## Solutions to Mathematical Techniques Exam - June 2003

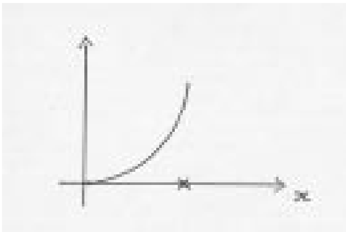
$1(a)$ Fourier sine series is

$$f(x) \sim \sum_{n=1}^{n=\infty} b_n \sin(nx)$$

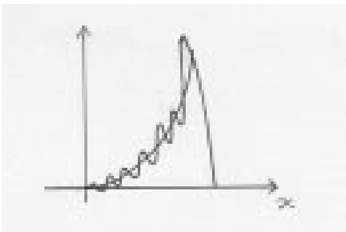
where

$$\begin{aligned}
b_n &= \tfrac{2}{\pi}\int_0^\pi x^2 \sin(nx)\,dx \\
&= -\tfrac{2}{n\pi}x^2\cos(nx)|_{x=0}^{x=\pi} + \tfrac{4}{n\pi}\int_0^\pi x\cos(nx)\,dx \\
&= -\tfrac{2\pi}{n}\cos(n\pi) + \tfrac{4}{n^2\pi}x\sin(nx)|_{x=0}^{x=\pi} - \tfrac{4}{n^2\pi}\int_0^\pi \sin(nx)\,dx \\
&= -\tfrac{2\pi}{n}\cos(n\pi) + 0 + \tfrac{4}{n^3\pi}\cos(nx)|_{x=0}^{x=\pi} \\
&= -\tfrac{2\pi}{n}\cos(n\pi) + \tfrac{4}{n^3\pi}(\cos(n\pi)-1) \\
&= \begin{cases} -\tfrac{2\pi}{n} \text{ if } n \text{ is even} \\ \tfrac{2\pi}{n} - \tfrac{8}{n^3\pi} \text{ if } n \text{ is odd} \end{cases}
\end{aligned}$$

On $[0,\pi]$ Fourier series converges to



Sum of first $N$ terms of Fourier series has graph



$1(b) \quad -y'' = \lambda y; \qquad y(0) = 0, \quad y(1) - \alpha y'(1) = 0.$

As we must first investigate positive eigenvalues we write $\lambda = k^2$ where $k > 0$.

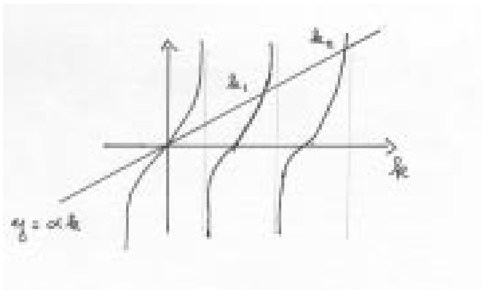
Then $-y'' = k^2 y$ has general solution $y = A\cos(kx) + B\sin(kx)$.

$y(0) = 0 \iff A = 0$. Hence we must have $y(x) = B\sin(kx)$ and so $y'(x) = Bk\cos(kx)$

Thus $y(1) - \alpha y'(1) = 0 \iff B\sin(k) - \alpha k B\cos(k) = 0$

$$\iff B = 0 \text{ or } \sin(k) - \alpha k\cos(k) = 0 \iff B = 0 \text{ or } \alpha k = \tan(k).$$

A graph shows that $\alpha k = \tan(k)$ has infinitely many positive solutions $k_1, k_2, k_3, \ldots,$



Thus we have infinitely many positive eigenvalues $k_1^2, k_2^2, k_3^2, \ldots,$

Suppose $\lambda < 0$; then we may write $\lambda = -k^2$ where $k > 0$.

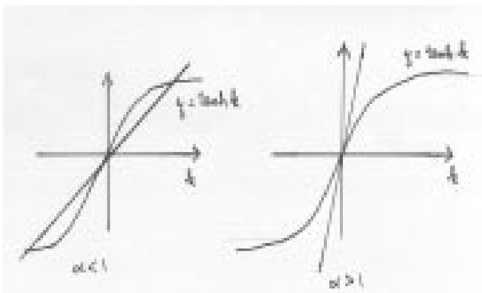
Then $y'' = k^2 y$ has general solution $y = A\ \cosh(kx) + B\ \sinh(kx)$. Then $y(0) = 0 \iff A = 0$. Thus $y(x) = B\ \sinh(kx)$ and $y'(x) = Bk\ \cosh(kx)$.

