Solutions to Mathematical Techniques Exam - June 2002

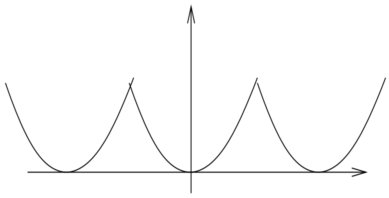
1. $a_0 = \frac{1}{1}\int_0^1 x^2\,dx = \frac{1}{3}$.

$a_n = \frac{2}{1}\int_0^1 x^2\cos(n\pi x)\,dx = 2\,[\,\frac{1}{n\pi}x^2\sin(n\pi x)\,|_{x=0}^{x=1} - \frac{2}{n\pi}\int_0^1 x\sin(n\pi x)\,dx\,]$
$= -\frac{4}{n\pi}\int_0^1 x\sin(n\pi x)\,dx = -\frac{4}{n\pi}\,[\,-\frac{1}{n\pi}x\cos(n\pi x)\,|_0^1 + \frac{1}{n\pi}\int_0^1\cos(n\pi x)\,dx\,]$
$= \frac{4}{n^2\pi^2}\cos(n\pi) = \frac{4}{n^2\pi^2}(-1)^n$.

Hence Fourier cosine series is

$$f(x) = \frac{1}{3} - \frac{4}{\pi^2}\,[\cos(\pi x) - \frac{1}{4}\cos(2\pi x) + \frac{1}{9}\cos(3\pi x) - \frac{1}{16}\cos(4\pi x) + \ldots]$$

Since Fourier cosine series converges to an even function of period 2, the series converges to the following limit on $[-3, 3]$



Hence series converges to 1 when $x = 1$ and so

$$1 = \frac{1}{3} - \frac{4}{\pi^2}\,[-1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \ldots]$$

Hence $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{4}\,.\,\frac{2}{3} = \frac{\pi^2}{6}$.

(b) By the Fourier Inversion Theorem

$$\begin{aligned} f(x) &= \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha\xi^2}\,e^{i\xi x}\,d\xi \\ &= \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha\xi^2}\,\cos(\xi x)\,dx + \tfrac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha\xi^2}\,\sin(\xi x)\,dx \\ &= \tfrac{1}{\sqrt{2\pi}}\,.\,\tfrac{\sqrt{\pi}}{\sqrt{\alpha}}\,e^{-\frac{x^2}{4\alpha}} = \tfrac{1}{\sqrt{2\alpha}}e^{-\frac{x^2}{4\alpha}} \end{aligned}$$

$$u_t = u_{xx} \qquad -\infty < x < \infty; \qquad t > 0; \qquad u(x,0) = g(x) \qquad -\infty < x < \infty.$$

Taking Fourier transforms with respect to x gives
$\frac{d}{dt}\,\hat{u}(\xi,t) = -\xi^2\,\hat{u}(\xi,t)$ and so $\hat{u}(\xi,t) = K(\xi)e^{-\xi^2 t}$.

Now $\hat{u}(\xi,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,0)e^{-ix\xi}\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{-ix\xi}\,dx = \hat{g}(\xi)$ and so $K(\xi) = \hat{g}(\xi)$.

Thus $\hat{u}(\xi,t) = \hat{g}(\xi)e^{-\xi^2 t} = \hat{g}(\xi)\hat{k}(\xi)$ where $k(x) = \frac{1}{\sqrt{2t}}e^{-\frac{x^2}{4t}}$ from the first part of the question.

Hence by the Convolution Theorem

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s) \frac{1}{\sqrt{2t}} e^{-\frac{(x-s)^2}{4t}}\, ds = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} g(s) e^{-\frac{(x-s)^2}{4t}}\, ds.$$

2. (a) $-y'' = \lambda y; \quad y(0) = 0, \ y(1) + 2y'(1) = 0.$

As we must investigate positive eigenvalues we write $\lambda = k^2$ where $k > 0$.

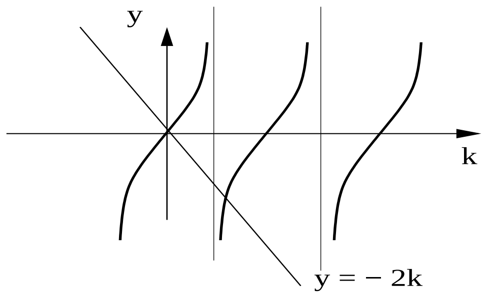
Then $y'' = k^2 y$ has general solution $y = A\cos(kx) + B\sin(kx)$.

