponding to the sum of the first N terms of the Fourier series when N is large.

(b) Consider the boundary value problem

$$-y'' = \lambda y, \quad y(0) = 0, \ \ y(\pi) + y'(\pi) = 0.$$

Show that there exist infinitely many eigenvalues λ_n and find the corresponding eigenfunctions.

Show that an eigenfunction expansion for $f(x) = 1, \ 0 < x < \pi$ in terms of these eigenfunctions is given by $\sum_{n=1}^{\infty} a_n \sin(\sqrt{\lambda_n} x)$ where

$$a_n = \frac{2(1 - \cos(\sqrt{\lambda_n}\pi))}{\sqrt{\lambda_n}\left(\pi + \cos^2(\sqrt{\lambda_n}\pi)\right)}.$$

Question 2

(a) Solve the PDE

$$u_x + 2xu_y = u^2; \quad u(0, y) = y.$$

(b) By using the method of separation of variables, find a solution of

$$u_x + 2xu_y = u$$

which depends non-trivially on both x and y.

continued overleaf

Question 3

(a) A string of unit length is fixed at its ends on the same horizontal level at $x = 0$ and $x = 1$. When the string vibrates, its displacement $u(x, t)$ satisfies the equation

$$u_{tt} = u_{xx}; \qquad 0 < x < 1, \; t > 0.$$

If the string has initial displacement $u(x, 0) = \sin(2\pi x)$ and initial velocity $u_t(x, 0) = x$ for $0 < x < 1$, determine $u(x, t)$.

(b) By considering the function

$$E(t) = \frac{1}{2}\int_0^1 \left[u_x^2(x,t) + u_t^2(x,t)\right] dx,$$

show that the only solution of

$$\begin{aligned} &u_{tt} = u_{xx}; \quad 0 < x < 1, \; t > 0; \\ &u(0,t) = u(1,t) = 0 \;\; \text{for} \;\; t > 0 \\ &u(x,0) = u_t(x,0) = 0 \;\; \text{for} \;\; 0 < x < 1 \end{aligned}$$

is $u \equiv 0$.

Question 4

(a) Solve the heat equation

$$u_t = u_{xx}\,; \qquad 0 < x < \pi, \quad t > 0$$

subject to the boundary conditions

$$u(0,t) = u(\pi,t) = 0 \text{ for all } t > 0$$

and the initial condition $u(x, 0) = \sin(x)\cos(x)$ for $0 < x < \pi$.

(b) Solve Laplace's equation

$$\nabla^2 u(x,y) = 0 \text{ for } 0 < x < 1, \; 0 < y < 2$$

subject to the boundary conditions

$$\begin{aligned} &u(x,0) = \cos^2(\pi x), \; u(x,2) = 0 \text{ for } 0 < x < 1; \\ &u_x(0,y) = u_x(1,y) = 0 \text{ for } 0 < y < 2 \end{aligned}$$

continued overleaf

Question 5

(a) By taking Laplace transforms, solve the initial value problem

$$x'(t) = x(t) - 2y(t); \quad y'(t) = 2x(t) - 3y(t); \qquad x(0) = 1; \quad y(0) = 0.$$

(b) By considering the eigenvalues and eigenvectors of an appropriate matrix, find the general solution of the system of equations

$$x'(t) = x(t) - 3y(t); \quad y'(t) = 2x(t) - 4y(t).$$

Sketch the phase plane for the system.

If x and y are the solutions of the above system satisfying $x(0) = y(0) = 1$, sketch, on the same diagram, the graphs of x and y as functions of t.

Question 6

(a) Draw the phase plane for the equation

$$x' = x - x^2$$

and, hence, sketch the graph of the solution of the equation satisfying the initial condition $x(0) = \frac{1}{2}$.

(b) Write the equation

$$x'' = x - x^2$$

as a system of first order equations, find the equations of the trajectories in the phase plane and hence sketch the phase plane for the system.

Sketch the graph of the solution of the equation satisfying $x(0) = \frac{1}{2}$ and $x'(0) = 0$.

Question 7

(a) Find all equilibrium points of the system

$$x' = x - y; \quad y' = 1 - 4xy$$

and determine the nature of each.

(b) By considering a Lyapunov function of the form $V(x, y) = ax^2 + by^2$ for an appropriate choice of a and b, show that $(0, 0)$ is a stable equilibrium point for the system

$$x' = y - xy^2; \qquad y' = -2x - x^2y.$$

(c) Show that $(0, 0)$ is an unstable equilibrium point for the system

$$x' = x - xy^2; \qquad y' = -2x + y^2.$$

END OF PAPER