Hence $y(1) - \alpha y'(1) = 0 \iff B\ \sinh\ (k) - \alpha Bk\ \cosh(k) = 0$

$\iff B = 0$ or $\sinh\ (k) - \alpha k\ \cosh\ (k) = 0 \iff B = 0$ or $\alpha k = \ \tanh\ (k)$.

Thus we have a negative eigenvalue if and only if $\alpha k = \ \tanh(k)$ has a positive solution.

The graphs below show that this occurs if and only if $\alpha < 1$.



2(a) $u_t - 4u_x = 1; \qquad u(t,t) = t^2.$

Characteristics are solutions of $\frac{dx}{dt} = -4$.

Now $\frac{dx}{dt} = -4 \Longrightarrow x = -4t + c$.

Let $(t_0, x_0) \in \mathbf{R}^2$.

The characteristic through $(t_0, x_0)$ is $x = -4t + (x_0 + 4t_0)$ which intersects the initial line $x = t$ where $x = -4x + (x_0 + 4t_0)$, i.e., $x = \frac{1}{5}(x_0 + 4t_0)$, i.e., at the point $(\frac{1}{5}(x_0 + 4t_0), \frac{1}{5}(x_0 + 4t_0))$.

Let $v(t) = u(t, x(t))$ where $x(t)$ is the characteristic above. Then

$$\frac{dv}{dt} = u_t + u_x \frac{dx}{dt} = u_t - 4u_x = 1$$

and so $v(t) = t + c$.

Also $v(\frac{1}{5}(x_0 + 4t_0)) = \left(\frac{1}{5}(x_0 + 4t_0)\right)^2$ and so $c = \frac{(x_0+4t_0)^2}{25} - \frac{x_0+4t_0}{5}$,

i.e., $v(t) = t + \frac{(x_0+4t_0)^2}{25} - \frac{x_0+4t_0}{5}$.

Hence $u(t_0, x_0) = v(t_0) = t_0 + \frac{(x_0+4t_0)^2}{25} - \frac{(x_0+4t_0)}{5}$.

Thus $u(t,x) = t + \frac{(x+4t)^2}{25} - \frac{x+4t}{5}$.

Consider

$$u_t - 4u_x = 1; \qquad u(t, -4t) = t^2.$$

As above any solution of the PDE must satisfy $u(t, x(t)) = t + c$ for some constant $c$ along the characteristic $x = -4t$. Thus we must have $u(t, -4t) = t^2$ and $u(t, -4t) = t + c$ for some constant $c$ and this is impossible.

2(b) $$u_{tt} = 4u_{xx} \qquad 0 < x < L, \quad t > 0$$

We seek solutions of the form $u(x,t) = X(x)\,T(t)$.

To ensure that $u(0,t) = u(L,t) = 0$, we require that $X(0) = 0 = X(L)$.

Then $u$ satisfies the equation if $X(x)\,T''(t) = 4\,X''(x)\,T(t)$ and so if

$$\frac{1}{4}\frac{T''}{T} = \frac{X''}{X} = -k$$

where $k$ is a constant.

Thus we require

$$-X'' = kX; \quad X(0) = 0 = X(L) \qquad (1)$$
$$-T'' = 4kT \qquad (2)$$

(1) has nonzero solutions if and only if $k = \frac{n^2\pi^2}{L^2}$ with corresponding solutions $\sin\frac{n\pi x}{L}$ for $n = 1, 2, \ldots$

If $k = \frac{n^2\pi^2}{L^2}$, (2) has general solution $T(t) = A_n \cos\frac{2n\pi t}{L} + B_n \sin\frac{2n\pi t}{L}$.

Hence any function of the form

$$u(x,t) = \sum_{n=1}^{\infty}\{A_n \cos\tfrac{2n\pi t}{L} + B_n \sin\tfrac{2n\pi t}{L}\}\sin\tfrac{n\pi x}{L}$$

satisfies the PDE and the boundary conditions.