$y(0) = 0 \iff A = 0$. Hence we must have $y(x) = B\sin(kx)$.

Thus $y(1) + 2y'(1) = 0 \iff B\sin(k) + 2Bk\cos(k) = 0$

$\iff B = 0$ or $\sin(k) + 2k\cos(k) = 0 \iff B = 0$ or $-2k = \tan(k)$.

A graph shows that $-2k = \tan(k)$ has infinitely many positive solutions $k_1, k_2, k_3, \ldots,$



where $\frac{\pi}{2} < k_1 < \frac{3\pi}{2}$, $\frac{3\pi}{2} < k_2 < \frac{5\pi}{2}$ etc. and that, for large n,
$k_n \approx \frac{\pi}{2} + (n-1)\pi = (n - \frac{1}{2})\pi$.

Thus we have infinitely many positive eigenvalues $\lambda_1, \lambda_2, \ldots$with

$$\lambda_n \approx (n - \frac{1}{2})^2 \pi^2.$$

(b) $x^2 u_x + y^2 u_y = 0; \qquad u(1,y) = y.$

Characteristics are solutions of $\frac{dy}{dx} = \frac{y^2}{x^2}$.

But $\frac{dy}{dx} = \frac{y^2}{x^2} \Longrightarrow \frac{1}{y^2}\, dy = \frac{1}{x^2}\, dx \Longrightarrow \frac{1}{y} = \frac{1}{x} + C \Longrightarrow x = y + Cxy$.

Let $(x_0, y_0) \in \mathbf{R}^2$.

The characteristic through (x_0, y_0) corresponds to C satisfying $x_0 = y_0 + Cx_0y_0$, i.e., $C = \frac{x_0 - y_0}{x_0 y_0}$.

This characteristic meets the line $x = 1$ when $1 = y + Cy$,
i.e., $y = \frac{1}{1+C} = \frac{1}{1 + \frac{x_0 - y_0}{x_0 y_0}} = \frac{x_0 y_0}{x_0 y_0 + x_0 - y_0}$, i.e., at $(1, \frac{x_0 y_0}{x_0 y_0 + x_0 - y_0})$.

If u is the required solution and $x \to y(x)$ is a characteristic, then

$$\frac{d}{dx}u(x,y(x)) = u_x + u_y\frac{dy}{dx} = u_x + \frac{y^2}{x^2}u_y = \frac{1}{x^2}(x^2u_x + y^2u_y) = 0,$$

i.e., the solution is constant along characteristics.
Hence $u(x_0,y_0) = u(1, \frac{x_0y_0}{x_0y_0+x_0-y_0}) = \frac{x_0y_0}{x_0y_0+x_0-y_0}$.
Thus $u(x,y) = \frac{xy}{xy+x-y}$.

3 (a) $u_t = ku_{xx}$; $\quad u(0,t) = 0 = u(L,t), \quad u(x,0) = \begin{cases} T_0 \text{ for } 0 \le x \le \frac{L}{2} \\ 0 \text{ for } \frac{L}{2} < x \le L \end{cases}$.

We seek solutions of the form $u(x,t) = X(x)\ T(t)$.
To ensure that $u(0,t) = 0 = u(L,t)$ we require that $X(0) = 0 = X(L)$.
Substituting into the equation we must have that

$$T'(t)X(x) = kX''(x)T(t)$$

Thus it is sufficient to have

$$\frac{X''(x)}{X(x)} = \frac{1}{k}\ \frac{T'(t)}{T(t)} = -\lambda$$

where λ is a constant,
i.e., we require

$$\begin{aligned} &-X'' = \lambda X, \quad X(0) = 0 = X(L) \qquad (1) \\ &T' = -k\lambda T \qquad\qquad\qquad\qquad\quad (2) \end{aligned}$$

(1) has nonzero solutions if and only if $\lambda = \frac{n^2\pi^2}{L^2}$ for $n = 1, 2, \ldots, \ldots,$ and the corresponding eigenfunctions are $\sin(\frac{n\pi x}{L})$.

