Question 1

(a) Find the Fourier sine series for the function $f(x) = x^2, \ \ 0 < x < \pi$.

Sketch the graph for the range $0 \leq x \leq \pi$ of

(i) the function to which the Fourier series above converges

(ii) the sum of the first N terms of the Fourier series when N is large.

(b) Suppose $\alpha > 0$. Show that the boundary value problem

$$-y'' = \lambda y; \quad y(0) = 0, \ \ y(1) - \alpha y'(1) = 0$$

(i) has infinitely many positive eigenvalues;

(ii) has a negative eigenvalue if and only if $\alpha < 1$.

Question 2

(a) Solve the PDE

$$u_t - 4u_x = 1; \quad u(t,t) = t^2.$$

Explain why the PDE

$$u_t - 4u_x = 1; \quad u(t,-4t) = t^2$$

does not have a solution.

(b) A string of length L is fixed at its ends at $x = 0$ and $x = L$. When the string vibrates, its displacement $u(x,t)$ satisfies the equation

$$u_{tt} = 4u_{xx}; \qquad 0 < x < L, \ t > 0.$$

If the string is initially at rest with initial displacement $u(x,0) = \sin(\frac{2\pi x}{L})$, determine $u(x,t)$.

continued overleaf

Question 3

(a) The steady state temperature $u(r,\theta)$ of a plate in the shape of a circle with centre the origin and radius 1 satisfies the equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

where (r,θ) denotes plane polar co-ordinates.
If the steady state solution satisfies the boundary condition that

$$u(1,\theta) = \begin{cases} 10 & \text{if } 0 \le \theta \le \pi \\ 0 & \text{if } \pi < \theta < 2\pi \end{cases},$$

find $u(r,\theta)$.

(b) Let D be a region in $\mathbf{R}^2$ with smooth boundary. Show that, if u and v are solutions of

$$\nabla^2\phi(\underline{x}) = 0 \ \text{ for } \underline{x} \in D; \qquad \phi(\underline{x}) = 0 \ \text{ for } \underline{x} \in \partial D$$

then $u(\underline{x}) = v(\underline{x})$ for all $\underline{x} \in D$.

(You may assume without proof that

$$\int_D \nabla^2 u(\underline{x}) v(\underline{x})\, d\underline{x} = \int_{\partial D} [\, \nabla u(\underline{x}) \,.\, \underline{n}(\underline{x}) \,]\, v(\underline{x})\, dS - \int_D \nabla u(\underline{x}) \,.\, \nabla v(\underline{x})\, d\underline{x}$$

for all smooth functions u and v.]

Question 4

(a) Solve the heat equation

$$u_t = u_{xx}\,; \qquad 0 < x < 2, \quad t > 0$$

subject to the boundary conditions

$$u_x(0,t) = u_x(2,t) = 0 \text{ for all } t > 0$$

and the initial condition $u(x,0) = \cos^2(\pi x)$ for $0 \le x \le 2$.

(b) By taking Fourier sine transforms in x, show that the equation

$$u_t = u_{xx} \ \text{ for } 0 < x < \infty, t > 0; \ \ u(0,t) = 0 \ \text{ for } t > 0; u(x,0) = e^{-x} \ \text{ for } x > 0$$

has solution

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\xi}{1+\xi^2}\, e^{-\xi^2 t} \sin(x\xi)\, d\xi$$

continued overleaf

Question 5

(a) By taking Laplace transforms, solve the initial value problem

$$x'(t) = x(t) - 2y(t); \quad y'(t) = x(t) + 3y(t); \qquad x(0) = 2; \quad y(0) = 1.$$

(b) By considering the eigenvalues and eigenvectors of an appropriate matrix, find the general solution of the system of equations

$$x'(t) = x(t) - 2y(t); \quad y'(t) = 2x(t) - 3y(t).$$

Sketch the phase plane for the system.

Question 6

(a) Write each of the following equations as a system of first order equations, find the equations of the trajectories in the phase plane and hence sketch the phase planes.

$$(i) \quad x'' = -x^3 \qquad\qquad (ii) \quad x'' = x^3.$$

(b) Find all the equilibrium points for the system

$$x' = x - y \qquad\qquad y' = 2x - 2y$$

Find also the equations of all trajectories in the phase plane and hence sketch the phase plane for the system.

Question 7

(a) For the system describing the interaction of two competing species

$$x' = x(3 - x - y); \quad y' = y(8 - 3x - 2y)$$

find all of the equilibrium points and determine the nature of each.

Hence sketch the phase plane for the system.

(b) By considering a Lyapunov function of the form $V(x, y) = ax^2 + by^2$ for an appropriate choice of a and b, determine whether or not $(0, 0)$ is a stable equilibrium point for the system

$$x' = x^3 - y^3; \qquad y' = 2xy^2 + 4x^2y + 2y^3.$$

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