We must choose the coefficients $A_n$ and $B_n$ so that the initial conditions are satisfied.

Since $u(x,0) = \sin\frac{2\pi x}{L}$, we require $\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = \sin\frac{2\pi x}{L}$. Thus we should choose $A_2 = 1$ and $A_n = 0$ for $n \neq 2$.

Also $u_t(x,0) = \sum_{n=1}^{\infty}\left(\frac{-2n\pi}{L} A_n \sin\frac{2n\pi t}{L} + \frac{2n\pi}{L} B_n \cos\frac{2n\pi t}{L}\right)\sin\frac{n\pi x}{L}$.

Since $u_t(x,0) = 0$, we must choose $B_n = 0$ for all $n$.

Hence we have the solution $u(x,t) = \cos\frac{4\pi t}{L} \sin\frac{2\pi x}{L}$.

3$(a)$ We seek solutions of the form $u(r,\theta) = R(r)\Theta(\theta)$. Clearly $\Theta$ must have period $2\pi$.

Also we must have that

$$R''\Theta + \tfrac{1}{r}R'\Theta + \tfrac{1}{r^2}R\Theta'' = 0$$

and so it is sufficient to have

$$r^2\tfrac{R''}{R} + r\tfrac{R'}{R} = -\tfrac{\Theta''}{\Theta} = k \text{ where } k \text{ is a constant.}$$

Thus

$$r^2R'' + rR' - kR = 0 \qquad (1)$$

$$-\Theta'' = k\Theta;\quad \Theta(0) = \Theta(2\pi);\quad \Theta'(0) = \Theta'(2\pi) \qquad (2)$$

(2) has nonzero solutions if and only if $k = n^2$ where $n = 0, 1, 2, \ldots$.

Corresponding to $n = 0$ we have a constant eigenfunction and for $n \geq 1$ we have eigenfunctions $\sin(n\theta)$ and $\cos(n\theta)$.

When $k = n^2$, (1) becomes

$$r^2R'' + rR' - n^2R = 0 \quad \text{- an Euler equation.}$$

When $n = 0$, we have solutions 1 and $\ln(r)$.

When $n \geq 1$, we have solutions $r^n$ and $r^{-n}$.

Since we require solutions to be bounded at $r = 0$ we do not make use of the solutions $\ln r$ or $r^{-n}$.

Thus equation has solutions 1, $r^n\cos(n\theta)$, $r^n\sin(n\theta)$.

Hence any function of the form

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty}[\,A_n\cos(n\theta) + B_n\sin(n\theta)\,]\,r^n$$

is also a solution.

Thus we require that

$$u(1,\theta) = A_0 + \sum_{n=1}^{\infty}[A_n\cos(n\theta) + B_n\sin(n\theta)] = \begin{cases} 10 \text{ if } 0 \leq \theta \leq \pi \\ 0 \text{ if } \pi \leq \theta \leq 2\pi. \end{cases}$$

and so we choose $A_0, A_n, B_n$ to be the Fourier coefficients

$$A_0 = \frac{1}{2\pi}\int_0^\pi 10\,d\theta = 5$$

$$A_n = \frac{1}{\pi}\int_0^\pi 10\cos(n\theta)\,d\theta = \frac{10}{n\pi}\sin(n\theta)|_0^\pi = 0$$

$$B_n = \frac{1}{\pi}\int_0^\pi 10\sin(n\theta)\,d\theta = -\frac{10}{n\pi}\cos(n\theta)|_0^\pi = \frac{10}{n\pi}[1-\cos(n\pi)] = \begin{cases} 0 \text{ if } n \text{ is even} \\ \frac{20}{n\pi} \text{ if } n \text{ is odd} \end{cases}$$

$3(b)$ Suppose $u$ and $v$ are solutions of the equation. Let $w = u - v$. Then

$$\nabla^2 w(x) = \nabla^2 u(x) - \nabla^2 v(x) = 0 \text{ on } D; \quad w(x) = u(x) - v(x) = 0 \text{ on } \partial D.$$