If $\lambda = \frac{n^2\pi^2}{L^2}$, (2) becomes $T' = -k\frac{n^2\pi^2}{L^2}T$ and so has solution $T = A_n e^{-k\frac{n^2\pi^2}{L^2}t}$

Hence any function of the form $u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L})e^{-k\frac{n^2\pi^2}{L^2}t}$ satisfies the PDE and the boundary conditions.

We now choose A_n to ensure that $u(x,0) = \begin{cases} T_0 \text{ for } 0 \le x \le \frac{L}{2} \\ 0 \text{ for } \frac{L}{2} < x \le L \end{cases}$.

Since $u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L})$, we choose A_n's as Fourier sine coefficients, i.e.,

$$A_n = \frac{2}{L}\int_0^{\frac{L}{2}} T_0 \sin(\frac{n\pi x}{L})\,dx = \frac{2T_0}{L}\,.\,\frac{L}{n\pi} - \cos(\frac{n\pi x}{L})|_{x=0}^{x=L/2} = \frac{2T_0}{n\pi}[\,1 - \cos(\frac{n\pi}{2})\,].$$

Equation describes temperature $u(x,t)$ of a metal bar, thermal diffusivity k, lying between $x = 0$ and $x = L$ with the ends of the bar maintained at 0°, the left hand half of the bar initially at T° and the right hand half initially at 0°.

(b) Suppose that u satisfies

$$u_t = u_{xx}\,; \qquad u(0,t) = 0 = u(1,t), \quad u(x,0) = 0 \quad 0 \le x \le 1.$$

Let $E(t) = \int_0^1 u^2(x,t)\,dx$.

Clearly $E(t) \ge 0$ for all t and since $u(x,0) \equiv 0$, $E(0) = 0$.

Also

$$\begin{aligned} \tfrac{d}{dt}\,[E(t)\,] &= 2\int_0^1 u(x,t)u_t(x,t)\,dx = 2\int_0^1 u(x,t)u_{xx}(x,t)\,dx \\ &= 2u(x,t)u_x(x,t)|_{x=0}^{x=1} - 2\int_0^1 u_x^2(x,t)\,dx = -2\int_0^1 u_x^2(x,t)\,dx \le 0. \end{aligned}$$

It follows that $E(t) \equiv 0$ and so $u(x,t) \equiv 0$.

4. We seek solutions of the form $u(x,y) = X(x)Y(y)$.
Thus we require $X''Y + XY'' = 0$.
Hence it is sufficient to have

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -k$$

where k is a constant.

To ensure that $u(0,y) = u(2,y) = u(x,0) = 0$, we require that $X(0) = X(2) = 0$ and $Y(0) = 0$.
Thus it is sufficient to have

$$\begin{aligned} -X'' &= kX; \qquad X(0) = 0 = X(2) \qquad (1) \\ Y'' &= kY; \qquad Y(0) = 0 \qquad (2) \end{aligned}$$

(1) has nonzero solutions if and only if $k = \frac{n^2\pi^2}{4}$ with corresponding solutions $X(x) = \sin(\frac{n\pi x}{2})$ for $n = 1, 2, \ldots$.
If $k = \frac{n^2\pi^2}{4}$, (2) has solution $Y(y) = A_n\,\sinh(\frac{n\pi y}{2})$.
Thus equation has solution

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{2})\,\sinh(\frac{n\pi y}{2})$$

and we must choose the A_n's to ensure that $u(x,1) = \sin(\pi x)$.
Thus we require

$$\sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{2})\,\sinh(\frac{n\pi}{2}) = \sin(\pi x)$$

and so we choose $A_n = 0$ for $n \ne 2$ and $A_2\,\sinh\pi = 1$, i.e., $A_2 = \frac{1}{\sinh\,\pi}$.
Thus we have solution $u(x,y) = \frac{\sin\,(\pi x)\,\sinh\,(\pi y)}{\sinh\,(\pi)}$.

(b) $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$

We seek solutions of the form $u(r,\theta) = R(r)\Theta(\theta)$.

Clearly Θ must have period 2π.

Also we have a solution to the equation if

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

Thus it is sufficient to have that

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = k$$

where k is a constant.