Hence $\int_D \nabla^2 w(x)\,w(x)\,dx = \int_D 0\,.\,w(x)\,dx = 0$ and so

$$\int_{\partial D}\frac{\partial w}{\partial n}(x)\,w(x)\,dS - \int_D \nabla w(x)\,.\,\nabla w(x)\,dx = 0.$$

Hence $\int_D |\nabla w(x)|^2\,dx = 0$ (as $w(x) \equiv 0$ on $\partial D$) and so $|\nabla w(x)|^2 \equiv 0$.
It follows that $w(x) \equiv c$ on $D$ where $c$ is a constant.
Since $w(x) \equiv 0$ on $\partial D$, it follows that $w(x) \equiv 0$ on $D$.
Thus $u(x) = v(x)$ for all $x \in D$.

$4(a)$ $u_t = u_{xx}$ for $0 < x < 2$, $t > 0$; $\quad u_x(0,t) = 0 = u_x(2,t)$
We seek solutions of the form $u(x,t) = X(x)\,T(t)$.
To ensure that $u_x(0,t) = 0 = u_x(2,t)$ we require that $X'(0) = 0 = X'(2)$.
Substituting into the equation we must have that

$$T'(t)X(x) = X''(x)T(t)$$

Thus it is sufficient to have

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = -k$$

where $k$ is a constant,
i.e., we require

$$-X'' = kX, \quad X'(0) = 0 = X'(2) \qquad (1)$$
$$T' = -kT \qquad (2)$$

(1) has nonzero solutions if and only if $k = \frac{n^2\pi^2}{4}$ for $n = 0, 1, 2, \ldots$ and the corresponding eigenfunctions are $1, \cos(\frac{\pi x}{2}), \cos(\frac{2\pi x}{2})\ \ldots$
If $k = \frac{n^2\pi^2}{4}$, (2) becomes $T' = -\frac{n^2\pi^2}{4}T$ and so has solution $T = A_n e^{-\frac{n^2\pi^2 t}{4}}$.
Hence any function of the form $u(x,t) = A_0 + \sum_{n=1}^\infty A_n\cos(\frac{n\pi x}{2})e^{-\frac{n^2\pi^2 t}{4}}$ satisfies the PDE and the boundary conditions.

We now choose $A_n$ to ensure that $u(x,0) = \cos^2(\pi x) = \frac{1}{2}(\cos(2\pi x) + 1)$.
Thus we choose $A_0 = \frac{1}{2}$, $A_4 = \frac{1}{2}$ and $A_n = 0$ otherwise.
Hence $u(x,t) = \frac{1}{2} + \frac{1}{2}\cos(2\pi x)e^{-4\pi^2 t}$.

4(b) $u_t = u_{xx}$ for $0 < x < \infty$, $t > 0$; $u(0,t) = 0$ , $u(x,0) = e^{-x}$.
Taking Fourier sine transforms in $x$, we obtain
$\frac{d}{dt}\mathcal{F}_s(u(x,t))(\xi) = \sqrt{\frac{2}{\pi}}\,\xi\, u(0,t) - \xi^2\mathcal{F}_s(u(x,t))(\xi)$,
i.e., $\frac{d}{dt}\mathcal{F}_s(u(x,t))(\xi) = -\xi^2\mathcal{F}_s(u(x,t))(\xi)$.
Hence $\mathcal{F}_s(u(x,t))(\xi) = A(\xi)e^{-\xi^2 t}$. Also

$$\mathcal{F}_s(u(x,0))(\xi) = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,0)\sin(x\xi)\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin(x\xi)\,dx = \sqrt{\frac{2}{\pi}}\frac{\xi}{1+\xi^2}$$

Thus $A(\xi) = \sqrt{\frac{2}{\pi}}\frac{\xi}{1+\xi^2}$ and so $\mathcal{F}_s(u(x,t))(\xi) = \sqrt{\frac{2}{\pi}}\frac{\xi}{1+\xi^2}\,e^{-\xi^2 t}$.
Hence

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{\xi}{1+\xi^2}\,e^{-\xi^2 t}\sin(x\xi)d\xi = \frac{2}{\pi}\int_0^\infty \frac{\xi}{1+\xi^2}\,e^{-\xi^2 t}\sin(x\xi)d\xi.$$

5$(a)$ $x' = x - 2y$, $y' = x + 3y$ ; $x(0) = 2$, $y(0) = 1$.
Taking Laplace transforms we obtain

$s\overline{x}(s) - x(0) = \overline{x}(s) - 2\overline{y}(s)$; i.e., $(s-1)\,\overline{x}(s) + 2\overline{y}(s) = 2$ (1)

$s\overline{y}(s) - y(0) = \overline{x}(s) + 3\overline{y}(s)$; i.e., $-\overline{x}(s) + (s-3)\,\overline{y}(s) = 1$ (2)

(1) + $(s-1)\times$(2) gives $[2 + (s-1)(s-3)]\,\overline{y}(s) = 2 + s - 1$,

i.e., $(s^2 - 4s + 5)\,\overline{y}(s) = s + 1$,

i.e., $\overline{y}(s) = \frac{s+1}{(s-2)^2+1} = \frac{s-2}{(s-2)^2+1} + \frac{3}{(s-2)^2+1}$.

Hence $y(t) = e^{2t}\cos(t) + 3e^{2t}\sin(t)$.

$(s-3)\times(1) - 2\times(2)$ gives $[(s-3)(s-1) + 2]\,\overline{x}(s) = 2(s-3) - 2$,

i.e., $\overline{x}(s) = \frac{2s-8}{(s-2)^2+1} = 2\frac{s-2}{(s-2)^2+1} - \frac{4}{(s-2)^2+1}$.

Hence $x(t) = 2e^{2t}\cos(t) - 4e^{2t}\sin(t)$.

5.$(b)$ $x' = x - 2y$, $y' = 2x - 3y$.

The system can be written as $\underline{x}' = A\underline{x}$ where $A = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}$.

$\lambda$ is an eigenvalue of $A$ if and only if det $\begin{bmatrix} 1-\lambda & -2 \\ 2 & -3-\lambda \end{bmatrix} = 0$,

i.e., $(1-\lambda)\,(-3-\lambda) + 4 = 0$, i.e., $\lambda^2 + 2\lambda + 1 = 0$, i.e., $(\lambda+1)^2 = 0$,
i.e., $\lambda = -1$.

$\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -1$ if and only if

$$\begin{array}{ll} x - 2y = -x & \text{i.e., } 2x - 2y = 0. \\ 2x - 3y = -y & \text{i.e., } 2x - 2y = 0 \end{array}$$

Hence all eigenvectors corresponding to $\lambda = -1$ are multiples of $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \underline{\xi}$.

We seek a second solution of the form $\underline{x}(t) = te^{-t}\underline{\xi} + e^{-t}\underline{\eta}$.
Thus we require $e^{-t}\underline{\xi} - te^{-t}\underline{\xi} - e^{-t}\eta = te^{-t}A\underline{\xi} + e^{-t}A\underline{\eta} = -te^{-t}\xi + e^{-t}A\eta$,
i.e., $(A + I)\underline{\eta} = \underline{\xi}$.

Thus, if $\underline{\eta} = \begin{pmatrix} x \\ y \end{pmatrix}$, we require $\begin{array}{rcl} 2x - 2y &=& 1 \\ 2x - 2y &=& 1 \end{array}$.

Thus we may choose $x = \frac{1}{2}, \quad y = 0$.

Thus we have solution $\underline{x}(t) = e^{-t}[t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}]$.

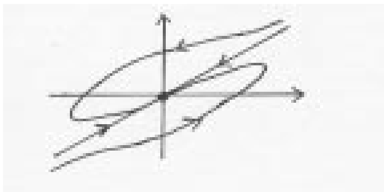
Hence we have general solution

$$\underline{x}(t) = e^{-t}\left[c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} t + \frac{1}{2} \\ t \end{pmatrix}\right] = e^{-t}\left[\begin{pmatrix} c_1 + \frac{1}{2}c_2 \\ c_1 \end{pmatrix} + c_2 t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]$$

As $t \to +\infty$, trajectory approaches $(0, 0)$ tangentially to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

As $t \to -\infty$, trajectory approaches $\infty$ in direction of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