Thus we require

$$r^2R'' + rR' - kR = 0 \qquad (1)$$

$$-\Theta'' = k\Theta; \qquad \Theta \text{ has period } 2\pi. \qquad (2)$$

(2) has non-zero solutions if and only if $k = n^2$ for $n = 0, 1, 2, \ldots$; when $n = 0$ eigenfunction is a constant and when $n \geq 1$, eigenfunctions are $\sin(n\theta)$ and $\cos(n\theta)$.

When $k = n^2$, (1) becomes

$$r^2R'' + rR' - n^2R = 0 \ \ - \ \text{an Euler equation}$$

When $n = 0$, we have solutions 1 and $\ln r$.

When $n = 1$, we have solutions r^n and r^{-n}.

Since we require solutions to be bounded at $r = 0$, we do not make use of the solutions $\ln r$ or r^{-n}.

Thus we seek a solution of the form

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} (\, A_n\cos(n\theta) + B_n\sin(n\theta)\,)r^n$$

such that $u(1,\theta) = \cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$.

Hence we choose $A_0 = \frac{1}{2}$, $A_2 = \frac{1}{2}$ and all the other coefficients $= 0$.

Thus we have solution

$$u(r,\theta) = \frac{1}{2}\,[\ 1 + \cos(2\theta)r^2\ ].$$

5.(a) $x' = 3x - y, \quad y' = 5x - y\ ; \qquad x(0) = 1, \quad y(0) = 2.$

Taking Laplace transforms we obtain

$s\overline{x}(s) - x(0) = 3\overline{x}(s) - \overline{y}(s); \qquad$ i.e., $(s-3)\,\overline{x}(s) + \overline{y}(s) = 1 \qquad (1)$

$s\overline{y}(s) - y(0) = 5\overline{x}(s) - \overline{y}(s); \qquad$ i.e., $-5\overline{x}(s) + (s+1)\,\overline{y}(s) = 2 \qquad (2)$

$5\times$ (1) + $(s-3)\times$(2) gives $[5 + (s+1)(s-3)]\,\overline{y}(s) = 5 + 2(s-3)$,

i.e., $(s^2 - 2s + 2))\,\overline{y}(s) = 2s - 1$,

i.e., $\overline{y}(s) = \frac{2s-1}{(s-1)^2+1} = 2\frac{s-1}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}$.

Hence $y(t) = 2e^t\cos(t) + e^t\sin(t)$.

$(s+1)\times$ (1) $-$ (2) gives $[(s-3)(s+1)+5]\,\overline{x}(s) = s + 1 - 2$,

i.e., $\overline{x}(s) = \frac{s-1}{s^2-2s+2} = \frac{s-1}{(s-1)^2+1}$.

Hence $x(t) = e^t\cos(t)$.

(b) $x' = x - 3y, \quad y' = 4x - 6y$.

The system can be written as $\underline{x}' = A\underline{x}$ where $A = \begin{pmatrix} 1 & -3 \\ 4 & -6 \end{pmatrix}$.

λ is an eigenvalue of A if and only if det $\begin{bmatrix} 1-\lambda & -3 \\ 4 & -6-\lambda \end{bmatrix} = 0$,

i.e., $(1-\lambda)\,(-6-\lambda) + 12 = 0$, i.e., $\lambda^2 + 5\lambda + 6 = 0$, i.e., $(\lambda+2)(\lambda+3) = 0$,
i.e., $\lambda = -3,\ -2$.

$\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -3$ if and only if

$$\begin{aligned} x - 3y &= -3x \qquad \text{i.e., } 4x - 3y = 0. \\ 4x - 6y &= -3y \qquad \text{i.e., } 4x - 3y = 0. \end{aligned}$$

Thus $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -3$.

$\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -2$ if and only if

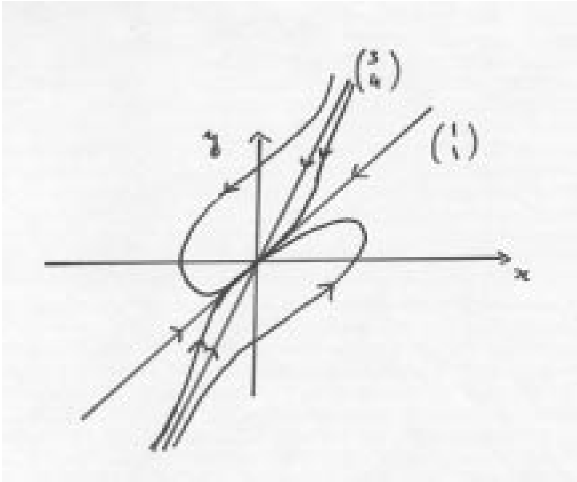
$$\begin{aligned} x - 3y &= -2x \qquad \text{i.e., } 3x - 3y = 0 \\ 4x - 6y &= -2y \qquad \text{i.e., } 4x - 4y = 0. \end{aligned}$$

Thus $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -2$.