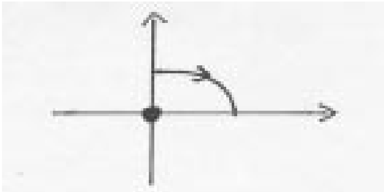
At $(0, 1)$, $\frac{dx}{dt} < 0$ and $\frac{dy}{dt} < 0$
Hence phase plane is



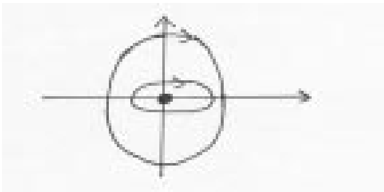
6(a) $\quad x'' = -x^3$.
The equation may be written as the system $x' = y\,;\, y' = -x^3$.
Clearly $(0, 0)$ is the only equilibrium point of the system.
Any trajectory of the form $y = y(x)$ must satisfy $\frac{dy}{dx} = \frac{dy}{dt}/\frac{dx}{dt} = \frac{-x^3}{y}$,
i.e., $y\,dy = -x^3\,dx$, i.e., $\frac{1}{2}y^2 + \frac{1}{4}x^4 = c$ i.e., $2y^2 + x^4 = c$.
Consider the trajectory passing through $(0, a)$ where $a > 0$, i.e., the trajectory with equation $2y^2 + x^4 = 2a^2$

As $x$ increases into the first quadrant, $y$ decreases until $y = 0$ when $x^4 = 2a^2$, i.e., $x = 2^{\frac{1}{4}}\sqrt{a}$.

Thus we obtain trajectories



As equations of trajectories are symmetric in $x$ and $y$ we have phase plane

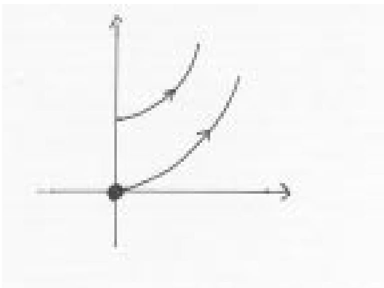


$(b)$ $\quad x'' = -x^3$.

Arguing as above equation of trajectories is $2y^2 - x^4 = c$.

Consider the trajectory passing through $(0, a)$ where $a \geq 0$, i.e., the trajectory with equation $2y^2 - x^4 = 2a^2$.

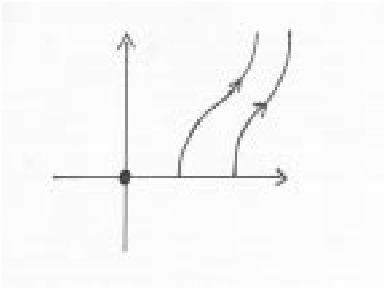
As $x$ increases into the first quadrant, $y$ increases - roughly speaking $y \approx \frac{1}{\sqrt{2}}x^2$ for large $x$ and $y$. Hence we obtain the trajectory



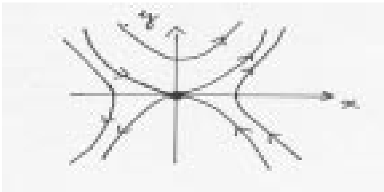
Consider the trajectory passing through $(a, 0)$ where $a \geq 0$, i.e., $2y^2 - x^4 = -a^4$.

As $y$ increases into the first quadrant, $x$ increases - again $y \approx \frac{1}{\sqrt{2}}x^2$ for large $x$ and $y$.

Hence we obtain the trajectory



As equations of trajectories are symmetric in $x$ and $y$ we have phase plane



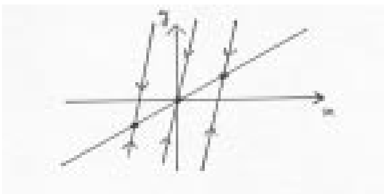
$(c)\qquad x' = x - y; \qquad y' = 2x - 2y$

Equilibrium points occur when $x - y = 0$ and so the set of all equilibrium points consists of the line $y = x$.

Any trajectory of the form $y = y(x)$ must satisfy $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{2x-2y}{x-y} = 2$.