Hence system has general solution

$$\underline{x}(t) = c_1\,e^{-3t}\begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2\,e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The phase plane is



Solution satisfying $x(0) = 2$, $y(0) = 2$, is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 2\begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix}$.

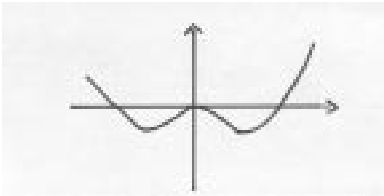
6(a) $x' = 3x - 4y + x^2 - y^2, \quad y' = x - 2y - y^3.$
The corresponding linearized system at $(0, 0)$ is

$$\begin{aligned} x' &= 3x - 4y \\ y' &= x - 2y \end{aligned} \qquad \text{i.e., } \underline{x}' = A\underline{x} \text{ where } A = \begin{pmatrix} 3 & -4 \\ 1 & -2) \end{pmatrix}$$

λ is an eigenvalue of A if and only if $\det \begin{vmatrix} 3-\lambda & -4 \\ 1 & -2-\lambda \end{vmatrix} = 0,$
i.e., $(3-\lambda)(-2-\lambda) + 4 = 0$, i.e., $\lambda^2 - \lambda - 2 = 0$, i.e., $(\lambda - 2)(\lambda + 1) = 0$, i.e., $\lambda = -1, 2$.
Thus $(0, 0)$ is an unstable saddle point for the linearized system and so is also an unstable saddle point for the original system.

(b) $\quad x'' = x - x^3$.
The equation may be written as the system $x' = y\,;\; y' = x - x^3$.
(x, y) is an equilibrium point if $y = 0$ and $x - x^3 = x(1 - x^2)$
$= x(1 - x)(1 + x) = 0$, i.e., equilibrium points are $(-1, 0)$, $(0, 0)$ and $(1, 0)$.
Any trajectory of the form $y = y(x)$ must satisfy $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{x - x^3}{y}$,
i.e., $y\,dy = (x - x^3)\,dx$, i.e., $\frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4 = c$.
We may write the equation of the trajectories as $\frac{1}{2}y^2 + f(x) = c$ where $f(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$. Then $f'(x) = x^3 - x = x(1 - x)(1 + x)$ and so f has turning points at $x = -1, 0, 1$.

Now $f(-1) = f(1) = \frac{1}{4}$ and f has graph

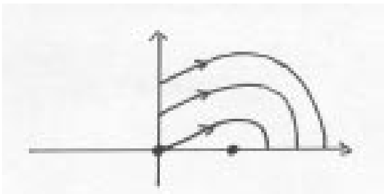


Consider the trajectory passing through $(0, a)$ where $a \geq 0$, i.e., the trajectory with equation $\frac{1}{2}y^2 + f(x) = \frac{1}{2}a^2$

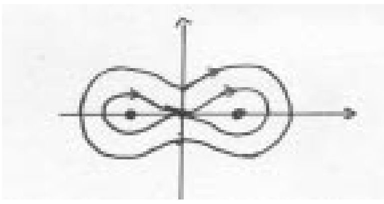
As x increases into the first quadrant from 0 to 1, $f(x)$ decreases. Hence as x increases from 0 to 1, y^2 and so y increases.

As x increases beyond 1, $f(x)$ increases and so y^2 decreases until $y = 0$ when $f(x) = \frac{1}{2}a^2$.

Thus we obtain trajectories



As equations of trajectories are symmetric in x and y we have phase plane



8.(a) $\quad \frac{dx}{dt} = x(3 - y) = f(x, y); \quad \frac{dy}{dt} = y(1 - y + x) = g(x, y)$

Equilibrium points occur when $x = 0, y = 0$: $\quad x = 0, y = 1$ and $y = 3, x = 2$, i.e., at $(0, 0)$, $(0, 1)$, $(2, 3)$.