Hence trajectories are lines of the form $y = 2x + c$

Hence we have phase plane



Note that $\frac{dx}{dt} < 0$ and $\frac{dy}{dt} < 0$ when $y > x$.

$7(a)\qquad \frac{dx}{dt} = x(3 - x - y) = f(x, y); \qquad \frac{dy}{dt} = y(8 - 3x - 2y) = g(x, y)$

Equilibrium points occur when $x = 0, y = 0$: $\quad x = 0, y = 4$ : $\quad x = 3, y = 0$ and where

$$\begin{aligned} 3 - x - y &= 0 \\ 8 - 3x - 2y &= 0 \end{aligned} \quad \text{i.e.,} \quad \begin{aligned} x + y &= 3 \\ 3x + 2y &= 8 \end{aligned} \quad \text{i.e., } x = 2, y = 1.$$

Since $f(x,y) = x(3-x-y)$, $\frac{\partial f}{\partial x}(x,y) = 3-2x-y$ and $\frac{\partial f}{\partial y}(x,y) = -x$.
Since $g(x,y) = y(8-3x-2y)$, $\frac{\partial g}{\partial x}(x,y) = -3y$ and $\frac{\partial g}{\partial y}(x,y) = 8-3x-4y$.
Hence linearized equation at $(x,y)$ is $\underline{x}' = A\underline{x}$ where $A = \begin{pmatrix} 3-2x-y & -x \\ -3y & 8-3x-4y \end{pmatrix}$.

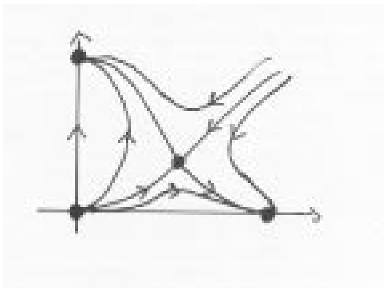
Hence linearized equation at $(0,0)$ is $\underline{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 8 \end{pmatrix} \underline{x} = A\underline{x}$.
Eigenvalues of $A$ are $\lambda = 3, 8$ and so $(0,0)$ is an unstable node.

Linearized equation at $(0,4)$ is $\underline{x}' = \begin{pmatrix} -1 & 0 \\ -12 & -8 \end{pmatrix} \underline{x} = A\underline{x}$.
Eigenvalues of $A$ are $\lambda = -1, -8$ and so $(0,4)$ is an asymptotically stable node.

Linearized equation at $(3,0)$ is $\underline{x}' = \begin{pmatrix} -3 & -3 \\ 0 & -1 \end{pmatrix} \underline{x} = A\underline{x}$.
Eigenvalues of $A$ are $\lambda = -3, -1$ and so $(3,0)$ is an asymptotically stable node.

Linearized equation at $(2,1)$ is $\underline{x}' = \begin{pmatrix} -2 & -2 \\ -3 & -2 \end{pmatrix} \underline{x} = A\underline{x}$.

$\lambda$ is an eigenvalue of $A$ if and only if det $\begin{vmatrix} -2-\lambda & -2 \\ -3 & -2-\lambda \end{vmatrix} = 0$,
i.e., $(-2-\lambda(-2-\lambda) - 6 = 0$, i.e., $(\lambda+2)^2 = 6$, i.e., $\lambda = -2 \pm \sqrt{6}$.
Hence $(2,1)$ is a saddle point.
Thus possible phase plane is



$(b)$ $x' = x^3 - y^3$; $\quad y' = 2xy^2 + 4x^2y + 2y^3$.
Let $V(x,y) = ax^2 + by^2$. Then if $x(t)$ and $y(t)$ are solutions of the system
$\frac{d}{dt}[V(x(t),y(t))] = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial t} = 2ax(x^3-y^3) + 2by(2xy^2+4x^2y+2y^3)$
$= 2ax^4 + (-2a+4b)xy^3 + 8bx^2y^2 + 4by^4$
Choosing $a = 2$, $b = 1$, we have $V(x,y) = 2x^2 + y^2$ and
$\frac{d}{dt}[V(x(t),y(t))] = 4x^4 + 8x^2y^2 + 4y^4 = 4(x^2+y^2)^2 > 0$.
Thus $(0,0)$ is an unstable equilibrium point.