Since $f(x, y) = x(3 - y)$, $\frac{\partial f}{\partial x}(x, y) = 3 - y$ and $\frac{\partial f}{\partial y}(x, y) = -x$.

Since $g(x, y) = y(1 - y + x)$, $\frac{\partial g}{\partial x}(x, y) = y$ and $\frac{\partial g}{\partial y}(x, y) = 1 - 2y + x$.

Hence linearized equation at (x, y) is $\underline{x}' = A\underline{x}$ where $A = \begin{pmatrix} 3 - y & -x \\ y & 1 - 2y + x \end{pmatrix}$.

Hence linearized equation at $(0, 0)$ is $\underline{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \underline{x} = A\underline{x}$.

Eigenvalues of A are $\lambda = 3, 1$ and so $(0,0)$ is an unstable node.

Linearized equation at $(0,1)$ is $\underline{x}' = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} \underline{x} = A\underline{x}$.

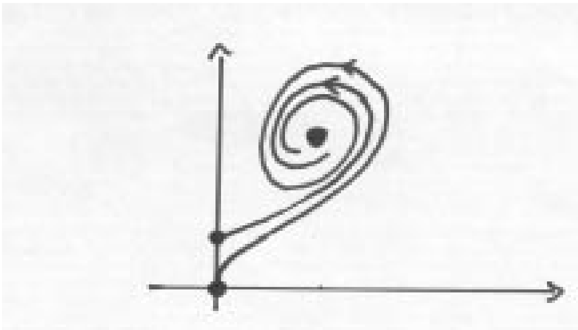
Eigenvalues of A are $\lambda = 2, -1$ and so $(0,1)$ is a saddle point.

Linearized equation at $(2,3)$ is $\underline{x}' = \begin{pmatrix} 0 & -2 \\ 3 & -3 \end{pmatrix} \underline{x} = A\underline{x}$.

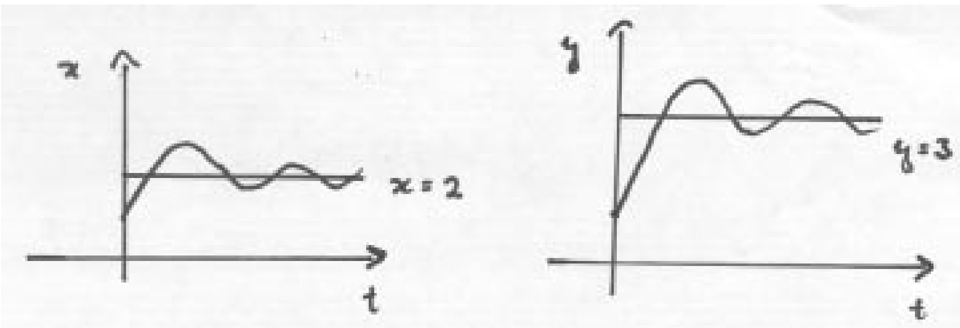
λ is an eigenvalue of A if and only if $\det \begin{vmatrix} -\lambda & -2 \\ 3 & -3-\lambda \end{vmatrix} = 0$,

i.e., $-\lambda\,(-3-\lambda)+6 = 0$, i.e., $\lambda^2+3\lambda+6 = 0$, i.e., $\lambda = \frac{-3\pm\sqrt{9-24}}{2} = -\frac{3}{2}\pm\frac{\sqrt{15}}{2}i$.

Hence $(2,3)$ is a stable spiral point. Thus possible phase plane is



When $x(0) = 1$ and $y(0) = 1$ we obtain



(b) $x' = 2y - xy^2$; $\quad y' = -x - y^3$.

Let $V(x,y) = ax^2 + by^2$. Then

$\frac{d}{dt}\left[V(x(t), y(t))\right] = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial t} = 2ax(2y - xy^2) + 2by(-x - y^3)$

$= (4a - 2b)xy - 2ax^2y^2 - 2by^4$

Choosing $a = 1$, $b = 2$, we obtain $V(x,y) = x^2 + 2y^2$ and

$\frac{d}{dt}[V(x(t), y(t))] = -2x^2y^2 - 4y^4 \leq 0$.

Thus $(0,0)$ is a stable equilibrium point.